

# Quantum Processes + Computation

Model solutions, sheet 3  
Oxford MT 2022

# Ex 3.1

$$\begin{array}{|c} \psi \\ \hline \end{array} \leftrightarrow \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} \quad \begin{array}{|c} \phi \\ \hline \end{array} \leftrightarrow (\phi_0 \ \phi_1)$$

$$(i) \quad \begin{array}{|c} \lambda \\ \hline \end{array} \begin{array}{|c} \psi \\ \hline \end{array} \leftrightarrow \begin{pmatrix} \lambda \psi^0 \\ \lambda \psi^1 \end{pmatrix}$$

$$(ii) \quad \begin{array}{|c} \psi \\ \hline \end{array} \begin{array}{|c} \phi \\ \hline \end{array} \leftrightarrow \begin{pmatrix} \psi^0 \phi_0 & \psi^0 \phi_1 \\ \psi^1 \phi_0 & \psi^1 \phi_1 \end{pmatrix}$$

$$(iii) \quad \begin{array}{|c} \psi \\ \hline \end{array} = \left( \begin{array}{|c} \psi \\ \hline \end{array} \right)^\dagger \leftrightarrow (\bar{\psi}^0 \ \bar{\psi}^1)$$

so:

$$\begin{array}{|c} \psi \\ \hline \end{array} \begin{array}{|c} \phi \\ \hline \end{array} \leftrightarrow (\bar{\psi}^0 \phi_0 \ \bar{\psi}^0 \phi_1 \ \bar{\psi}^1 \phi_0 \ \bar{\psi}^1 \phi_1)$$

$$(iv) \quad \begin{array}{|c} \phi \\ \hline \end{array} \begin{array}{|c} \psi \\ \hline \end{array} \leftrightarrow (\phi_0 \ \phi_1) \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} = \phi_0 \psi^0 + \phi_1 \psi^1$$

### Ex 3.2

$$\cup \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \cap \leftrightarrow (1001)$$

$$(i) (\cap \otimes 1) \circ (1 \otimes \cup)$$

$$\leftrightarrow \left[ (1001) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

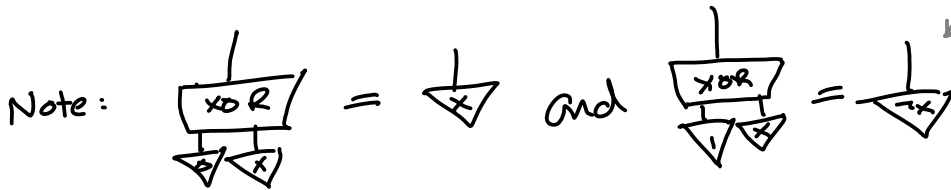
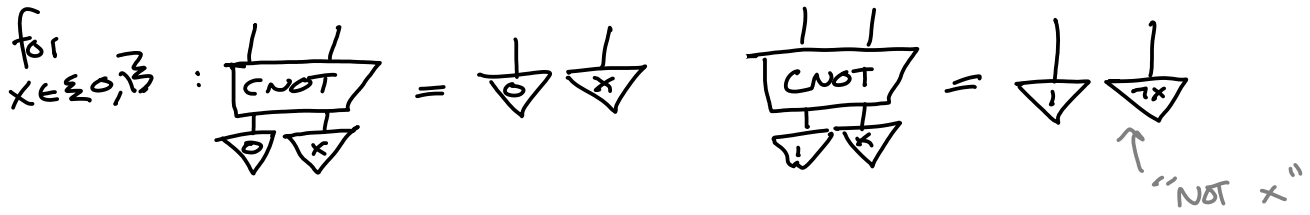
$$(ii) \cup = \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} + \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} + \begin{array}{|c|} \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline \end{array}$$

$$\cup \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{matrix} 00 \\ 01 \\ 02 \\ 10 \\ 11 \\ 12 \\ 20 \\ 21 \\ 22 \end{matrix}$$

