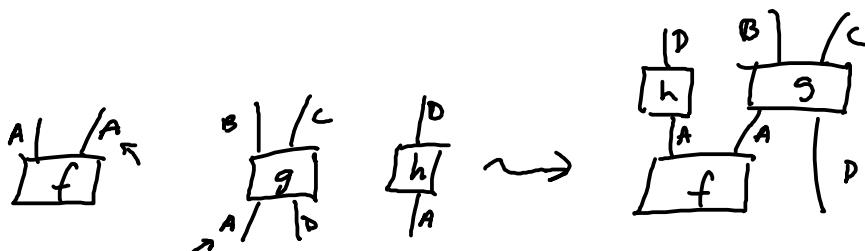
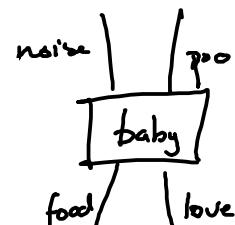
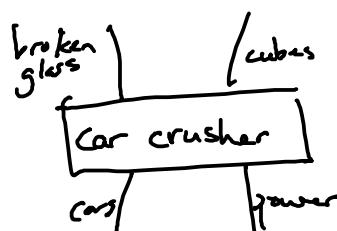
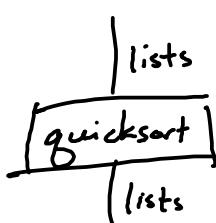
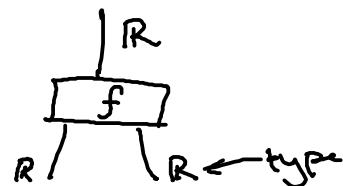


LECTURE 1

3.1 Processes

Def A process is anything with 0 or more inputs and outputs.

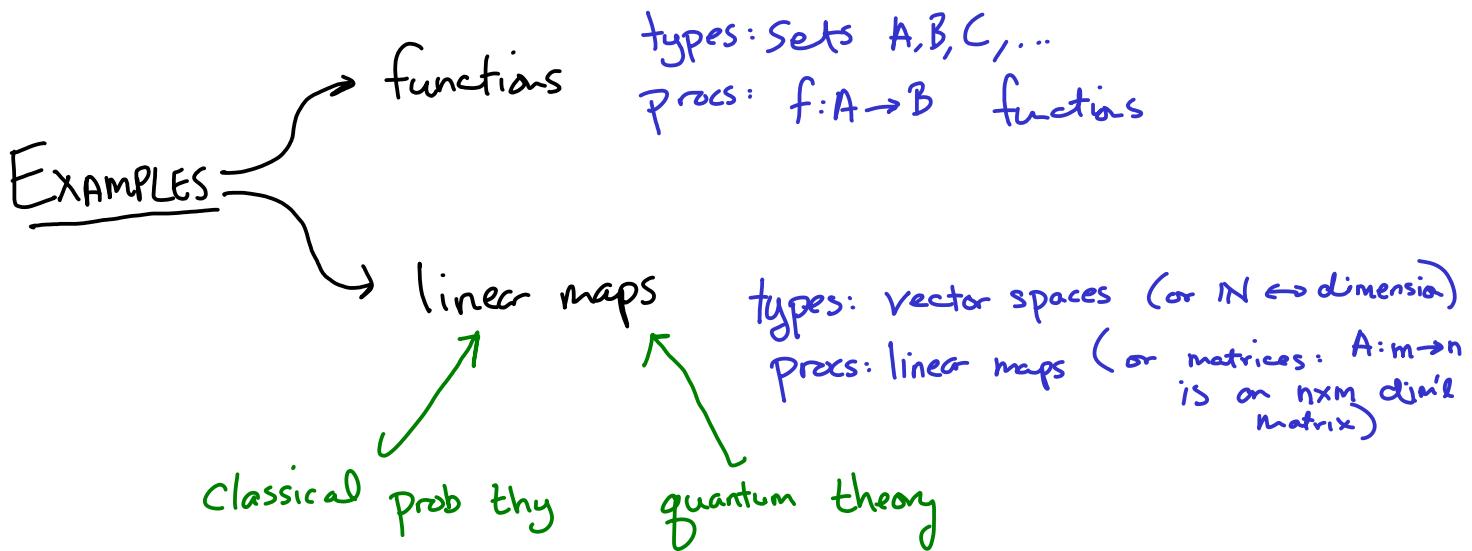
Ex $f(x, y) = x^2 + y$



Def A process theory consists of:

- (i) a collection T of system-types
- (ii) a collection P of processes
- (iii) a means of composing diagrams of processes.





The Golden Rule of Process Theories:

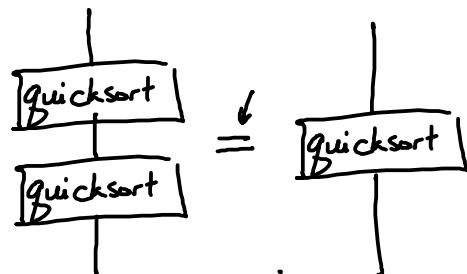
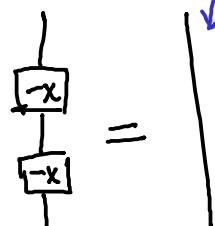
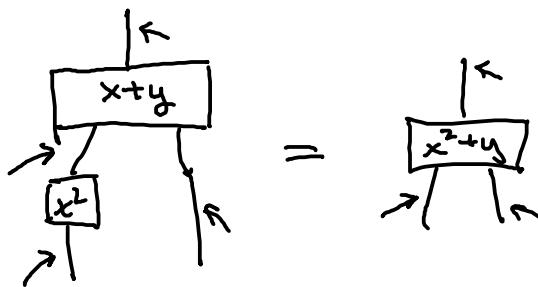
ONLY CONNECTIVITY MATTERS (ocm)



- These processes are equal because their diagrams are equal. However, we can also have multiple diagrams that describe the same process.

$$f(x,y) = x+y$$

"do nothing" / identity process

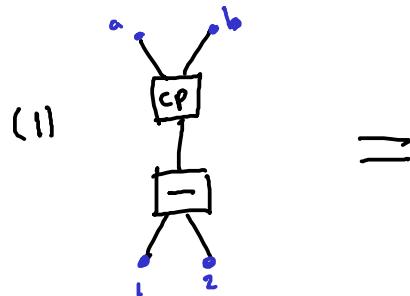


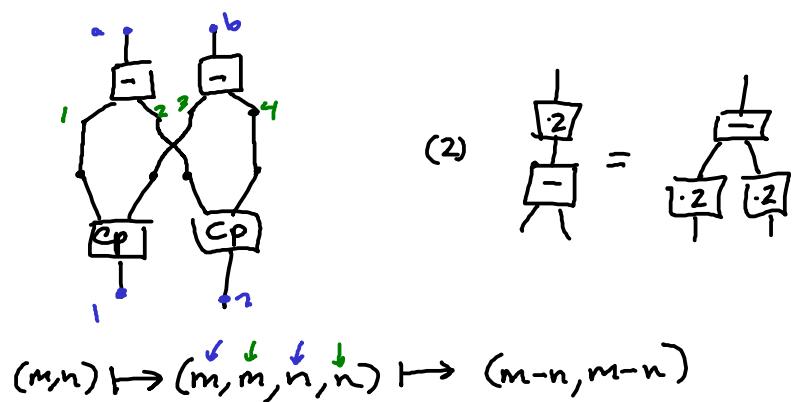
DIAGRAMMATIC REASONING := "using equations between diagrams to prove stuff"

