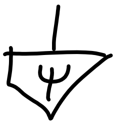
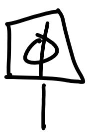
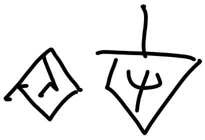
Quantum Processes & Computation

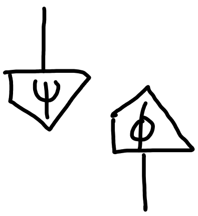
Model solutions, sheet 3

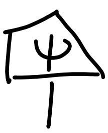
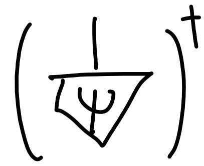
Oxford MT 2022

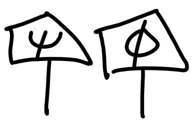
Ex 3.1

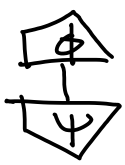
 $\longleftrightarrow \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix}$  $\longleftrightarrow \begin{pmatrix} \phi_0 & \phi_1 \end{pmatrix}$

(i)  $\longleftrightarrow \begin{pmatrix} \lambda\psi^0 \\ \lambda\psi^1 \end{pmatrix}$

(ii)  $\longleftrightarrow \begin{pmatrix} \psi^0\phi_0 & \psi^0\phi_1 \\ \psi^1\phi_0 & \psi^1\phi_1 \end{pmatrix}$

(iii)  $=$  $\longleftrightarrow \begin{pmatrix} \overline{\psi}^0 & \overline{\psi}^1 \end{pmatrix}$

so:  $\longleftrightarrow \begin{pmatrix} \overline{\psi}^0\phi_0 & \overline{\psi}^0\phi_1 & \overline{\psi}^1\phi_0 & \overline{\psi}^1\phi_1 \end{pmatrix}$

(iv)  $\longleftrightarrow \begin{pmatrix} \phi_0 & \phi_1 \end{pmatrix} \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} = \phi_0\psi^0 + \phi_1\psi^1$

Ex 3.2

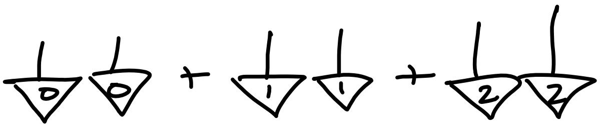
$$\cup \longleftrightarrow \begin{pmatrix}1\\0\\0\\1\end{pmatrix} \qquad \cap \longleftrightarrow (1\ 0\ 0\ 1)$$

(i) $(\cap \otimes 1) \circ (1 \otimes \cup)$

$$\longleftrightarrow \left[(1\ 0\ 0\ 1) \otimes \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right]\left[\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right]$$

$$= \begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\0 & 1 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\\0 & 0\\1 & 0\\0 & 1\\0 & 0\\0 & 0\\0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