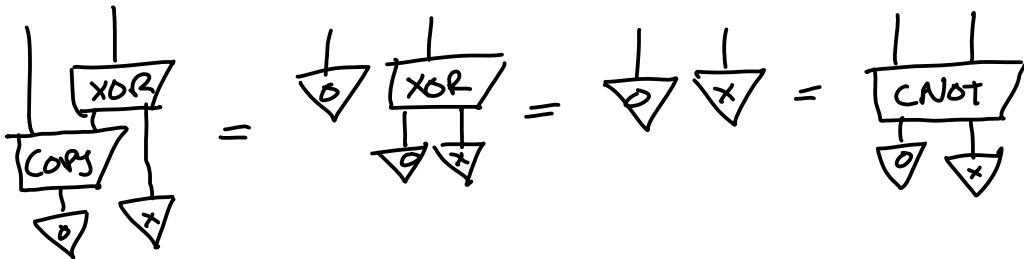
$$\cap \leftrightarrow \begin{matrix} 00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22 \\ (1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1) \end{matrix}$$

n.b. these are the cols/rows of a 3x3 id matrix.

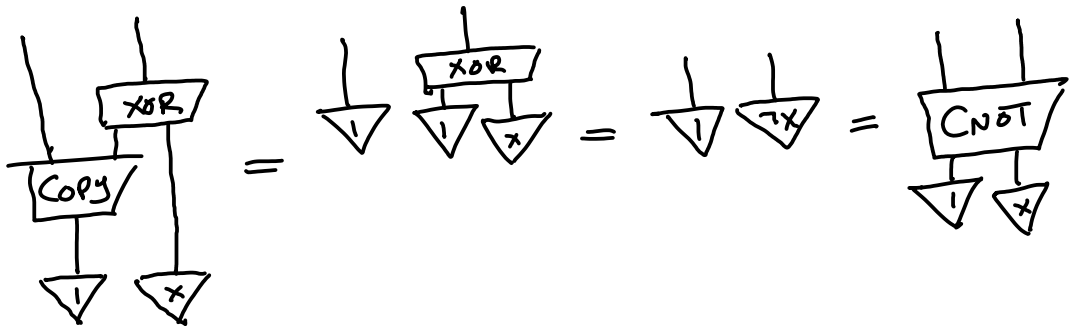
# Ex 3.3



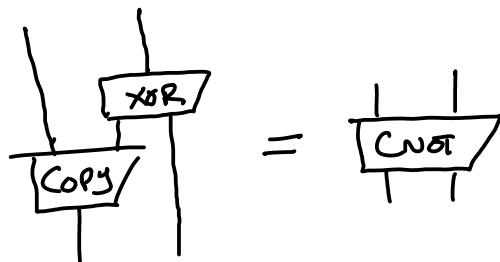
Then:

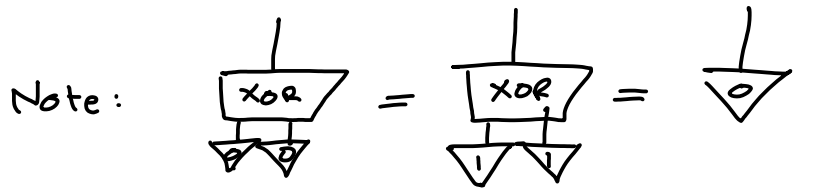


And

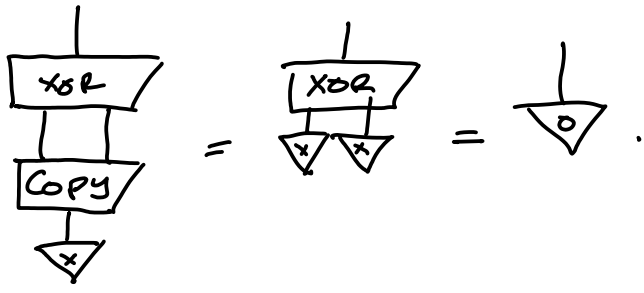


Hence :