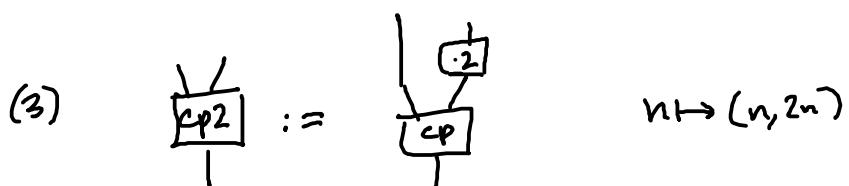
Ex $\therefore (m,n) \mapsto m-n$

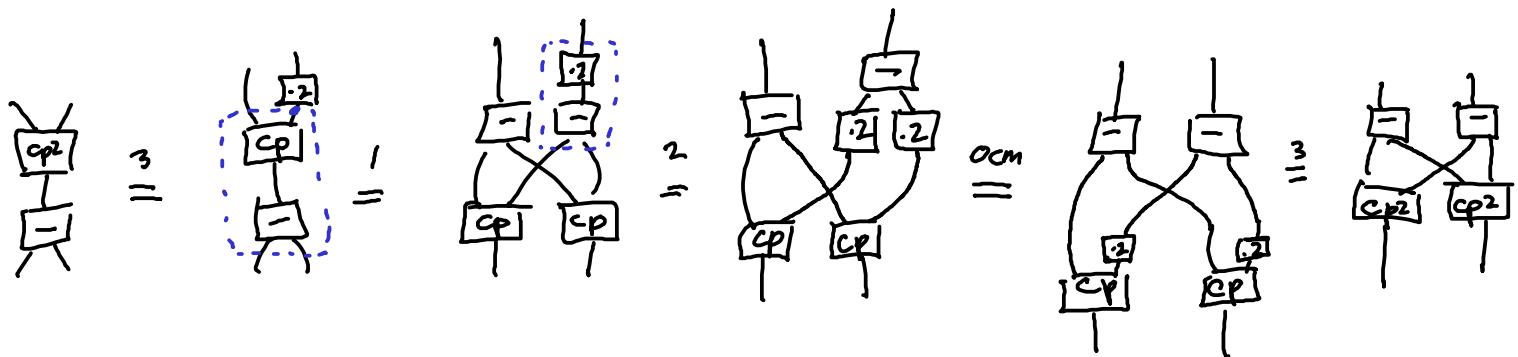
$\therefore m \mapsto 2m$

$\therefore n \mapsto (n,n)$



$$(m,n) \mapsto m-n \mapsto (m-n, m-n)$$


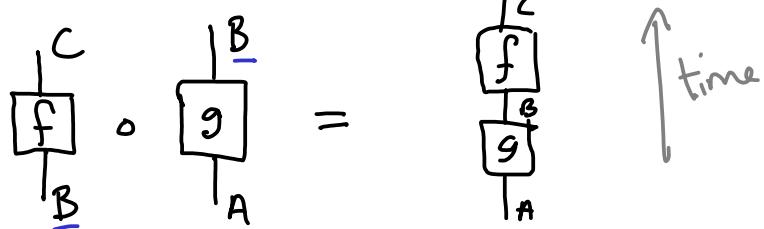
$$(m,n) \mapsto (m, m, n, n) \mapsto (m-n, m-n)$$


$$n \mapsto (n, 2^n)$$


3.2 Circuit diagrams

Sequential composition

$f \circ g :=$ "f after g"



- associative

$$\boxed{c} \circ (\boxed{b} \circ \boxed{a}) = \boxed{\begin{matrix} c \\ b \\ a \end{matrix}} = (\boxed{c} \circ \boxed{b}) \circ \boxed{a}$$

- Has a unit:

\boxed{a} ← "do nothing / identity" process

$$\boxed{b} \circ \boxed{\begin{matrix} B \\ f \\ A \end{matrix}} = \boxed{\begin{matrix} B \\ f \\ A \end{matrix}} = \boxed{\begin{matrix} B \\ f \\ A \end{matrix}} = \boxed{\begin{matrix} B \\ f \\ A \end{matrix}} \circ \boxed{a}$$

* For functions: $(f \circ g)(x) = f(g(x))$
function composition

* For matrices:

$$A \circ B = AB$$

matrix multiplication

Parallel composition : $f \otimes g$ "f while g"

$$\left(\begin{array}{c} f \\ \sqcap \\ \top \end{array} \right) \otimes \left(\begin{array}{c} g \\ \sqcap \\ k \end{array} \right) = \begin{array}{c} \top \\ g \\ f \\ k \end{array}$$

- * associative $(\begin{array}{c} \top \\ f \\ \sqcap \end{array} \otimes \begin{array}{c} \top \\ g \\ \sqcap \end{array}) \otimes \begin{array}{c} \top \\ h \\ \sqcap \end{array} = \begin{array}{c} \top \\ \top \\ f \\ g \\ h \\ \sqcap \end{array} = \begin{array}{c} \top \\ f \\ \top \\ g \\ \sqcap \\ h \end{array} = \begin{array}{c} \top \\ f \\ \otimes \\ (\begin{array}{c} \top \\ g \\ \sqcap \end{array} \otimes \begin{array}{c} \top \\ h \\ \sqcap \end{array}) \end{array}$

- * unit $\begin{array}{c} \top \\ f \\ \otimes \\ \sqcap \end{array} = \begin{array}{c} \top \\ f \end{array} = \begin{array}{c} \sqcap \\ \otimes \\ \top \\ f \end{array}$

- * The order matters. $\begin{array}{c} \top \\ f \\ \otimes \\ g \end{array} = \begin{array}{c} \top \\ f \\ g \end{array} \neq \begin{array}{c} \top \\ g \\ f \end{array} =: \begin{array}{c} \top \\ g \\ \otimes \\ f \end{array}$



- * We can also form joint systems.

