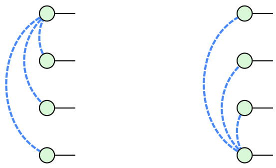
Quantum Software
Assignment 6, Hilary 2026

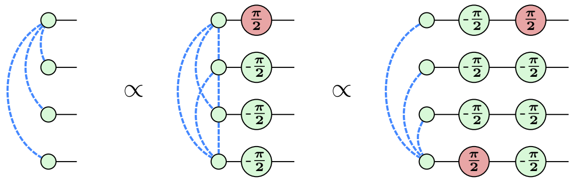
Solutions are shown after each question. Note some solutions are marked Sketch. These are intended to be instructions on how to work out the solution yourself, rather than an example of how you should answer this question on an exam.

Exercise 1: We say two n-qubit quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ are equivalent under local operations when $U|\psi_1\rangle = |\psi_2\rangle$ for a local quantum circuit $U = U_1 \otimes U_2 \otimes \cdots \otimes U_n$ consisting of just single-qubit gates. Show that the following two graph states are equivalent under local operations.

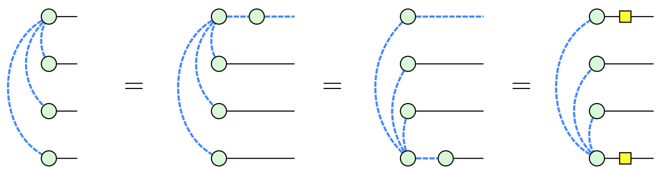


Hint: Use the fact that a local complementation can be done using just local unitaries.

Solution: . . . .
This can be accomplished with two local complementations:

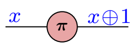


Alternatively, it is possible to show this be using H-cancellation and deforming the diagram:



End Solution . . . .

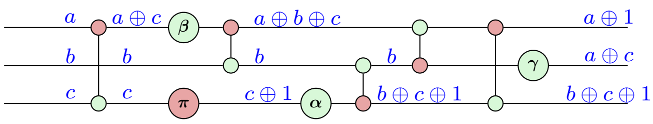
Exercise 2: Show that the phase polynomial representation also works for circuits made of CNOT, Z-phase, and X gates, using the fact that an X gate updates wire labels as follows:



More specifically, give an example circuit, compute its parity map and phase polynomial by wire-labelling, and re-synthesise a smaller circuit.

Finally, show that further simplifications are possible: namely expressions of the form $\alpha \cdot y + \beta \cdot (y \oplus 1)$, where y is some XOR of input variables, can be simplified in the phase polynomial, up to global phases.

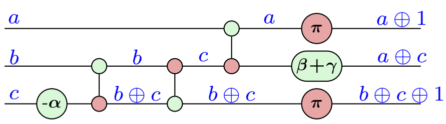
Solution: . . . .
Here's an example CNOT + X + phase circuit, with the wires labelled:



It's unitary is:

$$U :: |a, b, c\rangle \mapsto e^{i[\alpha\cdot(c\oplus 1)+(\beta+\gamma)\cdot(a\oplus c)]}|a \oplus 1, a \oplus c, b \oplus c \oplus 1\rangle$$

Here's a simpler circuit that does the same thing, up to a global phase:



For this we have actually already used the second part of the question. Note that for bits, $(c \oplus 1) = (1 - c)$. Hence,

$$e^{i\alpha(c\oplus 1)} = e^{i\alpha(1-c)} = e^{i(\alpha - c\cdot\alpha)} = e^{i\alpha}e^{-ic\cdot\alpha}$$

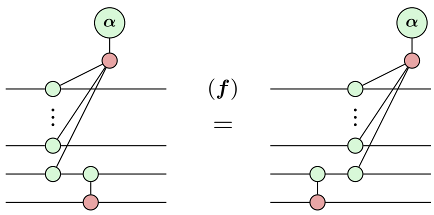
This implies that replacing a term $\alpha \cdot (c \oplus 1)$ in a phase polynomial with $-c \cdot \alpha$ only introduces a global phase. Hence, we can do the reduction:

$$\alpha \cdot y + \beta \cdot (y \oplus 1) \mapsto (\alpha - \beta) \cdot y$$

in the phase polynomial and only introduce a global phase.

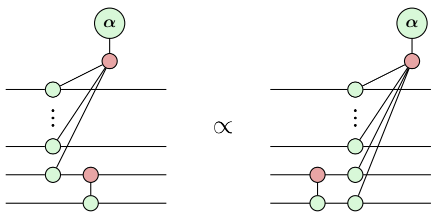
End Solution . . . .