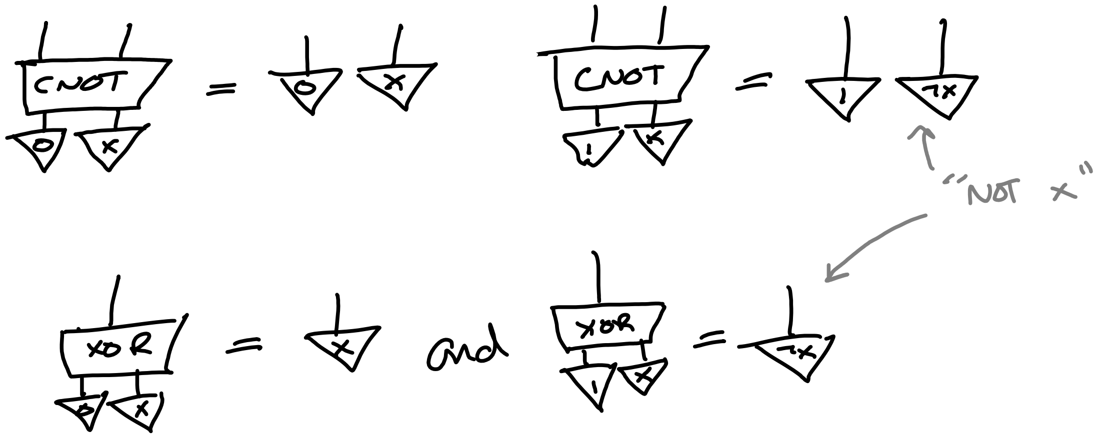
(ii) $\cup =$ 

$$\cup \longleftrightarrow \begin{array}{cl} \begin{pmatrix}1\\0\\0\\0\\1\\0\\0\\0\\1\end{pmatrix} & \begin{array}{l}00\\01\\02\\10\\11\\12\\20\\21\\22\end{array}\end{array} \qquad \cap \longleftrightarrow \begin{array}{c} \begin{array}{ccccccccc}00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22\end{array} \\ (1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1)\end{array}$$

n.b. these are the cols/rows of a 3x3 id matrix.

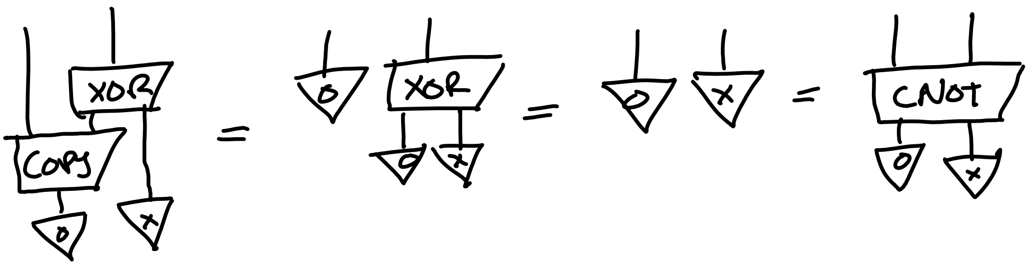
Ex 3.3

for $x \in \{0,1\}$:

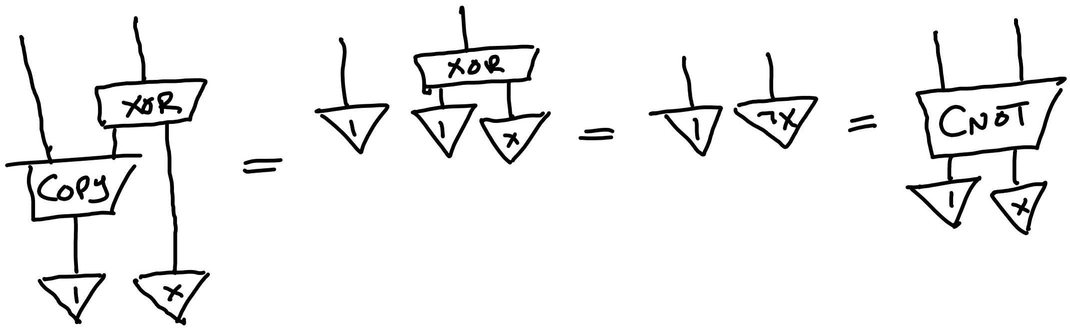


Note: and

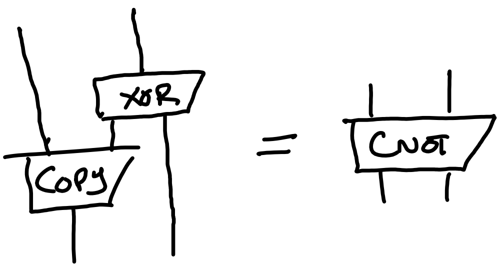
Then:



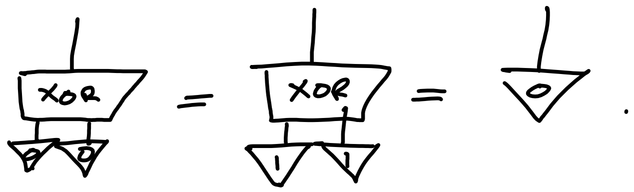
and



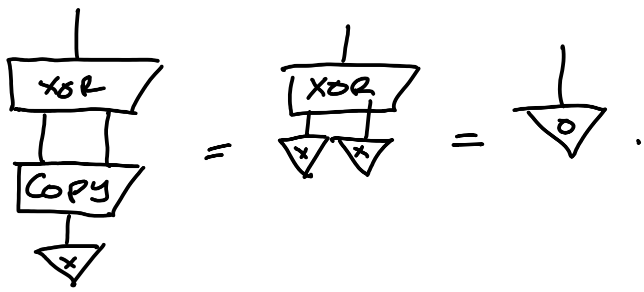
Hence:

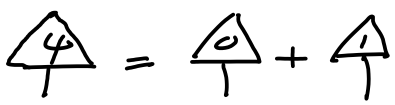
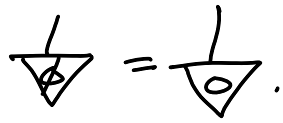


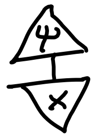
.

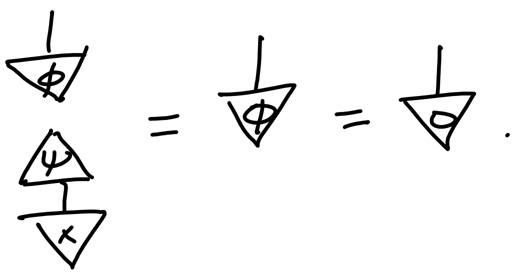
Note: 

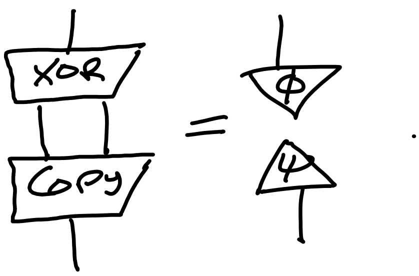
So, for $x \in \{0,1\}$:



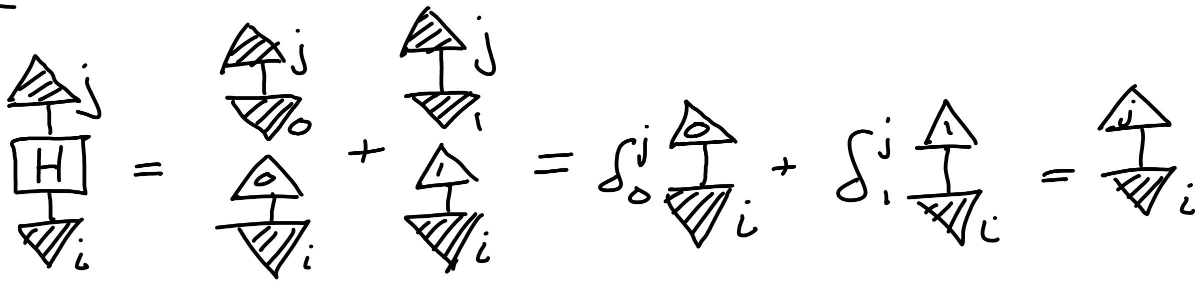
Let  and 

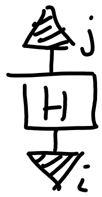
Then $\forall x \in \{0,1\}$.  $= 1$. So:



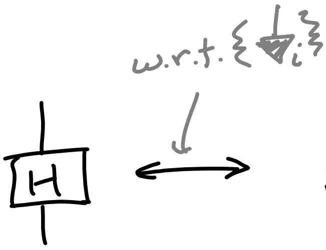
Hence 

Ex 3.4



So 

$$= \begin{cases} -\frac{1}{\sqrt{2}} & \text{if } i=j=1 \\ \frac{1}{\sqrt{2}} & \text{otherwise.} \end{cases}$$

Hence : 

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Ex 3.5

$$\bigtriangledown_0 = \text{double}\left(\tfrac{1}{\sqrt{2}}\left(\bigtriangledown_{0} + \bigtriangledown_{1}\right)\right) = \text{double}\Big(\underbrace{\cos\tfrac{\pi}{4}}_{=1/\sqrt{2}}\bigtriangledown_{0} + \underbrace{\sin\tfrac{\pi}{4}}_{=1/\sqrt{2}}\bigtriangledown_{1}\Big)$$

$$= \text{double}\left(\cos\tfrac{\theta}{2}\bigtriangledown_{0} + e^{i\phi}\cos\tfrac{\theta}{2}\bigtriangledown_{1}\right)$$

$$\text{for}\begin{cases}\theta = \pi/2\\ \phi = 0\end{cases}.$$

$$\bigtriangledown_1 = \text{double}\left(\cos\tfrac{\pi}{4}\bigtriangledown_{0} - \sin\tfrac{\pi}{4}\bigtriangledown_{1}\right) = \text{double}\left(\cos\tfrac{\theta}{2}\bigtriangledown_{0} + e^{i\phi}\sin\tfrac{\theta}{2}\bigtriangledown_{1}\right)$$

$$\text{for}\begin{cases}\theta = \pi/2\\ \phi = \pi\end{cases} \quad (e^{i\pi} = -1)$$

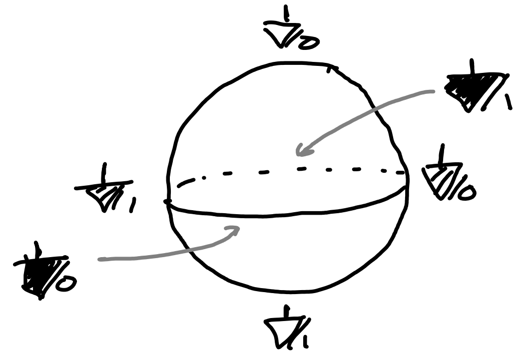
$$\blacktriangledown_0 = \text{double}\left(\tfrac{1}{\sqrt{2}}\left(\bigtriangledown_{0} + i\bigtriangledown_{1}\right)\right) = \text{double}\left(\cos\tfrac{\theta}{2}\bigtriangledown_{0} + e^{i\phi}\sin\tfrac{\theta}{2}\bigtriangledown_{1}\right)$$

$$\text{for}\begin{cases}\theta = \pi/2\\ \phi = \pi/2\end{cases} \quad (e^{i\pi/2} = i)$$