B, C system-types $\Rightarrow B \otimes C$ is a system-type

$$\begin{array}{c} \top \\ f \\ \otimes \\ A \end{array} = \begin{array}{c} \top \\ B \\ / \\ C \\ f \\ A \end{array}$$

- * We write the trivial system as I . $A \otimes I = A = I \otimes A$

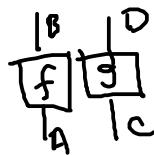
$$\begin{array}{c} \top \\ f \\ \otimes \\ A \end{array} = \begin{array}{c} \top \\ f \\ A \end{array}$$

$$\begin{array}{c} \top \\ I \\ \otimes \\ A \end{array} = \begin{array}{c} \top \\ A \end{array}$$

$$\begin{array}{c} \top \\ f \\ \otimes \\ A \end{array} = \begin{array}{c} \top \\ f \\ A \end{array} = \begin{array}{c} \top \\ B \\ / \\ A \end{array}$$

Lecture 2

Parallel composition:



$$f \otimes g : A \otimes C \rightarrow B \otimes D$$

For functions, parallel composition is Cartesian product:

types (sets) : $A \otimes B = A \times B = \{(a,b) \mid a \in A, b \in B\}$
 $I = \{\ast\} \leftarrow \text{one-element set. (not } \emptyset \text{!)}$

$$(A \times B) \times C \cong A \times (B \times C) \cong \{(a,b,c) \mid a \in A, b \in B, c \in C\}$$

$$A \times \{\ast\} = \{(a,\ast) \mid a \in A\} \cong A \cong \{\ast\} \times A$$

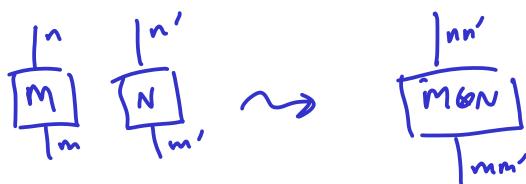
procs (functions) :
 $f \times g : A \times C \rightarrow B \times D$
 $(f \times g)(a,c) = (f(a), g(c))$

For linear maps, \otimes is the tensor product \otimes .

Let \mathbb{C}^n be the n-dimensional space $\left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mid v_1, \dots, v_n \in \mathbb{C} \right\}$

types (dimension) : $\mathbb{C}^m \otimes \mathbb{C}^n := \mathbb{C}^{m \cdot n}$ $I = \mathbb{C}^1 = \mathbb{C}$ (not 0!)

$$\begin{aligned} M \otimes N &= \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{2 \times 2} \otimes \underbrace{\begin{pmatrix} e & f \\ g & h \end{pmatrix}}_{2 \times 2} \\ &= \begin{pmatrix} aM & bM \\ cM & dM \end{pmatrix} = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}_{4 \times 4} \end{aligned}$$



TOGETHER :

$$\left(\begin{array}{c} f_2 \\ \square \end{array} \circ \begin{array}{c} f_1 \\ \square \end{array} \right) \otimes \left(\begin{array}{c} g_2 \\ \square \end{array} \circ \begin{array}{c} g_1 \\ \square \end{array} \right) = \begin{array}{c} f_2 \\ \square \end{array} \otimes \begin{array}{c} g_2 \\ \square \end{array} = \begin{array}{c} f_2 \\ \square \end{array} \begin{array}{c} g_2 \\ \square \end{array}$$

|| \curvearrowleft interchange law

$$\left(\begin{array}{c} f_2 \\ \square \end{array} \otimes \begin{array}{c} g_2 \\ \square \end{array} \right) \circ \left(\begin{array}{c} f_1 \\ \square \end{array} \otimes \begin{array}{c} g_1 \\ \square \end{array} \right) = \left(\begin{array}{c} f_2 \\ \square \end{array} \begin{array}{c} g_2 \\ \square \end{array} \right) \circ \left(\begin{array}{c} f_1 \\ \square \end{array} \begin{array}{c} g_1 \\ \square \end{array} \right) = \begin{array}{c} f_2 \\ \square \end{array} \begin{array}{c} g_2 \\ \square \end{array}$$

Non-trivial equation in 1D: $(f_2 \circ f_1) \otimes (g_2 \circ g_1) = (f_2 \otimes g_2) \circ (f_1 \otimes g_1)$

is free (ocm) in 2D:

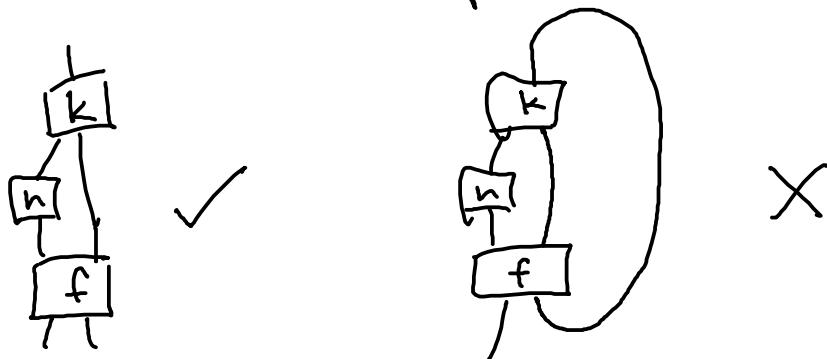
$$\begin{array}{c} f_2 \\ \square \end{array} \begin{array}{c} g_2 \\ \square \end{array} = \begin{array}{c} f_2 \\ \square \end{array} \begin{array}{c} g_2 \\ \square \end{array}$$

DEF A circuit diagram is any diagram built from

- boxes $\begin{array}{c} f \\ \square \end{array}, \begin{array}{c} g \\ \square \end{array}, \dots$
- (identity) wires $|_A, |_B, \dots, |_I = \text{---}$
- Swap processes $\begin{array}{c} B \\ \diagup \quad \diagdown \\ A & B \end{array}$

using only \otimes and \circ .

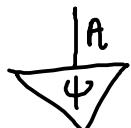
Equivalently circuit diagrams are any diagram that doesn't have feedback loops.



D.A.G.

3.4 Special processes

States



$\Psi: I \rightarrow A$ "preparation"

For functions states are the same as elements of a set.



$\Psi: \Sigma^* \rightarrow A$ $\Psi(*) = a \in A$.

For linear maps: $\Psi: \underbrace{\mathbb{C}}_{\text{dim}=1} \rightarrow \underbrace{\mathbb{C}^n}_{\text{dim}=n}$ } $n \times 1$ matrix
(column vector)

$$\Psi = \begin{pmatrix} \Psi^1 \\ \vdots \\ \Psi^n \end{pmatrix}$$

For classical thy \subseteq linear maps, states are prob distrs:

$$\begin{array}{l} \downarrow \\ p \end{array} = \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix} \quad \begin{aligned} \text{Prob}(0) &= 2/3 \\ \text{Prob}(1) &= 1/3 \\ \text{Prob}(2) &= 0. \end{aligned}$$

Effects

$$\begin{array}{c} \triangle \\ \diagup \quad \diagdown \\ A \quad I \end{array} : A \rightarrow I \quad \text{"question" or a "test"}$$

For functions, effects are.... trivial! $\pi_i : A \rightarrow \{*\}$

$$\begin{array}{c} \uparrow \\ \pi_i \end{array} \quad \begin{array}{c} \uparrow \\ \text{only one} \end{array}$$

For linear maps, effects $\pi_i : \underbrace{\mathbb{C}^n}_{\text{dim}=n} \rightarrow \underbrace{\mathbb{C}}_{\text{dim}=1}$ are $1 \times n$ matrices:
row vectors!

