Lecture 1

Quantum theory (should be a recap)

(S) <u>States</u> are descr. by vectors* in a Hilbert space

(C) <u>Compound</u> systems are decribed by the tensor product.

(U) Time-evolution is <u>unitary</u>.

(M) <u>Measurement</u> probabilities come from the Born rule.

STATES

<u>Def</u> A (finite-dim'l) Hilbert space $\mathcal{H}$ is a vector space with an inner product $\langle \psi | \phi \rangle \in \mathbb{C}$, $\psi, \phi \in \mathcal{H}$

Main example: $H = \mathbb{C}^n$ $\quad \psi = \begin{pmatrix} \psi^0 \\ \psi^1 \\ \vdots \\ \psi^{n-1} \end{pmatrix}$, $\psi^i \in \mathbb{C}$.

$$\langle \psi | \phi \rangle = \left(\overline{\psi^0}\ \overline{\psi^1} \cdots \overline{\psi^{n-1}} \right) \begin{pmatrix} \phi^0 \\ \vdots \\ \phi^{n-1} \end{pmatrix} = \sum_{i=0}^{n-1} \overline{\psi^i}\, \phi^i \in \mathbb{C}$$

ADJOINTS

<u>Def</u> For a linear map $M : \mathcal{H} \to \mathcal{K}$, $M^\dagger : \mathcal{K} \to \mathcal{H}$ is the unique map where: $\langle \psi | M \phi \rangle = \langle M^\dagger \psi | \phi \rangle$.

<u>Ex</u> $M : \mathbb{C}^m \to \mathbb{C}^n$ $\qquad\qquad$ $M^\dagger : \mathbb{C}^n \to \mathbb{C}^m$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \longmapsto \begin{pmatrix} \overline{\cdot} & \overline{\cdot} \\ \overline{\cdot} & \overline{\cdot} \\ \overline{\cdot} & \overline{\cdot} \end{pmatrix}$$

$n \times m$ matrix $\qquad\qquad$ $m \times n$ conjugate-transpose.

2 NOTATIONS:

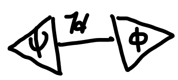
BRA-KET | **DIAGRAM**

$\psi \rightsquigarrow |\psi\rangle$ ket

$\psi^\dagger \rightsquigarrow \langle\psi|$ bra

INNER PRODUCT

$\langle\phi||\psi\rangle = \underbrace{\langle\phi|\psi\rangle}_{\text{bra-ket}}$



$M|\psi\rangle$

$\langle\phi|ZYX|\psi\rangle$

$M \rightsquigarrow M^\dagger$

Standard (computational/Z) basis

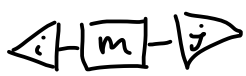
$e_i \rightsquigarrow \underbrace{\{|i\rangle\}_{i=0}^{d-1}}_{\text{ONB:}}$

$\langle i|j\rangle = \delta_i^j := \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

$= \delta_i^j$

Matrix elements:

$m_i^j := \langle j|M|i\rangle$



Def A quantum pure state is a normalised vector ($\langle\psi|\psi\rangle = 1$) $|\psi\rangle \in H$, upto a global phase: $|\psi\rangle \sim e^{i\alpha}\cdot|\psi\rangle$.

Lecture 2

Compound Systems

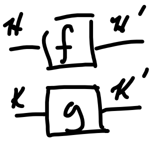
Tensor product:

$$\underset{m\text{-dim'l}}{\mathcal{H}}, \quad \underset{n\text{-dim'l}}{\mathcal{K}} \quad \rightsquigarrow \quad \underset{m \cdot n \text{ dim'l}}{\mathcal{H} \otimes \mathcal{K}}$$