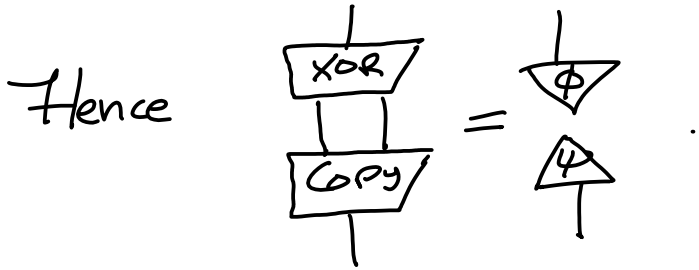
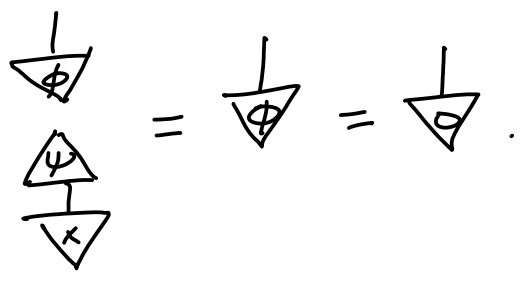


So, for  $x \in \{0, 1\}$ :



Let  $\uparrow_{\psi} = \uparrow_0 + \uparrow_1$  and  $\downarrow_{\phi} = \downarrow_0$ .

Then  $\forall x \in \{0, 1\}$ .  = 1. So:



### Ex 3.4

$$\begin{array}{c} \triangle^j \\ | \\ \square H \\ | \\ \nabla_i \end{array} = \begin{array}{c} \triangle^j \\ | \\ \nabla_0 \\ | \\ \triangle^j \\ | \\ \nabla_i \end{array} + \begin{array}{c} \triangle^j \\ | \\ \nabla_i \\ | \\ \triangle^j \\ | \\ \nabla_i \end{array} = \delta_0^j \begin{array}{c} \triangle^j \\ | \\ \square \\ | \\ \nabla_i \end{array} + \delta_i^j \begin{array}{c} \triangle^j \\ | \\ \square \\ | \\ \nabla_i \end{array} = \begin{array}{c} \triangle^j \\ | \\ \square \\ | \\ \nabla_i \end{array}$$

So

$$\begin{array}{c} \triangle^j \\ | \\ \square H \\ | \\ \nabla_i \end{array} = \begin{cases} -\frac{1}{\sqrt{2}} & \text{if } i=j=1 \\ \frac{1}{\sqrt{2}} & \text{otherwise.} \end{cases}$$

Hence :

$$\begin{array}{c} | \\ \square H \\ | \end{array} \xleftrightarrow{\text{w.r.t. } \left\{ \begin{array}{c} \triangle^j \\ | \\ \nabla_i \end{array} \right\}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

### Ex 3.5

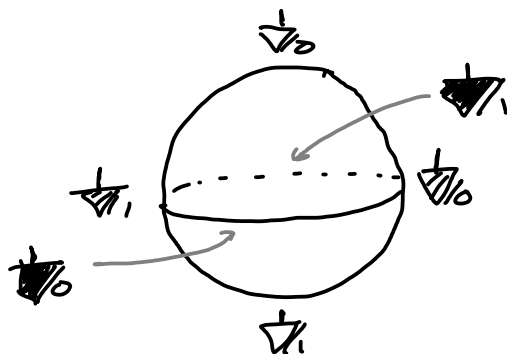
$$\begin{aligned}
 \downarrow_0 &= \text{double} \left( \frac{1}{\sqrt{2}} (\downarrow_0 + \downarrow_1) \right) = \text{double} \left( \overset{1/\sqrt{2}}{\cos \frac{\pi}{4}} \downarrow_0 + \overset{1/\sqrt{2}}{\sin \frac{\pi}{4}} \downarrow_1 \right) \\
 &= \text{double} \left( \cos \frac{\Theta}{2} \downarrow_0 + e^{i\phi} \sin \frac{\Theta}{2} \downarrow_1 \right) \\
 &\quad \text{for } \begin{cases} \Theta = \pi/2 \\ \phi = 0 \end{cases} .
 \end{aligned}$$

