**Quantum Processes and Computation**

**Assignment 1, Friday 13 Oct 2023**

**Deadline:** Class in week 3 (Ask your teacher for weekly marking deadline.)

**Goals:** After completing these exercises successfully you should be able to perform simple diagrammatic computations and work with the process theories of **functions** and **relations**.

**Note:** Many of these exercises also appear in *Picturing Quantum Processes*, but sometimes they have been slightly modified for the problem sheet. The corresponding exercise number from the book is shown in brackets.

**Exercise 1 (3.4):** We saw in the lecture that **functions** and **relations** are examples of process theories. Give two other examples of a process theory. For each one answer the following questions:

1. What are the system-types?
2. What are the processes?
3. What does it mean to compose them sequentially or in parallel?
4. When should two processes be considered equal?

**Hint:** Note that a single process is not a process theory. In particular, almost any process theory will have an infinite amount of system types (e.g. $A, A \otimes A, A \otimes A \otimes A, \ldots$). Also: Be creative! You don’t have to restrict yourself to mathematics.

**Exercise 2 (3.10):** Read Section 3.1.3 about diagrams as diagram formulas. Draw the diagrams corresponding to the following diagram formulas:

1. $f_{B_3C_2}^{C_4}g_{C_4}^{D_3}$
2. $f_{A_3}^{A_1}$
3. $g_{B_3}^{A_1}f_{A_3}^{A_1}$
4. $1_{A_3}^{A_1}1_{A_3}^{A_1}1_{A_3}^{A_1}$

Use the convention that inputs and outputs are numbered from left-to-right.

**Exercise 3 (3.12):** Give the diagrammatic equations of a process * taking two inputs and one output that express the algebraic properties of being

1. associative: $x \ast (y \ast z) = (x \ast y) \ast z$
2. commutative: $x \ast y = y \ast x$
3. having a unit: there exists a process $e$ (with no inputs) such that $x \ast e = e \ast x = x$

**Note:** $x, y$ and $z$ should not appear in your final diagrams. They are however useful in trying to figure out what the diagrammatic equation should be.

**Exercise 4 (3.15):** Using the copy operation:

```
\begin{verbatim}
\hline
| cp | \hline
\end{verbatim}
```

$:: n \mapsto (n, n)$
write down the diagram representing distributivity: \((x + y) \cdot z = (x \cdot z) + (y \cdot z)\)? Here, + and \(*\) are processes that take two inputs and one output.

**Exercise 5 (3.30):** First compute the values of the following functions, then give the commonly used names of these functions:

\[(a)\] 
\[
\text{NOT} \quad \text{NOT} \\
\text{CNOT} \\
\text{NOT}
\]

\[(b)\] 
\[
\text{CNOT} \\
\text{CNOT}
\]

where:

\[
\text{NOT} :: \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 0 \end{cases} 
\]

and

\[
\text{CNOT} :: \begin{cases} (0,0) \mapsto (0,0) \\ (0,1) \mapsto (0,1) \\ (1,0) \mapsto (1,1) \\ (1,1) \mapsto (1,0) \end{cases} 
\]

**Exercise 6 (3.31):** Suppose \(A, B, C,\) and \(D\) are sets and \(P\) is a relation given by:

\[
A = \{a_1, a_2, a_3\} \\
B = \mathbb{B} \\
C = \{\text{red, green}\} \\
D = \mathbb{N}
\]

Compute \(P\) first for \(R, S, T\) given by:

\[
R :: \begin{cases} 1 \mapsto (a_1, a_1) \\ 1 \mapsto (a_1, a_2) \end{cases} \\
S :: \begin{cases} (a_1, 5) \mapsto (0, \text{red}) \\ (a_1, 5) \mapsto (1, \text{red}) \\ (a_2, 6) \mapsto (1, \text{green}) \end{cases} \\
T :: \begin{cases} a_1 \mapsto 200 \\ a_3 \mapsto 5 \end{cases}
\]

and then for \(R, S, T\) given by:

\[
R :: \begin{cases} 0 \mapsto A \times \{a_2, a_3\} \\ 1 \mapsto A \times \{a_2, a_3\} \end{cases} \\
S :: \begin{cases} (a_1, 0) \mapsto \mathbb{B} \times \{\text{red, green}\} \\ (a_1, 1) \mapsto \mathbb{B} \times \{\text{red, green}\} \\ (a_1, 2) \mapsto \mathbb{B} \times \{\text{red, green}\} \\ : \end{cases} \\
T :: \begin{cases} a_1 \mapsto \mathbb{N} \\ a_2 \mapsto \mathbb{N} \\ a_3 \mapsto \mathbb{N} \end{cases}
\]

**Hint:** This exercise is in fact well-defined, and does not contain typos. Please read Section 3.3.3 if you are confused.

**Exercise 7 (3.38 & 3.40):** Suppose that there is a zero process \(0 : A \to B\) for all possible types \(A\) and \(B\) (see Section 3.4.2).

(a) Show that the family of zero processes is unique. That is, show that if there exists another family of zero processes \(0' : A \to B\) for all types \(A, B\) such that \(0' \circ f = 0' = f \circ 0'\) for all processes \(f\), then for all \(A, B, 0 : A \to B,\) and \(0' : A \to B\) we have \(0 = 0'\).

(b) We call two processes \(f\) and \(g\) with the same inputs and outputs equal up to a number (written \(f \approx g\)) if there exist non-zero numbers \(\lambda, \mu\) such that \(\lambda f = \mu g\). Suppose a process theory has no zero divisors. That is, it satisfies the following property: \(\lambda f = 0\) if and only if \(\lambda = 0\) or \(f = 0\). Show that \(f \approx 0\) if and only if \(f = 0\).