Quantum Processes and Computation
Assignment 3, Friday, 27 Oct, 12:00

Deadline: Monday Week 5

Goals: After completing these exercises you should know about (orthonormal) bases, composition of linear maps, encoding logic gates as linear maps, pure quantum states and maps, and $\otimes$-positivity.

Note: Many of these exercises also appear in Picturing Quantum Processes, but sometimes they have been modified for the problem sheet. The corresponding exercise number from the book is shown in brackets. If you are stuck, try looking up the exercise number in the book. Usually the definitions or equations you need are nearby.

Exercise 1 (5.54): Let
\[ \psi \rightarrow \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \quad \text{and} \quad \phi \rightarrow (\phi_0 \quad \phi_1) \]
be respectively a 2-dimensional state, and 2-dimensional effect. Let $\lambda$ be a number. Write the matrices for the processes

(i) \[ \begin{array}{c}
\begin{array}{c}
\psi \\
& \uparrow
\end{array}
\end{array} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

Exercise 2 (5.58): The matrices for cups and caps in 2 dimensions are:
\[ \begin{array}{c}
\begin{array}{c}
\cup \\
& \uparrow
\end{array}
\end{array} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
\cap \\
& \uparrow
\end{array}
\end{array} \leftrightarrow (1 \quad 0 \quad 0 \quad 1) \]

(i) First, verify the yanking equation
\[ \begin{array}{c}
\begin{array}{c}
\cup \\
& \uparrow
\end{array}
\end{array} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
\cap \\
& \uparrow
\end{array}
\end{array} \leftrightarrow (1 \quad 0 \quad 0 \quad 1) \]

(ii) Second, give the matrices for the cup and cap in 3 dimensions.

The next exercise is about encoding classical logic gates in the theory of linear maps, as explained in Section 5.3.4. Recall that a classical logic gate $F$ can be encoded as a linear map
Using this encoding, we defined:

\[
\begin{align*}
\text{XOR} & = 0 \quad 0 \quad 0 \quad 0 + 0 \quad 1 \quad 1 \quad 1 + 1 \quad 0 \quad 0 \quad 0 \quad 0 \\
\text{CNOT} & := 0 \quad 0 \quad 0 \quad 0 + 0 \quad 1 \quad 1 \quad 0 + 1 \quad 0 \quad 0 \quad 0 \\
\text{COPY} & := 0 \quad 0 \quad 0 \quad 0 + 1 \quad 0 \quad 0 \quad 0 \\
\end{align*}
\]

**Exercise 3 (5.86):** Show that

\[
\begin{array}{c}
\text{CNOT} \\
\text{COPY} \\
\end{array}
\]

(Hint: try comparing the LHS to the RHS on all basis states, rather than writing out a big sum.)

Next, find \(\psi\) and \(\phi\) such that the following equation holds:

\[
\begin{align*}
\text{XOR} & = 0 \\
\text{COPY} & = 0 \\
\end{align*}
\]

Although it might not look like much now, this equation will turn out to lie at the heart of the notion of *complementarity* which we will cover in great depth in the coming lectures.

**Exercise 4 (5.93):** In the proof of proposition 5.92, we see the Hadamard process written in matrix form with respect to the Z basis:

\[
\begin{align*}
H & = \left( \begin{array}{cc}
0 \quad 1 \\
1 \quad 0 \\
\end{array} \right) \\
& = \frac{1}{\sqrt{2}} \left( \begin{array}{cccc}
0 \quad 1 \\
1 \quad 0 \\
1 \quad 0 \\
0 \quad 1 \\
\end{array} \right)
\end{align*}
\]

From this we can conclude the matrix of \(H\) (with respect to the Z-basis) is:

\[
\begin{align*}
H & \leftrightarrow \frac{1}{\sqrt{2}} \left( \begin{array}{cc}
1 \quad 0 \\
1 \quad -1 \\
\end{array} \right)
\end{align*}
\]
What is the matrix of $H$ in the $X$-basis?

In section 6.1.2, it was shown that 2D quantum pure states correspond to points on a sphere.

**Exercise 5 (6.7):** Show that the following points:

\[
\begin{align*}
\downarrow & := \text{double} \left( \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ 1 \\
\end{array} \right) \right) \\
\downarrow & := \text{double} \left( \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ -1 \\
\end{array} \right) \right) \\
\uparrow & := \text{double} \left( \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ i \\
\end{array} \right) \right) \\
\downarrow & := \text{double} \left( \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -i \\
\end{array} \right) \right)
\end{align*}
\]

are located on the Bloch sphere as follows:

**Exercise 6 (6.10 & 6.22):**

(i) Show that doubling preserves parallel composition:

\[
\text{double} \left( \begin{array}{c} f \\ g \\
\end{array} \right) = \begin{array}{c} f \\ g \end{array}
\]

(ii) Show that doubling preserves normalisation: that a state $\psi$ is normalised if and only if its doubled state $\hat{\psi}$ is normalised.

(iii) Show that doubling preserves orthogonality: that states $\psi$ and $\phi$ are orthogonal if and only if $\hat{\psi}$ and $\hat{\phi}$ are orthogonal.

**Hint:** Use theorem 6.17 for the latter two points.

The transpose of a positive process is again a positive process and by bending some wires we can also take the ‘transpose’ of a $\otimes$-positive state, i.e. of a quantum state (see **Corollary 6.36**). This transpose acts as a swap of wires on the doubled system:
and it indeed sends quantum states to quantum states:

In the next exercise we will show that nevertheless, this swap of wires is not a quantum operation.

**Exercise 7:** In this exercise we will show that a swap applied to one pair of the wires of the doubled cup state will result in a state that is no longer ⋆-positive, and therefore not a quantum state. We will do this by contradiction. So suppose:

(1)

for some process $f$.

(i) Let $\psi$ be a normalised state. Show that the equation above implies that

and hence, by Proposition 5.74, that there exist states $a$ and $b$ such that:

(2)

(ii) Plug $\psi$ into equation 1 and use equation 2 to show that the identity wire disconnects. Conclude that therefore the swap can’t be a quantum map.

**Note:** In proposition 6.48 it is also shown that the swap is not a quantum operation, but it uses a specific counter-example found in linear maps. The proof above only uses string diagrams and the property implied by proposition 5.74.