

# Quantum Software

## Assignment 3, Hillary Week 4, 2023

In these exercises we will look at Pauli gadgets in a slightly more abstract way, motivating why they work the way they do.

First, let's define the following.

**Definition 1.** Let  $f : A \rightarrow \mathbb{C}^2 \otimes A$  be a linear map. We say it is a *measure-box* when it satisfies the following identities:

$$\begin{array}{c} \square f \end{array} \begin{array}{c} \circ \\ \text{---} \end{array} = \text{---} \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \square f \\ \square f \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \square f \\ \circ \end{array} \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \square f^\dagger \end{array} = \begin{array}{c} \square f \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

We call this map a measure-box because it allows us to define a von Neumann measurement (with 2 outcomes) on system  $A$ . Note that, while the first output wire is always a 2D space, the system  $A$  which appears on the input wire and the second output wire can be more general. As we will see, this is often a tensor product of multiple qubits.

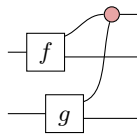
**Exercise 1:** Let  $f$  be a measure-box.

- Show that the following is unitary for any choice of  $\alpha$ :  $\begin{array}{c} \square f \\ \circ \end{array} \begin{array}{c} \alpha \\ \text{---} \end{array}$ . Recall that taking the adjoint of spiders flips the phase.
- Show that the following is a projector for  $b \in \{0, 1\}$ :

$$f_b := \frac{1}{\sqrt{2}} \cdot \begin{array}{c} \square f \\ \circ \end{array} \begin{array}{c} b\pi \\ \text{---} \end{array}$$

- Show that the above projectors for  $b = 0$  and  $b = 1$  are in fact *orthogonal* projectors, i.e. that  $f_0 \circ f_1 = f_1 \circ f_0 = 0$ .
- Show that  $f_0$  and  $f_1$  form a *resolution of the identity*, namely  $f_0 + f_1 = I$ .

**Exercise 2:** Let  $f : A \rightarrow \mathbb{C}^2 \otimes A$  and  $g : B \rightarrow \mathbb{C}^2 \otimes B$  be two measure-boxes. Show that the following combined process is also a measure-box from  $A \otimes B$  to  $\mathbb{C}^2 \otimes A \otimes B$ :



So now that we have these abstract boxes with nice properties that can be combined together, let's make it more concrete. We define the *Pauli boxes* as follows:

$$\begin{array}{c} \square I \end{array} := \frac{1}{\sqrt{2}} \begin{array}{c} \circ \\ \text{---} \end{array} \quad \begin{array}{c} \square X \end{array} := \begin{array}{c} \square \end{array} \begin{array}{c} \circ \\ \text{---} \end{array} \begin{array}{c} \square \end{array} \\
 \begin{array}{c} \square Y \end{array} := \begin{array}{c} \circ \\ \text{---} \end{array} \begin{array}{c} \frac{\pi}{2} \end{array} \begin{array}{c} \circ \\ \text{---} \end{array} \begin{array}{c} \frac{\pi}{2} \end{array} \quad \begin{array}{c} \square Z \end{array} := \begin{array}{c} \circ \\ \text{---} \end{array}$$

We've drawn the wires coming out at the top, since it will turn out not to matter if we treat them as inputs or outputs, due to the next exercise. But for now you can treat them as being output wires.

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**Exercise 3:**

- a) Show that each of the Pauli boxes is a measure-box. *Hint:* you can prove the result for  $X$ ,  $Y$ , and  $Z$  in one go with a good choice of lemma.
- b) Show that if we plug  $-\pi$  into each of the Pauli boxes' top wires that the result is the Pauli that it is named after.

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Okay, so the Pauli boxes are measure-boxes that give back the Pauli's when we plug in the right thing to their top wire. Using the construction of Exercise 2 we can then combine these Pauli boxes into larger measure-boxes, so that for any Pauli string  $P_1 \otimes P_2 \otimes \cdots \otimes P_n$  we can build an associated measure-box. Then, by plugging in a  $-\alpha$  it becomes a unitary by Exercise 1a).

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**Exercise 4:** Show that the unitaries we get by combining Pauli boxes using the constructions of Exercise 2 and then 1a), are Pauli exponentials, by showing that the diagrams agree with those found in the lecture.

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When we have two *different* Pauli's  $P, Q \in \{X, Y, Z\}$ , they always *anti-commute*:  $PQ = -QP$  (two copies of the same Pauli of course commute with one another). This means that when we have Pauli strings  $\vec{P}$  and  $\vec{Q}$ , they will either commute or anti-commute depending on how many of its factors commute or anti-commute. This (lack of) commutation translates onto the associated Pauli exponentials as well.

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**Exercise 5:**

- a) Prove diagrammatically that the Pauli exponentials  $ZZ(\alpha)$  and  $XX(\beta)$  commute for any value of  $\alpha$  and  $\beta$ .
  - b) Try doing the same rewrite strategy with the (non-commuting)  $ZZ(\alpha)$  and  $ZX(\beta)$ . What goes wrong?
  - c) Describe the condition on general Pauli strings that is required to allow you to commute the associated Pauli exponentials past each other.
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