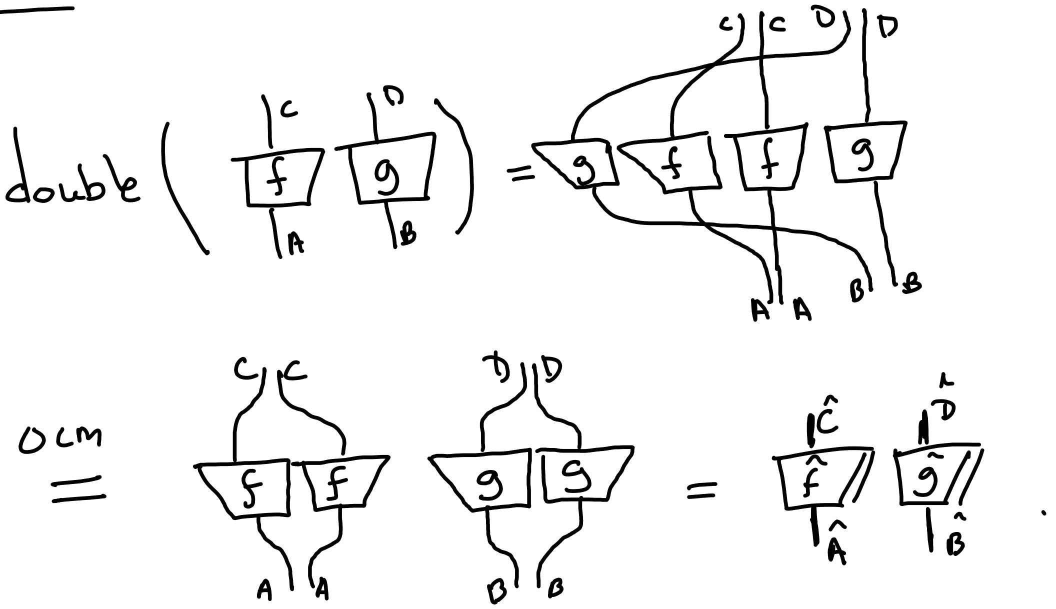
$$\blacktriangledown_1 = \text{double}\left(\tfrac{1}{\sqrt{2}}\left(\bigtriangledown_{0} - i\bigtriangledown_{1}\right)\right) = \text{double}\left(\cos\tfrac{\theta}{2}\bigtriangledown_{0} + e^{i\phi}\sin\tfrac{\theta}{2}\bigtriangledown_{1}\right)$$

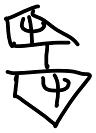
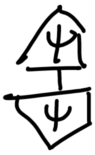
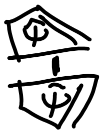
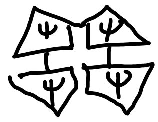
$$\text{for}\begin{cases}\theta = \pi/2\\ \phi = -\pi/2\end{cases} \quad (e^{-i\pi/2} = -i)$$

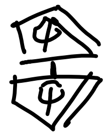
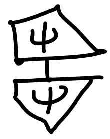
Plotting in spherical coordinates, we obtain:

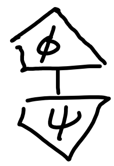
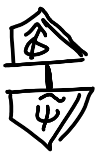
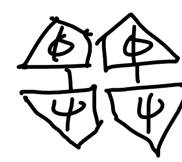


Ex 3.6

(i) 

(ii) Suppose  $= 1$. Then, by conjugating both sides,  $= 1$. So:  $=$  $= 1 \cdot 1 = 1$.

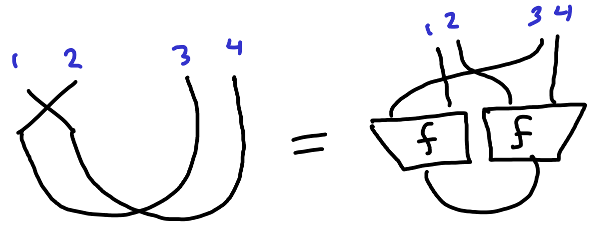
Conversely, suppose  $= 1$. Then by Theorem 6.17,  $= e^{i\alpha}$. But  is a positive number, so the only possibility is $\alpha = 0$, so  $= 1$.

(iii) Suppose  $= 0$, then  $=$  $= 0 \cdot 0 = 0$.

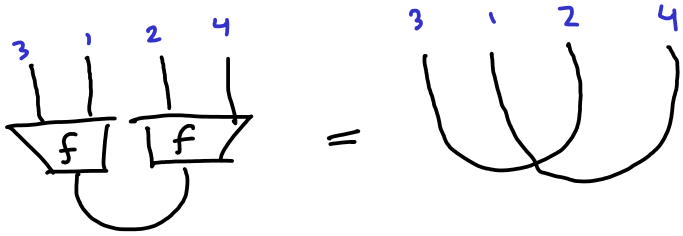
Conversely, if  $= 0$ then by 6.17,  $= e^{i\alpha} \cdot 0 = 0$.