In classical thy, these are row vectors w/ entries $0 \leq r \leq 1$.
(sometimes called "fuzzy predicates")

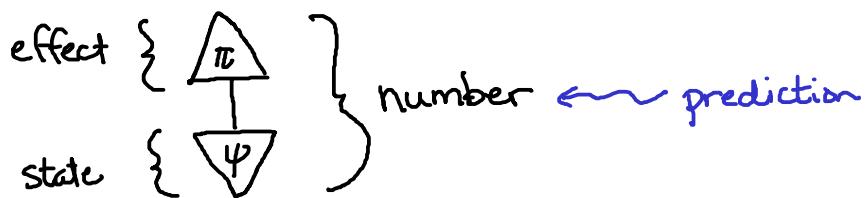
$$\begin{array}{c} \uparrow \\ 0 \end{array} = (1 \ 0 \ 0) \quad \text{"in state 0"}$$

$$\begin{array}{c} \uparrow \\ 0 \vee 1 \end{array} = (1 \ 1 \ 0) \quad \text{"in state 0 or 1"}$$

$$\begin{array}{c} \uparrow \\ \text{info} \end{array} = \left(\frac{99}{100} \ \frac{1}{100} \ \frac{1}{100} \right) \quad \text{"got pretty reliable info that the state is 0"}$$

Numbers: processes $\lambda: I \rightarrow I$

... for linear maps: $\lambda: \underbrace{\mathbb{C}}_{\text{dim}=1} \rightarrow \underbrace{\mathbb{C}}_{\text{dim}=1}$ 1×1 matrix (λ)



... in classical thy:

$$\text{predicate} \left\{ \begin{array}{c} \nearrow \pi \\ \searrow p \end{array} \right\} = p \leadsto \text{probability. "classical Born rule"}$$

prob.
dist.

$$(1 \ 0 \ 0) \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix} = 2/3$$

$$(1 \ 1 \ 0) \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix} = 1$$

$$\left(\frac{99}{100} \ \frac{1}{100} \ \frac{1}{100} \right) \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix} = \frac{199}{300} \approx 2/3$$

Chapter 4: String diagrams

"separable vs. non-separable"

4.1

DEF a \otimes -separable state $\Psi: I \rightarrow A \otimes B$ is a state s.t.
there exist $\psi_1: I \rightarrow A$, $\psi_2: I \rightarrow B$ where:

$$\begin{array}{c} |A \\ \text{\scriptsize } \downarrow \\ \text{\scriptsize } \varphi \end{array} \quad |B = \begin{array}{c} |A \\ \text{\scriptsize } \downarrow \\ \text{\scriptsize } \psi_1 \end{array} \quad \vdots \quad \begin{array}{c} |B \\ \text{\scriptsize } \downarrow \\ \text{\scriptsize } \psi_2 \end{array}$$

In functions, states are elements. \Rightarrow all states are \otimes -sep'l.

$$\begin{array}{c} |A \\ \text{\scriptsize } \downarrow \\ (a,b) \end{array} = \begin{array}{c} |A \\ \text{\scriptsize } \downarrow \\ a \end{array} \quad \begin{array}{c} |B \\ \text{\scriptsize } \downarrow \\ b \end{array} .$$

In linear maps, some states are separable:

$$\begin{array}{c} |\mathbb{C}^2 \\ \text{\scriptsize } \downarrow \\ \text{\scriptsize } \psi \end{array} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{array}{c} \downarrow \\ \text{\scriptsize } \psi \end{array} \quad \begin{array}{c} \downarrow \\ \text{\scriptsize } \psi \end{array}$$

... but some are not!

$$\begin{array}{c} \downarrow \\ \text{\scriptsize } \psi \end{array} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix}$$

$$ac=1 \Rightarrow a \neq 0, b \neq 0 \dots \text{but } ad=0! \\ bd=1 \Rightarrow b \neq 0, d \neq 0 \dots \text{but } bc=0!$$

Lecture 3

$$\begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{pmatrix} 1 & & & \\ 0 & 0 & & \\ & & 1 & \\ & & 0 & \end{pmatrix} \neq \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \otimes \begin{pmatrix} c & d \\ d & d \end{pmatrix}$$

In fact, $\begin{array}{c} \diagup \\ \diagdown \end{array}$ is a special state, called the Bell State, or simply the Cup.

Its transpose $\begin{array}{c} \cap \\ \cup \end{array} = (1 \ 0 \ 0 \ 1)$ is called the Cap.

$$\begin{array}{c} \cap \\ \cup \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} \Big| \begin{array}{c} c^2 \\ c^2 \\ c^2 \\ c^2 \end{array} = \Big| \begin{array}{c} c^2 \\ c^2 \end{array}$$

New notation: $\begin{array}{c} \diagup \\ \diagdown \end{array} = U \quad \begin{array}{c} \cap \\ \cup \end{array} = \cap$

Def A process theory admits string diagrams if every system A has a special state $\begin{array}{c} \cap \\ \cup \end{array}_A$ and effect $\begin{array}{c} \cap \\ \cup \end{array}_A$ such that:

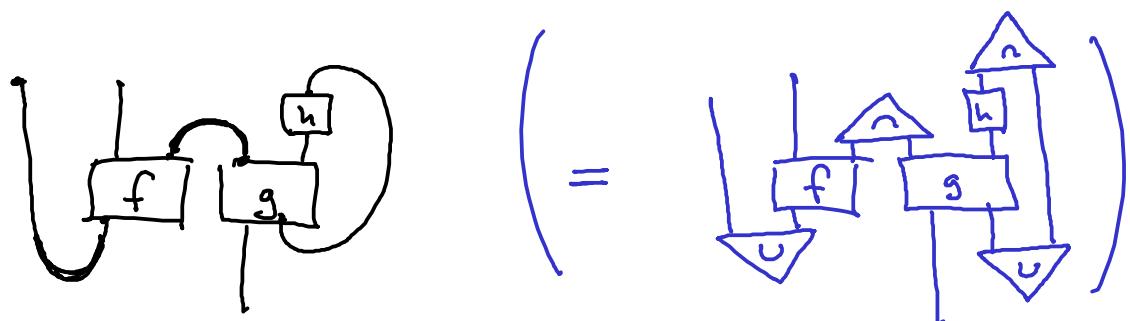
$$\begin{array}{c} \times \\ \times \end{array} = U \quad \begin{array}{c} \cap \\ \cup \end{array} = \cap$$

... and $\begin{array}{c} \cap \\ \cup \end{array} = \Big| = \cap$.

$$\begin{array}{c} \uparrow \quad \leftarrow \quad \leftarrow \\ (\cap \otimes 1_A) \circ (1_A \otimes U) = 1_A = (1_A \otimes \cap) \circ (U \otimes 1_A) \end{array}$$

Def A string diagram is a circuit diagram with cups & caps.