$$\begin{aligned}
 \downarrow_1 &= \text{double} \left( \cos \frac{\pi}{4} \downarrow_0 - \sin \frac{\pi}{4} \downarrow_1 \right) = \text{double} \left( \cos \frac{\Theta}{2} \downarrow_0 + e^{i\phi} \sin \frac{\Theta}{2} \downarrow_1 \right) \\
 &\quad \text{for } \begin{cases} \Theta = \pi/2 \\ \phi = \pi \quad (e^{i\pi} = -1) \end{cases}
 \end{aligned}$$

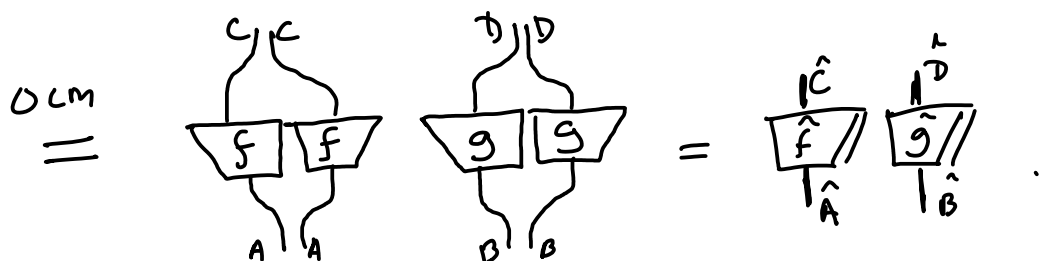
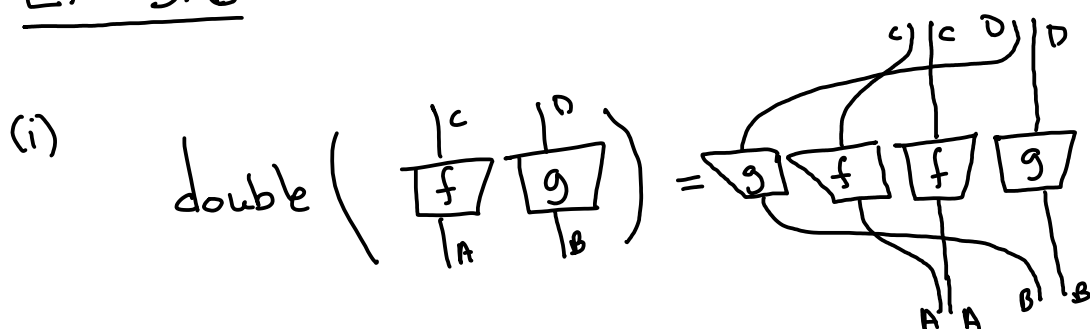
$$\begin{aligned}
 \downarrow_0 &= \text{double} \left( \frac{1}{\sqrt{2}} (\downarrow_0 + i \downarrow_1) \right) = \text{double} \left( \cos \frac{\Theta}{2} \downarrow_0 + e^{i\phi} \sin \frac{\Theta}{2} \downarrow_1 \right) \\
 &\quad \text{for } \begin{cases} \Theta = \pi/2 \\ \phi = \pi/2 \quad (e^{i\pi/2} = i) \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 \downarrow_1 &= \text{double} \left( \frac{1}{\sqrt{2}} (\downarrow_0 - i \downarrow_1) \right) = \text{double} \left( \cos \frac{\Theta}{2} \downarrow_0 + e^{i\phi} \sin \frac{\Theta}{2} \downarrow_1 \right) \\
 &\quad \text{for } \begin{cases} \Theta = \pi/2 \\ \phi = -\pi/2 \quad (e^{-i\pi/2} = -i) \end{cases}
 \end{aligned}$$

Plotting in spherical coordinates, we obtain:



# Ex 3.6



(ii) Suppose  $\begin{array}{c} \boxed{\phi} \\ \downarrow \\ \boxed{\psi} \end{array} = 1$ . Then, by conjugating both sides,  $\begin{array}{c} \boxed{\psi} \\ \downarrow \\ \boxed{\psi} \end{array} = 1$ . So:  $\begin{array}{c} \boxed{\phi} \\ \downarrow \\ \boxed{\psi} \end{array} = \begin{array}{c} \boxed{\psi} \boxed{\psi} \\ \downarrow \downarrow \\ \boxed{\psi} \boxed{\psi} \end{array} = 1 \cdot 1 = 1$ .