Ex 3.7

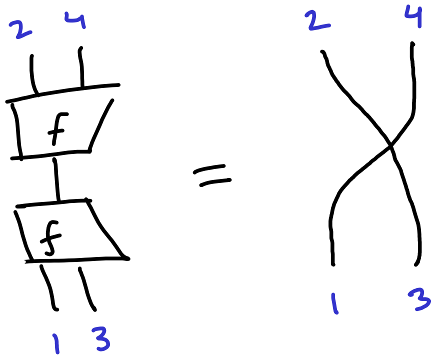
(i)



$\Rightarrow$

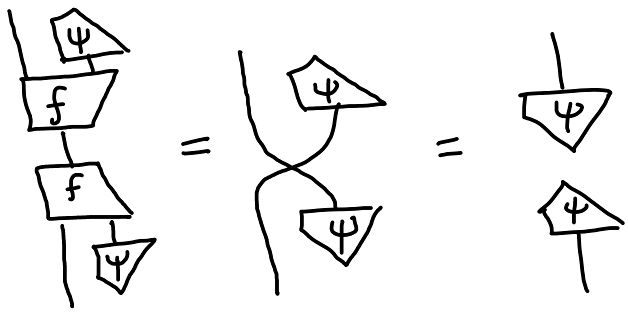


$\Rightarrow$

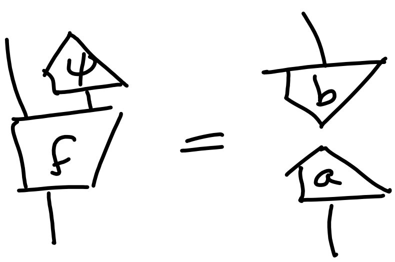


$(*)$

$\Rightarrow$



Hence, by Prop. 5.74, $\exists a, b$.



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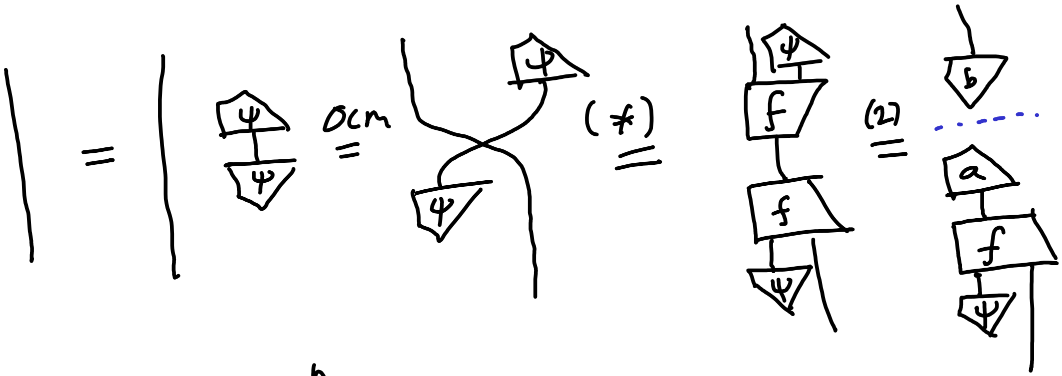
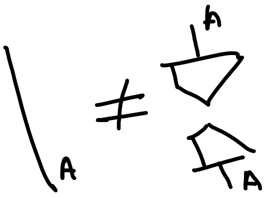
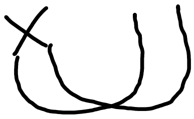
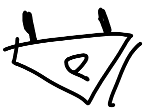
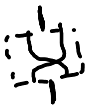
(ii) We'll use eq. (*) from the previous part, which follows from eq. (1).



Since  for $\dim A > 1$, 

is not ⊗-positive. Hence we have a quantum state  where  is not a quantum state, so  cannot be a quantum map.