Equivalently, it is a diagram where loops and in-in/out-out connections are allowed.



T_{Hm} String diagrams satisfy OCM!

Two equations illustrating OCM identities:

Left: A box labeled 'f' with a vertical line above it and a horizontal line below it is equal to a diagram where a vertical line labeled 'g' enters a box 'f' from the left, and a horizontal line exits from the right.

Right: A box labeled 'g' with a vertical line above it and a horizontal line below it is equal to a box labeled 'f' with a vertical line above it and a horizontal line below it, enclosed in a circle. The text "etc." follows.

(Pf uses OCM for circuit diagrams, plus cap/cup rules.)

n.b. functions does **NOT** admit string diagrams!
(but relations does, see Ex 4.16 in **Dodo**)

PROCESS- STATE DUALITY

$$\left\{ \begin{array}{c} |B \\ f \\ |A \end{array} \right\} \xrightleftharpoons{\cong} \left\{ \begin{array}{c} |A & |B \\ \Psi \\ |A \end{array} \right\}$$

(in Q.T. Choi-Jamiołkowski isomorphism)

Hm In a process theory that admits string diagrams, the processes $f: A \rightarrow B$ are in 1-to-1 corresp. with states $\Psi: I \rightarrow A \otimes B$.

Pf

Let: $ps: \begin{array}{c} |B \\ f \\ |A \end{array} \mapsto \begin{array}{c} |A & |B \\ \Psi_f \\ |A \end{array} := \begin{array}{c} |A \\ \cup \\ f \\ |B \end{array}$

and: $sp: \begin{array}{c} |A & |B \\ \Psi \\ |A \end{array} \mapsto \begin{array}{c} |B \\ f_\Psi \\ |A \end{array} := \begin{array}{c} |B \\ \cap \\ \Psi \\ f \\ |A \end{array}$

Then:

$$\begin{array}{c} |B \\ f \\ |A \end{array} \xrightarrow{ps} \begin{array}{c} |A & |B \\ \cup \\ f \\ |B \end{array} \xrightarrow{sp} \begin{array}{c} |A \\ \cap \\ f \\ |B \end{array} = \begin{array}{c} |B \\ f \\ |A \end{array}$$

$$\begin{array}{c} |A & |B \\ \Psi \\ |A \end{array} \xrightarrow{sp} \begin{array}{c} |B \\ \cap \\ \Psi \\ f \\ |A \end{array} \xrightarrow{ps} \begin{array}{c} |A \\ \cup \\ \Psi \\ f \\ |B \end{array} = \begin{array}{c} |A & |B \\ \Psi \\ |A \end{array}$$

$ps = sp^{-1}$, so $\{f: A \rightarrow B\} \cong \{\Psi: I \rightarrow A \otimes B\}$. \square

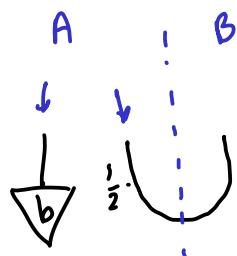
"Classical" teleportation.

A has a bit \downarrow_b that she wants to send to B.

- They also share a random bit:

$$\frac{1}{2} \cdot \bigcup^{\text{two bits}} = \begin{pmatrix} 00 \\ 01 \\ 10 \\ 11 \end{pmatrix} \begin{matrix} \leftarrow 50\% \text{ chance } 00 \\ \leftarrow 50\% \text{ chance } 11 \end{matrix}$$

- A will observe her 2 bits:



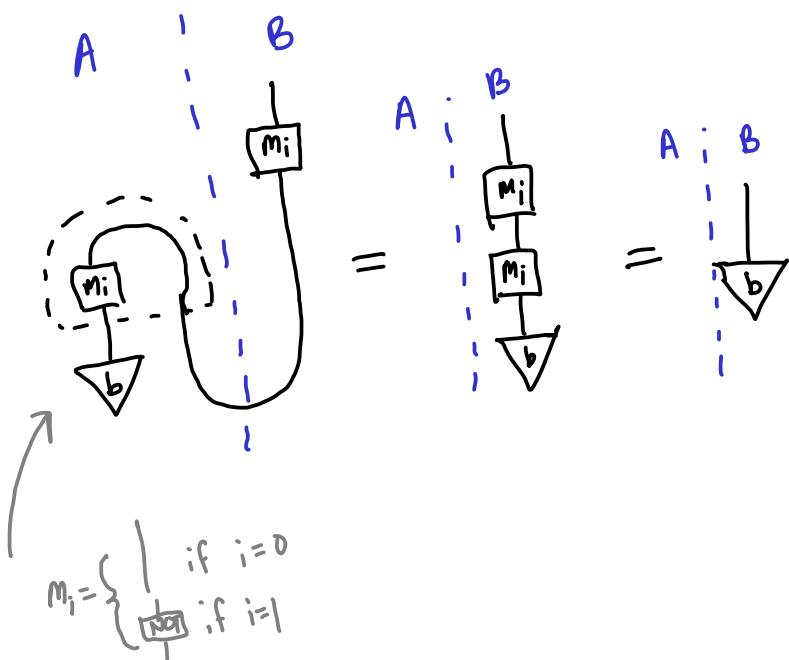
And make one of 2 obs:

(0) the bits are the same $\stackrel{\text{TO}}{\triangle} = \cap = (1 \ 0 \ 0 \ 1)$

(1) the bits are different $\stackrel{\text{TI}}{\triangle} = \boxed{\text{NOT}} = (0 \ 1 \ 1 \ 0)$

$$\left(\boxed{\text{NOT}} = (0 \ 1) \right)$$

- She tells B to flip his bit only if they are different.

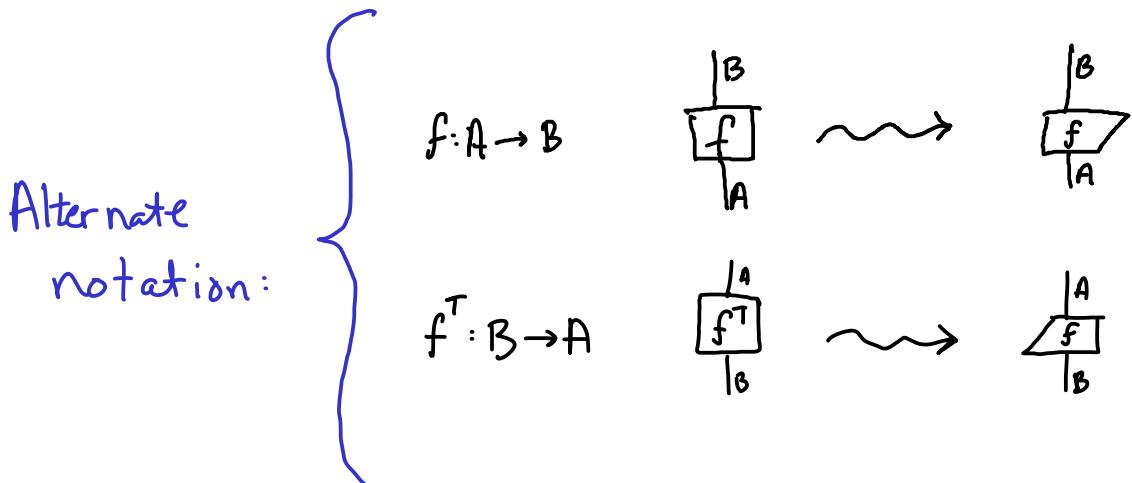


This is known as one-time pad crypto.