$\{|i\rangle\}_{i=1}^{m}$ ONB $\qquad$ $\{|j\rangle\}_{j=1}^{n}$ ONB $\qquad$ $\{|i,j\rangle := |i\rangle \otimes |j\rangle\}_{i=1,j=1}^{m,n}$ product ONB

$$f : \mathcal{H} \to \mathcal{H}' \qquad g : \mathcal{K} \to \mathcal{K}'$$

$$f \otimes g : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H}' \otimes \mathcal{K}'$$

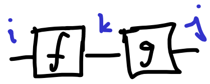


$f \otimes g$ has matrix elements:

$$(f \otimes g)_{i,j}^{k,l} := f_i^k \cdot g_j^l$$

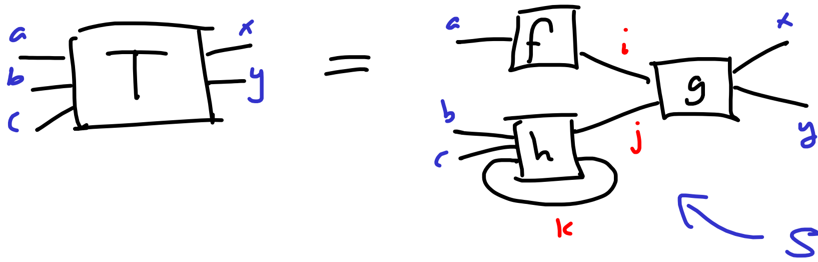
e.g.

$$\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{M} \otimes \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{N} = \begin{pmatrix} x \cdot N \\ y \cdot N \end{pmatrix} = \begin{pmatrix} xa & xb \\ xc & xd \\ ya & yb \\ yc & yd \end{pmatrix}$$

Whereas $g \circ f =$  has matrix elems:

$$(g \circ f)_i^j = \sum_k f_i^k g_k^j .$$

More generally:



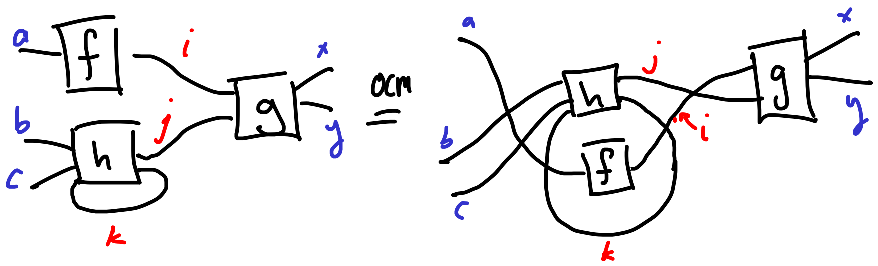
string diagram

has matrix elems:

$$T_{a,b,c}^{x,y} = \sum_{ijk} f_a^i \, g_{ij}^{xy} \, h^j_{bck}$$ ← tensor network

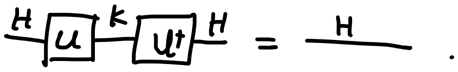
String diagram motto:

ONLY CONNECTIVITY MATTERS!
(OCM)



Unitaries:

Def An *isometry* $U: \mathcal{H} \to \mathcal{K}$ is a linear map such that $U^\dagger U = I$.



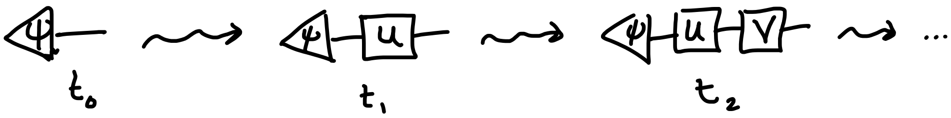
$|\psi\rangle$ $\qquad$ $|\phi\rangle = U|\psi\rangle$ $\qquad$ $\langle\phi|\phi\rangle$ $\qquad$ $\langle\psi|\psi\rangle$

$(\Rightarrow \dim H \leq \dim K)$

Def A unitary $U : \mathcal{H} \to \mathcal{K}$ is a linear map where:

$(\Rightarrow \dim \mathcal{H} = \dim \mathcal{K})$

Quantum state

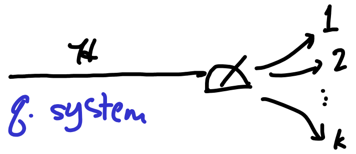


$$\left[\text{Schrödinger eq.: } \; i\hbar \frac{d}{dt}|\psi_t\rangle = \underset{\substack{\uparrow \\ \text{Hamiltonian} \\ H = H^\dagger}}{H}|\psi_t\rangle \Rightarrow |\psi_t\rangle = \underbrace{e^{-i\frac{t}{\hbar}H}}_{\substack{\uparrow \\ \text{unitary} \\ U}}|\psi_0\rangle \right]$$

Lecture 3

Measurements

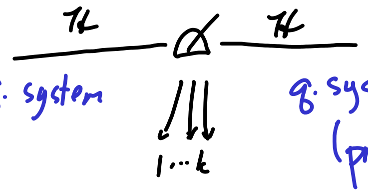
... are the only way to get info out of a q. system.



(nothing afterwards)

classical outcome

OR:



classical outcome

q. system (probably in a different state)

Def A quantum (von Neumann) measurement is a set of projectors:

$$\mathcal{M} = \{M_1, \ldots, M_k\} \quad \text{where} \quad \sum_i M_i = I.$$

When we measure $|\psi\rangle$ with $\mathcal{M}$, prob. of outcome i is:

$$\text{Prob}(i \,|\, |\psi\rangle) := \langle\psi| M_i |\psi\rangle$$

(The Born rule.)

Aside

[This is a special case of the Born rule from QPC: let

then

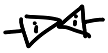
$= \langle\psi| M_i^\dagger M_i |\psi\rangle = \langle\psi| M_i |\psi\rangle$.]

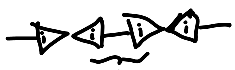
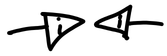
n.b. $\text{Prob}(i \,|\, e^{i\alpha} |\psi\rangle) = (e^{-i\alpha}\langle\psi|) M_i (e^{i\alpha}|\psi\rangle) = \cancel{e^{-i\alpha} e^{i\alpha}} \cdot \langle\psi|M_i|\psi\rangle = \text{Prob}(i \,|\, |\psi\rangle)$

Ex: ONB measurements:

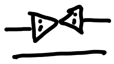
ONB $= \{$  $\}_i$

$\rightsquigarrow$

Projectors $:= \{ M_i =$  $\}_i$ and $\sum_i |i\rangle\langle i| = I$

$M_i^2 = M_i M_i =$  $=$  $= M_i = M_i^\dagger$

Ex: Measuring 1 system:

$\{ M_i :=$  $\}$

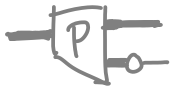
Ex: Distinguishing subspaces: $\mathrm{span}\{|0\rangle, |1\rangle\} \leq \mathbb{C}^3$ vs. $\mathrm{span}\{|2\rangle\} \leq \mathbb{C}^3$

$$\mathcal{M} = \{ M_1 = |0\rangle\langle 0| + |1\rangle\langle 1| \,,\; M_2 = |2\rangle\langle 2| \}$$