Exercise 3: We computed the phase gadget form by appealing to the simplification strategy for CNOT circuits from Chapter 3. However, there is a more direct way we can translate CNOT+phase circuits into circuits of phase gadgets, by studying the way that phase gadgets commute with CNOT gates. There are a few cases to think about here. First, is the trivial case where a phase gadget and CNOT gate share no qubits. Obviously these commute. Only slightly harder to see is if a phase gadget appears on the control qubit (i.e. the Z spider) of a CNOT gate. Then, commutation follows from spider fusion:



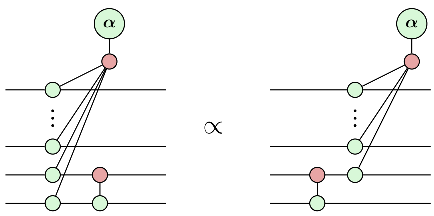
(1)

If the phase gadget overlaps with the target qubit (i.e. the X spider) of a CNOT gate, we can still push a phase gadget through, but it picks up an extra leg:



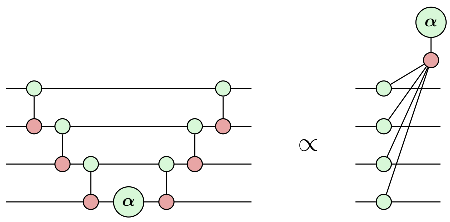
(2)

Just by reading the equation above in reverse, we can also see what happens when a phase gadget overlaps with both qubits of the CNOT gate. It loses a leg:



(3)

Prove equations (2) and (3). Use the three "phase-gadget walking" equations (1), (2), and (3), as well as the fact that Z-phase gates are equivalent to 1-legged phase gadgets, to show that an n-legged phase gadget has the following decomposition:

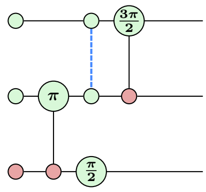


Solution: . . . .

Sketch: The "walking" rules follow from strong complementarity. To prove the phase gadget decomposition rule, one just needs to "walk" the phase from the middle of the circuit to the left or the right side, producing a 4-legged phase gadget. The CNOTs in the original circuit will then all cancel out.

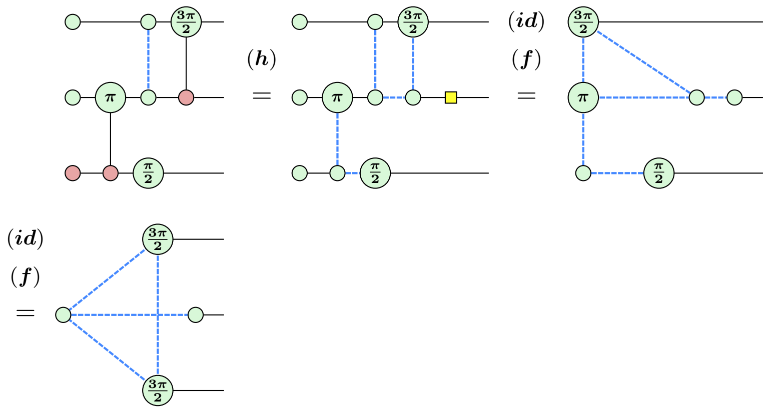
End Solution . . . .

Exercise 4: Reduce the following diagram to AP-form:



What is its parity matrix and its phase polynomial?

Solution: . . . .

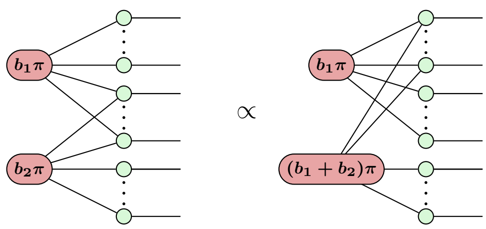


This state is: $\sum_{\vec{b}\in\mathcal{A}} e^{i\frac{\pi}{2}\phi(\vec{b})}$, where the affine subspace is $\mathcal{A} = \{(x, y, z) \mid x \oplus y \oplus z = 0\}$. The phase polynomial is $\phi(x, y, z) = 3x + 3z + 2xz$.

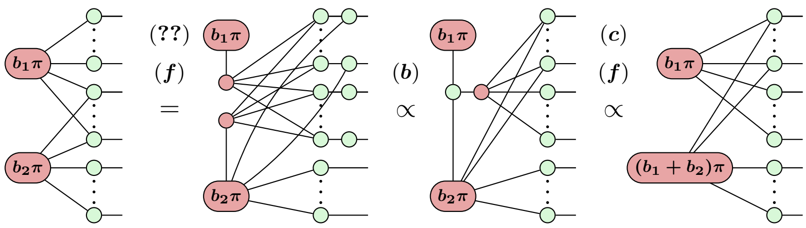
It is also acceptable to incorporate the π and/or the $\frac{1}{2}$ into ϕ. These are just different conventions for writing the phase polynomial.

End Solution . . . .

Exercise 5: The bit strings appearing in the superposition in an AP state are described by the solutions to the affine system of equations $M\vec{x} = \vec{b}$. When we do row operations on M and $\vec{b}$ (as in a Gaussian elimination of the linear system) this does not change the solutions, and so this transformed system $(M', \vec{b}')$ should still describe the same state. As the matrix M corresponds to the connectivity of the internal spiders to the boundary spiders, show that these row operations can be implemented diagrammatically:

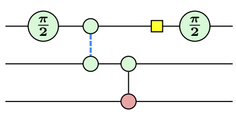


Solution: . . . .



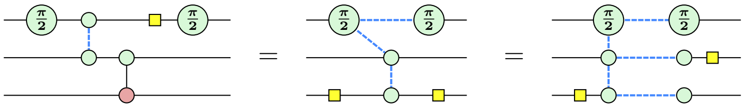
End Solution . . . .

Exercise 6: Reduce the following Clifford circuit to GSLC form:



Solution: ....

This is very nearly GSLC already, we just need to make it graph-like and insert some dummy nodes to make each spider connect to exactly one input/output:



End Solution ....