4.2 Transpose & adjoint of a process

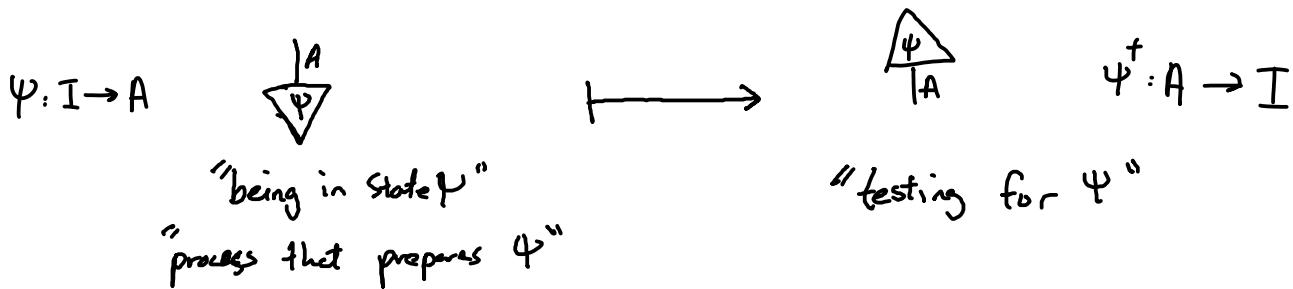
DEF (Transpose) $\boxed{\begin{array}{c} A \\ f \\ B \end{array}} := \boxed{\begin{array}{c} A \\ f \\ B \end{array}}_{\boxed{\begin{array}{c} B \\ f \\ A \end{array}}}$

Tfm For linear maps, the matrix of $\boxed{\begin{array}{c} A \\ f \\ B \end{array}}_{\boxed{\begin{array}{c} B \\ f \\ A \end{array}}}$ is the transpose of the matrix of $\boxed{\begin{array}{c} A \\ f \\ B \end{array}}$.
Pf (later).



$$B \xrightarrow{\quad \text{f} \quad} = \xrightarrow{\quad \text{f}^T \quad} = \xrightarrow{\quad \text{f} \quad}$$

4.3.1 ADJOINTS.



EXTENDS TO PROCESSES:

For linear maps, the adjoint is defined as the conjugate-transpose matrix.

Q: why not the transpose?

A: if we test a non-zero state Ψ for itself, the result should be > 0 . (positive-definiteness)

$$\begin{array}{c} \uparrow \\ \Psi \\ \downarrow \\ \Psi \end{array} > 0.$$

- The transpose does not have this property, e.g.

$$\begin{array}{c} \downarrow \\ \Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \end{array} \Rightarrow \begin{array}{c} \uparrow \\ \Psi^\dagger = \frac{1}{2} (1-i) \begin{pmatrix} 1 \\ i \end{pmatrix} = 1+i^2 = 0. \end{array} \quad \ddot{\cup}$$

whereas:

$$\begin{array}{c} \uparrow \\ \Psi \\ \downarrow \\ \Psi \end{array} = \Psi^\dagger \circ \Psi = \frac{1}{2} (1-i) \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} (1-i^2) = 1. \quad \ddot{\cup}$$

Def A state is called normalised if $\begin{array}{c} \uparrow \\ \Psi \\ \downarrow \\ \Psi \end{array} = 1$.

- Generally we will assume states are normalised (we'll see why next week.)

New (old) notation: $\Psi: I \rightarrow A$

$$\Psi^*: A \rightarrow I$$

$$\phi^* \circ \Psi$$

$$\begin{array}{c} \uparrow \\ \Psi \\ \downarrow \\ \Psi \end{array}^A$$

$$\begin{array}{c} \uparrow \\ \Psi \\ \downarrow \\ \Psi \end{array}^A$$

$$\begin{array}{c} \uparrow \\ \phi \\ \downarrow \\ \Psi \end{array}^A$$

$$|\Psi\rangle \quad \text{ket}$$

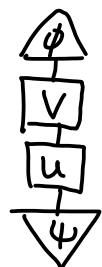
$$\langle \Psi | \quad \text{bra}$$

$$\langle \phi | \Psi \rangle \quad \text{bra-ket}$$

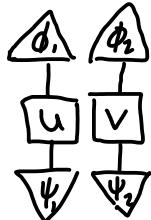
inner product

(a.k.a "Dirac")

Bra-ket notation is used extensively in QC literature. It is just graphical notation sideways:



$$\langle \phi | V U | \psi \rangle$$



$$\langle \phi_1, \phi_2 | (U \otimes V) | \psi_1, \psi_2 \rangle$$

\parallel

$$(\langle \phi_1 | \otimes \langle \phi_2 |)(U \otimes V)(|\psi_1\rangle \otimes |\psi_2\rangle)$$

interchange law

$$\langle \phi_1 | U | \psi_1 \rangle \langle \phi_2 | V | \psi_2 \rangle$$

shorthand
for \otimes states/effects

Ch 5. Hilbert spaces, ... or doing linear algebra with pictures.

Q: How much do we need to add to diagrams to calculate everything we can w/ matrices?

A: 2 extra things.

1. (Orthonormal) bases

DEF A basis for a type A is a minimal set of states:

$$\mathcal{B} := \left\{ \begin{smallmatrix} |A\rangle \\ \downarrow \\ 1 \end{smallmatrix}, \dots, \begin{smallmatrix} |A\rangle \\ \downarrow \\ n \end{smallmatrix} \right\}$$

such that:

if for all $\begin{smallmatrix} |A\rangle \\ \downarrow \\ i \end{smallmatrix} \in \mathcal{B}$. $\begin{smallmatrix} |B\rangle \\ \downarrow \\ f \end{smallmatrix} = \begin{smallmatrix} |B\rangle \\ \downarrow \\ g \end{smallmatrix}$ then $f = g$.

Intuition basis states are "reference points" for a process

DEF The dimension of a system A is the size of any basis for A.

T_{Hm} (Dimension theorem) For linear maps, all bases are the same size.

DEF Two states are orthogonal if $\langle \uparrow \downarrow | \downarrow \uparrow \rangle = 0$.