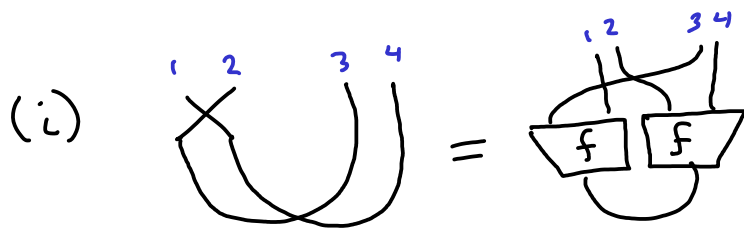
Conversely, suppose  $\begin{array}{c} \boxed{\phi} \\ \downarrow \\ \boxed{\phi} \end{array} = 1$ . Then by Theorem 6.17,  $\begin{array}{c} \boxed{\psi} \\ \downarrow \\ \boxed{\psi} \end{array} = e^{i\alpha}$ . But  $\begin{array}{c} \boxed{\psi} \\ \downarrow \\ \boxed{\psi} \end{array}$  is a positive number, so the only possibility is  $\alpha=0$ , so  $\begin{array}{c} \boxed{\psi} \\ \downarrow \\ \boxed{\psi} \end{array} = 1$ .

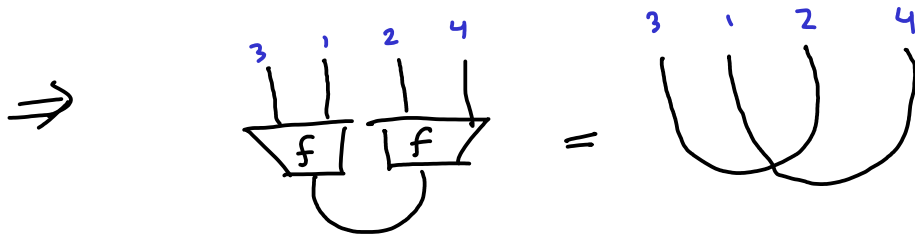
(iii) Suppose  $\begin{array}{c} \boxed{\phi} \\ \downarrow \\ \boxed{\psi} \end{array} = 0$ , then  $\begin{array}{c} \boxed{\phi} \\ \downarrow \\ \boxed{\psi} \end{array} = \begin{array}{c} \boxed{\psi} \boxed{\phi} \\ \downarrow \downarrow \\ \boxed{\psi} \boxed{\psi} \end{array} = 0 \cdot 0 = 0$

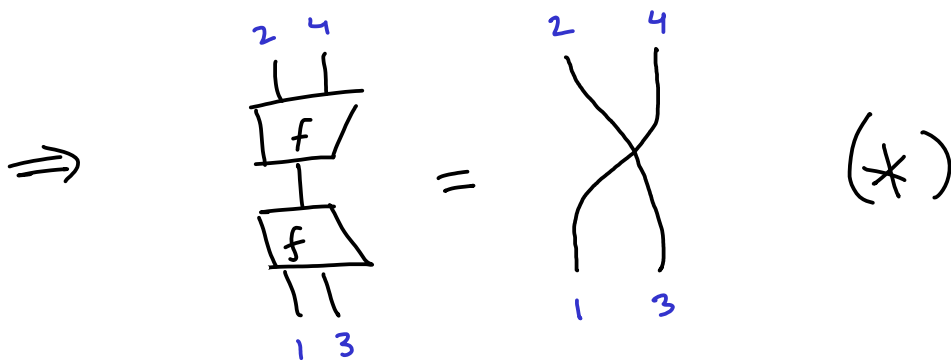
Conversely, if  $\begin{array}{c} \boxed{\psi} \\ \downarrow \\ \boxed{\psi} \end{array} = 0$  then by 6.17,  $\begin{array}{c} \boxed{\phi} \\ \downarrow \\ \boxed{\psi} \end{array} = e^{i\alpha} \cdot 0 = 0$ .

