Quantum Processes and Computation

Assignment 2, Friday, 20 Oct, 12:00

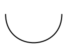
Deadline: Monday Week 4

Goals: After completing these exercises you should know how to reason with the transpose, adjoints, and the conjugate, and work with projections, unitaries and isometries. Material covered in book: Chapter 4.

Note: Many of these exercises also appear in *Picturing Quantum Processes*, but sometimes they have been slightly modified for the problem sheet. The corresponding exercise number from the book is shown in brackets.

Exercise 1 (4.10 & 4.16):

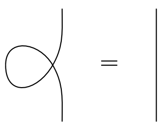
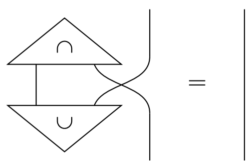
(a) Prove that in **relations**, the following relations on a set A:

 $:: * \mapsto \{(a,a) \mid a \in A\}$  $:: \forall a \in A : (a,a) \mapsto *$

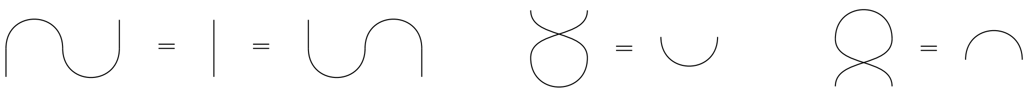
satisfy the *yanking equations* (eq. 4.11 in the book), and thus that **relations** has process-state duality.

(b) Show that process-state duality does not hold for **functions**.

Exercise 2 (4.12): Prove that

 or written differently: 

follows from the following 4 equations:

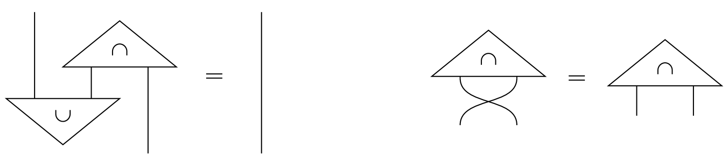


Hint: Use the second notation (with the boxes) as it prevents you from accidentally cheating.

Exercise 3 (4.14 in online version of PQP): Show that the following are equivalent:

(i) a state and an effect satisfying:

(ii) a state and an effect satisfying:

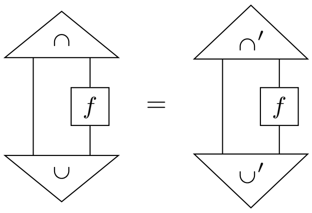


So in particular, if either eqs. (i) or eqs. (ii) hold, then all equations hold.

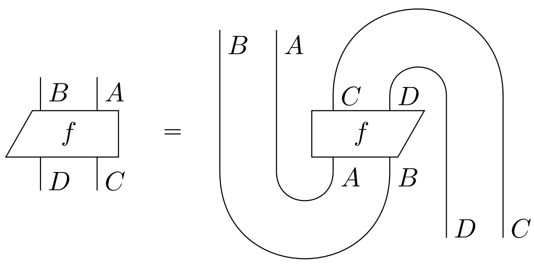
Exercise 4 (4.59): An *inverse* for a process $f : A \to B$ is a process $f^{-1} : B \to A$ such that $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$. Show that for a process f the following are equivalent:

- f is unitary.
- f is an isometry and has an inverse.
- $f^\dagger$ is an isometry and has an inverse.

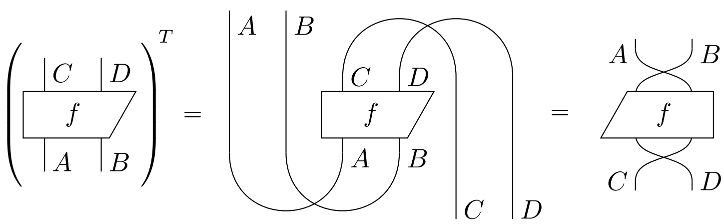
Exercise 5 (4.37): Show that the trace of a process is independent of the particular choice of cup and cap, i.e. that if $\cup$ and $\cap$ satisfy the yanking equations, but $\cup'$ and $\cap'$ also satisfy it that then:



In the lecture we saw the transpose. If a system has multiple inputs and outputs then there are two choices for the transpose. There is the *diagrammatic* transpose:



and the *algebraic* transpose:



(for more details see section 4.2.2 of the book)

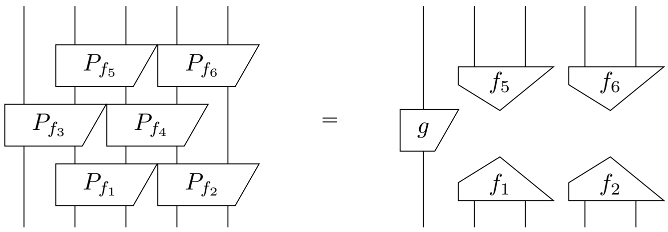
Exercise 6 (4.39): The adjoint is an operation which acts like a vertical reflection. Show that the algebraic transpose is in this regard an adjoint, i.e. show diagrammatically that:

1. It is an involution: $(f^T)^T = f$,
2. It preserves parallel composition: $(f \otimes g)^T = f^T \otimes g^T$,
3. It preserves sequential composition: $(f \circ g)^T = g^T \circ f^T$,

4. It preserves the identities: $\text{id}^T = \text{id}$,
5. It sends cups to caps: $\cup^T = \cap$,
6. It reverse the direction of swaps: $\text{SWAP}_{A,B}^T = \text{SWAP}_{B,A}$

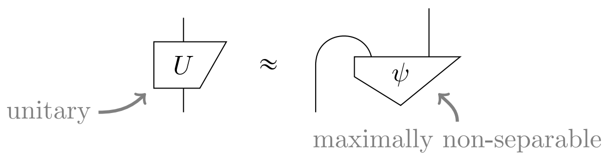
For a process $f : A \to A$ we define its *separable projector* by

Exercise 7 (4.73): Given processes $f_i : A \to A$ find the process g such that:

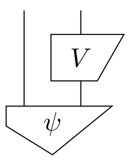


Write g as a sequential composition of the conjugates, transposes and adjoints of the f_i's.
Hint: Doing exercise 4.73 from the book first might reveal whether you understand the concept.

Exercise 8 (4.82): A state ψ is *maximally non-separable* if it corresponds to a unitary U by process-state duality, up to a number:



Show (i) that if one applies a unitary V to one side of a maximally non-separable state:



that one again obtains a maximally non-separable state, and (ii) that this unitary can always be chosen such that the resulting state is the cup (up to a number).