DEF An orthonormal basis $B = \{ |\downarrow_i\rangle\}_{i=1...n}^{(ONB)}$ is a basis

where

$$\langle \downarrow_i | \downarrow_j \rangle = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \quad \hookrightarrow \delta_i^j \leftarrow \text{Kronecker } \delta.$$

In bra-ket:

$$B = \{ |1\rangle, |2\rangle, \dots, |n\rangle \} \quad \langle j | i \rangle = \delta_i^j$$

For \mathbb{C}^2 , we normally write the standard basis $\{ |\downarrow_0\rangle, |\downarrow_1\rangle \} = \{ |0\rangle, |1\rangle \}$
(a.k.a computational basis)

Using ONB's, we can recover the matrix of a process:

$$m_i^j := \left. \begin{array}{c} \downarrow_j \\ M \\ \downarrow_i \end{array} \right\} \text{number in the } j\text{-th row and } i\text{-th column.}$$

e.g. in 2D: $|\downarrow_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\downarrow_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \boxed{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{array}{c} \downarrow_1 \\ \boxed{M} \\ \downarrow_0 \end{array} = (0 \ 1) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (0 \ 1) \begin{pmatrix} a \\ c \end{pmatrix} = c.$$

2. Sums (of diagrams)

DEF A process theory has sums if, for any two processes of the same type

$$\begin{array}{c} \text{B} \\ \boxed{f} \\ \text{A} \end{array}, \begin{array}{c} \text{B} \\ \boxed{g} \\ \text{A} \end{array} \rightsquigarrow \begin{array}{c} \text{B} \\ \boxed{f+g} \\ \text{A} \end{array}$$

1. associative, commutative, and has unit \emptyset .

$$f + (g + h) = f + (g + h) \quad f + g = g + f \quad \begin{array}{c} \text{B} \\ \boxed{f} \\ \text{A} \end{array}, \begin{array}{c} \text{B} \\ \boxed{\emptyset} \\ \text{A} \end{array} \rightsquigarrow f + \emptyset = f.$$

$$\text{Let } \sum_i \begin{array}{c} \text{B} \\ \boxed{f_i} \\ \text{A} \end{array} := f_1 + \dots + f_n$$

2. Sums distribute over diagrams

$$\left(\sum_i \begin{array}{c} \text{B} \\ \boxed{h_i} \\ \text{A} \end{array} \right) \begin{array}{c} \text{B} \\ \boxed{g} \\ \text{A} \end{array} = \sum_i \begin{array}{c} \text{B} \\ \boxed{h_i} \\ \text{A} \end{array} \begin{array}{c} \text{B} \\ \boxed{g} \\ \text{A} \end{array}$$

3. Sums preserve adjoints

$$\left(\sum_i \begin{array}{c} \text{B} \\ \boxed{f_i} \\ \text{A} \end{array} \right)^+ = \sum_i \begin{array}{c} \text{A} \\ \boxed{f_i^+} \\ \text{B} \end{array}$$

Examples of sums distr. over diagrams.

(i) linearity.

Linear combinations: $\sum_i \lambda_i |\Psi_i\rangle = \sum_i \boxed{\lambda_i} \downarrow \Psi_i$

$$f\left(\sum_i \lambda_i |\Psi_i\rangle\right) = \begin{array}{c} f \\ \downarrow \\ \left(\sum_i \boxed{\lambda_i} \downarrow \Psi_i\right) \end{array} = \sum_i \boxed{\lambda_i} \begin{array}{c} f \\ \downarrow \\ \Psi_i \end{array} = \sum_i \lambda_i f(|\Psi_i\rangle)$$

(ii) linearity of $\langle - | - \rangle$

$$\langle \omega | v_1 + v_2 \rangle = \langle \omega | v_1 \rangle + \langle \omega | v_2 \rangle$$

$$\langle \omega | \sum_i v_i \rangle = \sum_i \langle \omega | v_i \rangle$$

(iii) conjugate-linearity of $\langle - | - \rangle$

$$\langle \sum_i v_i | \omega \rangle = \sum_i \langle v_i | \omega \rangle$$

$$(\diamondsuit)^+ = \square \implies \langle \sum_i \lambda_i v_i | \omega \rangle = \sum_i \bar{\lambda}_i \langle v_i | \omega \rangle$$

(iv) bilinearity of \otimes . $(\sum_i \frac{1}{f_i}) \otimes \frac{1}{g} = \sum_i \left[\frac{1}{f_i} \otimes \frac{1}{g} \right]$

$$\frac{1}{f} \otimes (\sum_i \frac{1}{g_i}) = \sum_i \left[\frac{1}{f} \otimes \frac{1}{g_i} \right]$$

Def A (finite-dimensional) Hilbert space is a vector space with an operation $\langle - | - \rangle : H \times H \rightarrow \mathbb{C}$ that is:

* linear in the 2^{nd} arg.

* conj-linear in the 1^{st} arg.

* conj-symmetric $\overline{\langle \psi | \phi \rangle} = \langle \phi | \psi \rangle$

* positive semi-def. $\langle \psi | \psi \rangle \geq 0$ and $\langle \psi | \psi \rangle = 0 \Rightarrow \psi = 0$.

Hm The types of linear maps are Hilbert

5.2 Matrix calculations with diagrams.

$$\text{THm} \quad |_A = \sum_i \begin{array}{c} \nearrow \\ \square \\ \searrow \\ i \end{array}$$

Pf The matrix of $\begin{array}{c} \nearrow \\ \square \\ \searrow \\ i \end{array}$ is $\begin{pmatrix} 0 & & & \\ 0 & 1 & & \\ 0 & & 0 & \\ \vdots & & & 0 \end{pmatrix} (0 \dots 0 1 0 \dots 0) = \begin{pmatrix} 0 & & & \\ 0 & 0 & 1 & \\ 0 & & 0 & \\ \vdots & & & 0 \end{pmatrix} \leftarrow i^{\text{th}} \text{ row}$

$$\Rightarrow \sum_i \begin{array}{c} \nearrow \\ \square \\ \searrow \\ i \end{array} = \begin{pmatrix} 0 & & & \\ 0 & 1 & & \\ 0 & & 0 & \\ \vdots & & & 0 \end{pmatrix} = | \quad \blacksquare$$

Cle For any linear map f , we have:

$$\frac{|B|}{|A|} := \sum_{ij} f_i^j \begin{array}{c} \nearrow \\ \square \\ \searrow \\ j \end{array} \quad \text{where } f_i^j = \begin{array}{c} \nearrow \\ \square \\ \searrow \\ i \\ j \end{array} \text{ are the matrix entries.}$$

