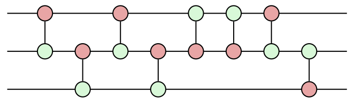
Quantum Software

Assignment 5, Hilary 2026

Exercise 1: Consider the following CNOT circuit:

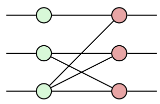


Cut the circuit in two halves containing the first 4 CNOT gates and the last 4 CNOT gates. Simplify to parity normal form in two phases. First, form D_1 by simplifying each half individually to parity normal form. Then form D_2 by simplifying all of D_1 the rest of the way to parity normal form. What can you say about:

1. The number of forward-directed paths from each input to each output in C, D_1, and D_2?
2. The relationship between the bi-adjancency matrices of each half of D_1 and the bi-adjacency matrix of D_2?
3. The relationship between facts 1 and 2?

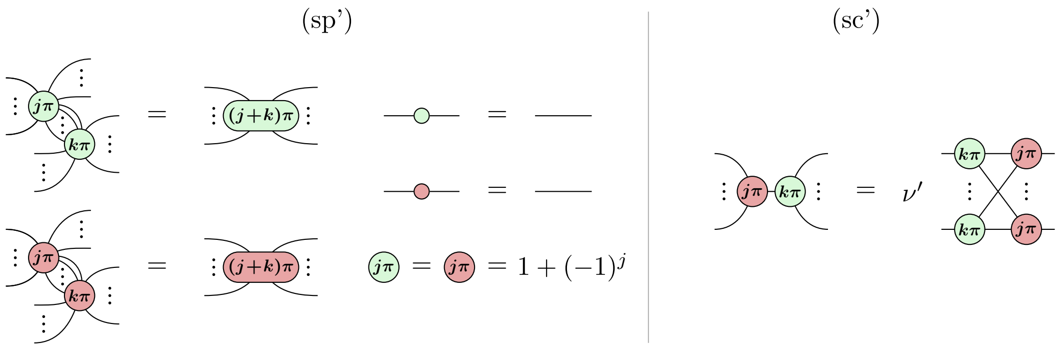
Note there is some ambiguity of what counts as a "forward-directed path", especially for C. This can be resolved as follows: for each diagram, choose a direction for all of the wires such that each Z spider has at most input and each X spider has at most one output. For CNOT gates, this means the wire connecting the two spiders should go from Z to X, not vice-versa.

Exercise 2: In the lecture we saw how to reduce a parity normal form diagram to a CNOT circuit if its biadjacency matrix is invertible. Apply the Gaussian elimination procedure to the following diagram which does *not* have an invertible matrix to see what it reduces to:



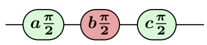
Use this to argue that the diagram is not unitary.

Exercise 3: In *Picturing Quantum Software*, an algorithm is given for reducing a phase-free ZX-diagram to generalised parity form using just the phase-free ZX rules. Consider the following extension to the rules accounting for π phases:



Prove (sc') using the rules of the (full) ZX-calculus. Describe a variation of the algorithm for phase-free diagrams that allows Z and X phases that are integer multiples of π. What do the normal form(s) look like?

Exercise 4: A single-qubit Clifford circuit is constructed out of just Hadamard and S gates. Show that any single-qubit Clifford circuit can be rewritten to the form



for some integers a, b and c. *Hint: We know that a single S or Hadamard can be brought to this form. So you just need to show that when you compose this normal form with an additional S or Hadamard gate that the resulting circuit also be brought to this normal form. You probably will want to make a case distinction on the value of b.*

Exercise 5: Show that the following states are Clifford states. That is, construct a Clifford circuit C that when applied to $|0 \cdots 0\rangle$ gives the desired state.

a) $|1\rangle$

b) $|+\rangle$

c) $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

d) $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

e) $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$