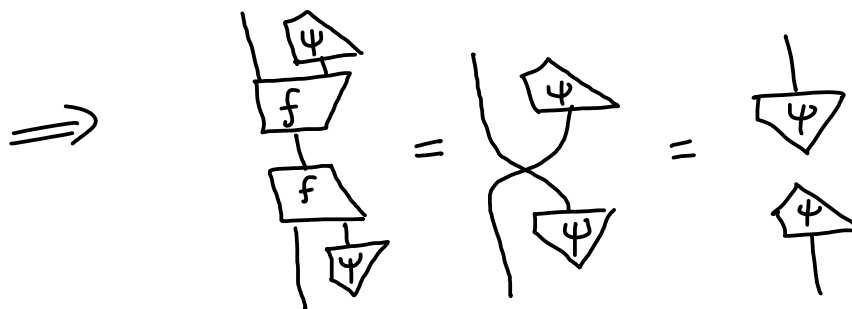


# Ex 3.7

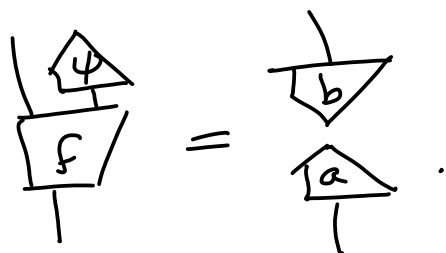
(i) 

$\Rightarrow$  

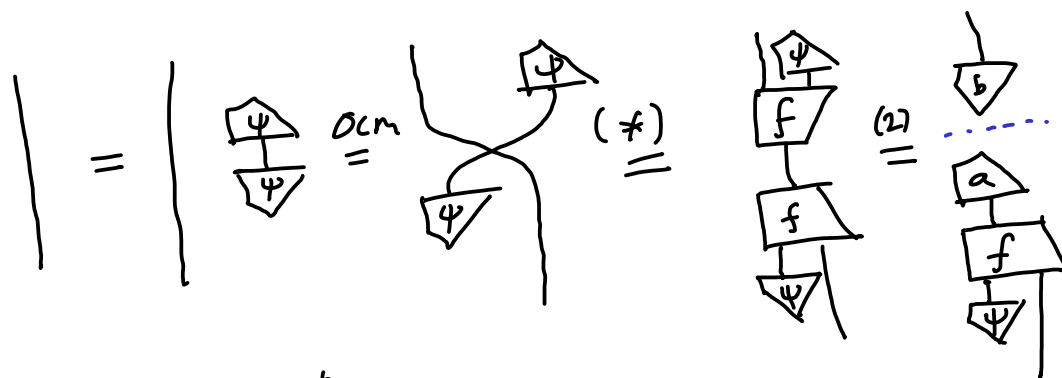
$\Rightarrow$   (\*)

$\Rightarrow$  

Hence, by Prop. 5.74,  $\exists a, b$ .



(ii) We'll use eq. (\*) from the previous part, which follows from eq. (1).



Since  $\downarrow_A \neq \begin{array}{c} \triangle^A \\ \triangle_A \end{array}$  for  $\dim A > 1$ ,  $\cup$

is not  $\otimes$ -positive. Hence we have a

quantum state  $\begin{array}{c} \triangle \\ \triangle \end{array}$  where  $\begin{array}{c} \triangle \\ \triangle \end{array}$  is not

a quantum state, so  $\begin{array}{c} \triangle \\ \triangle \end{array}$  cannot be a

quantum map.