Pf

$$\frac{|B|}{|A|} = \frac{\left(\sum_j \begin{array}{c} \nearrow \\ \square \\ \searrow \\ j \end{array} \right)}{\left(\sum_i \begin{array}{c} \nearrow \\ \square \\ \searrow \\ i \end{array} \right)} = \sum_{ij} \begin{array}{c} j \\ \square \\ i \end{array} = \sum_{ij} f_i^j \begin{array}{c} \nearrow \\ \square \\ \searrow \\ j \\ i \end{array} \quad \underbrace{\quad}_{\text{"matrix form" of } f.} \quad \blacksquare$$

e.g. $\boxed{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left[\begin{array}{c} \downarrow \begin{array}{c} \textcircled{1} \\ \textcircled{2} \end{array} \\ \uparrow \begin{array}{c} \textcircled{1} \\ \textcircled{2} \end{array} \end{array} + \begin{array}{c} \downarrow \begin{array}{c} \textcircled{1} \\ \textcircled{2} \end{array} \\ \uparrow \begin{array}{c} \textcircled{1} \\ \textcircled{2} \end{array} \end{array} - \begin{array}{c} \downarrow \begin{array}{c} \textcircled{1} \\ \textcircled{2} \end{array} \\ \uparrow \begin{array}{c} \textcircled{1} \\ \textcircled{2} \end{array} \end{array} \right] = (*)$

n.b. $H_i^j = \frac{1}{\sqrt{2}} \cdot (-1)^{ij}$, so $(*) = \frac{1}{\sqrt{2}} \sum_{ij} (-1)^{ij} \begin{array}{c} \downarrow \begin{array}{c} i \\ j \end{array} \\ \uparrow \begin{array}{c} i \\ j \end{array} \end{array}$.

In bra-ket : $f = \sum_{ij} f_i^j |j\rangle \langle i|$
 $\underbrace{\hspace{10em}}$ ket-bra

Calculations involving ket-bras are very common. These are essentially expanding a map in terms of it's matrix.

Other consequences:

$$(g \circ f)_i^j = \underbrace{\begin{array}{c} \uparrow \begin{array}{c} j \\ g \\ f \\ i \end{array} \end{array}}_{\text{matrix of } g \circ f} = \sum_k \begin{array}{c} \uparrow \begin{array}{c} j \\ g \\ k \\ f \\ i \end{array} \end{array} = \sum_k g_k^j f_i^k$$

$\underbrace{\hspace{10em}}$ product of
matrix of g w/
matrix of f .

For the standard basis: $(\downarrow_i)^\top = (\uparrow_i)^\dagger$, i.e.

$$\text{---} \downarrow_i = \uparrow_i .$$

Consequence:

$$(f^\top)_j^i = \begin{array}{c} \uparrow_j \\ f^\top \\ \downarrow_i \end{array} = \begin{array}{c} \uparrow_j \\ f \\ \downarrow_i \end{array} = \begin{array}{c} \uparrow_i \\ f \\ \downarrow_j \end{array} = f_j^i$$

$$f_i^j \xrightarrow{\text{transpose}} f_j^i$$

Other bases:

- Standard basis, a.k.a. Computational basis or \mathbb{Z} -basis.

$$|0\rangle = \begin{array}{c} \downarrow \\ \square \end{array} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{array}{c} \uparrow \\ \square \end{array} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- "Plus" basis, or X -basis

$$|+\rangle = \begin{array}{c} \downarrow \\ \square_0 \end{array} = \frac{1}{\sqrt{2}} \left(\begin{array}{c} \downarrow \\ \square \end{array} + \begin{array}{c} \uparrow \\ \square \end{array} \right) \quad |-\rangle = \begin{array}{c} \downarrow \\ \square_1 \end{array} = \frac{1}{\sqrt{2}} \left(\begin{array}{c} \downarrow \\ \square \end{array} - \begin{array}{c} \uparrow \\ \square \end{array} \right)$$

- Y -basis:

$$|+i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad |-i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

(Careful! $\text{---} \downarrow_{+i} = \frac{1}{\sqrt{2}} \cdot (|+i\rangle = \begin{array}{c} \uparrow \\ \square \end{array})$)

adjoint

Product bases:

- If $\left\{ \begin{smallmatrix} |A \\ j \end{smallmatrix} \right\}_{j=1}^m$ and $\left\{ \begin{smallmatrix} |B \\ k \end{smallmatrix} \right\}_{k=1}^n$ are ONB's then
- $\left\{ \begin{smallmatrix} |A & |B \\ j & k \end{smallmatrix} \mid j=1\dots m, k=1\dots n \right\}$ is an ONB for $A \otimes B$.
- $\dim(A \otimes B) = m \cdot n$.

$$\begin{array}{c|cc}
\begin{smallmatrix} |c^2 & |c^2 \\ m & \end{smallmatrix} \\
\hline
\begin{smallmatrix} |c^2 & |c^2 \\ m & \end{smallmatrix} < 4 \times 4 \text{ matrix with entries } M_{ij}^{kl} := \begin{smallmatrix} |k & |l \\ m & \end{smallmatrix}
\end{array}$$

$$M = \begin{pmatrix} M_{00}^{00} & M_{01}^{00} & M_{10}^{00} & M_{11}^{00} \\ M_{00}^{01} & M_{01}^{01} & M_{10}^{01} & M_{11}^{01} \\ M_{00}^{10} & M_{01}^{10} & M_{10}^{10} & M_{11}^{10} \\ M_{00}^{11} & M_{01}^{11} & M_{10}^{11} & M_{11}^{11} \end{pmatrix}$$

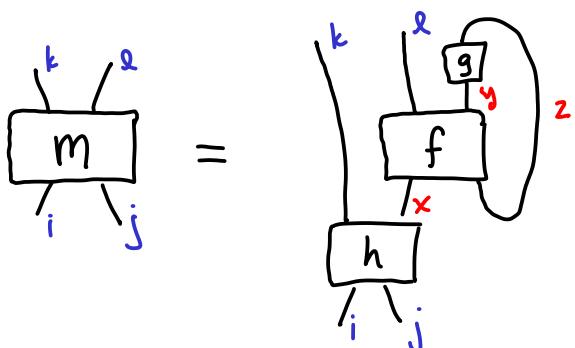
Thm The matrix of $\begin{smallmatrix} f & g \\ i & j \end{smallmatrix}$ is the Kronecker product of the matrices of $f \circ g$.

Pf $(f \otimes g)_{ij}^{kl} := \begin{smallmatrix} |k & |l \\ f & g \\ i & j \end{smallmatrix} = f_i^k \cdot g_j^l$

$\underbrace{\quad \quad \quad}_{\text{Kron. product}}$

Tensor contraction

There is a recipe for computing the matrix of any string diagram in a single calculation.



1. label every wire with a unique index
2. multiply the matrix entries of every box
3. sum over the **Connected** wires.

$$m_{ij}^{kl} = \sum_{xyz} f_x^{ly} g_y^z h_{ij}^{kx}$$

Special cases:

$$\begin{array}{c} f \\ \boxed{i} \end{array} \begin{array}{c} g \\ \boxed{j} \end{array} \rightsquigarrow f_i^k g_j^l \quad \begin{array}{c} \boxed{j} \\ \boxed{k} \\ f \\ \boxed{i} \end{array} \rightsquigarrow \sum_k f_i^k g_k^j$$

$$\begin{array}{c} f \\ \circ \\ \boxed{i} \end{array} \rightsquigarrow \sum_i f_i^i$$