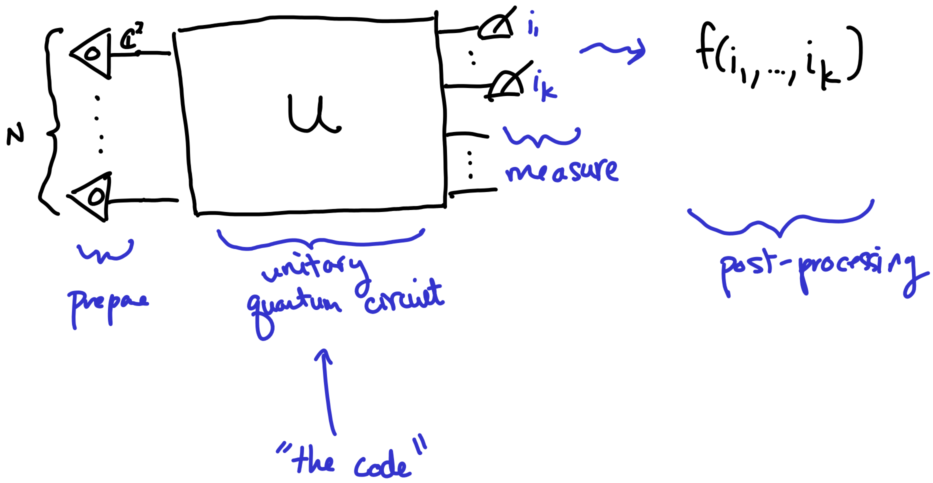
$|\psi\rangle \rightsquigarrow \mathcal{M} \rightsquigarrow \frac{1}{\| M_i |\psi\rangle \|} \cdot$ 

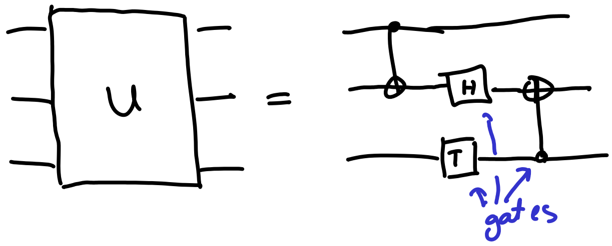
$\rightsquigarrow i$

classical outcome

[In QPC: ]

The Quantum Circuit Model.





CNOT (0,1)
H(1)
T(2)
CNOT (2,1)
...

Q: Where do they come from?

(I) classical computations:

$$\pi : \mathbb{B}^n \to \mathbb{B}^n \rightsquigarrow U_\pi :: |\vec{x}\rangle \longmapsto |\pi(\vec{x})\rangle$$

reversible fn (aka permutation) — unitary

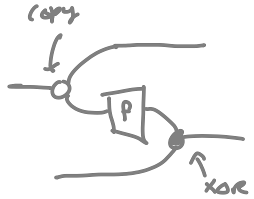
Ex

$$\text{NOT} : \mathbb{B} \to \mathbb{B} \rightsquigarrow X :: \begin{matrix} |0\rangle \longmapsto |1\rangle \\ |1\rangle \longmapsto |0\rangle \end{matrix}$$

$$\text{CNOT} : \mathbb{B}^2 \to \mathbb{B}^2 \rightsquigarrow \text{CNOT} :: |x, y\rangle \mapsto |x, x \oplus y\rangle$$

$$f : \mathbb{B}^n \to \mathbb{B} \quad \rightsquigarrow \quad U_f :: |\vec{x}, y\rangle \mapsto |\vec{x}, f(\vec{x}) \oplus y\rangle$$

any function $\qquad\qquad$ unitary ("Bennett trick")

[in PQP : ]

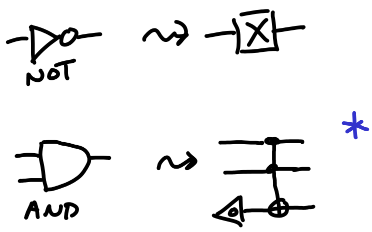
$\text{AND} : \mathbb{B}^2 \to \mathbb{B}$

Ex: $\quad \text{AND}(x,y) := xy \quad \rightsquigarrow \quad \text{TOF} :: |x,y,z\rangle \mapsto |x,y,(xy)\oplus z\rangle$

Toffoli / CCNOT

* classical (reversible) circuits $\quad C \rightsquigarrow C' \rightsquigarrow U$

AND + NOT $\qquad$ Toffoli + NOT + ancillas



(II) "quantum tricks"

(a) change of basis:

$$-\boxed{H}- \; := \; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Hadamard

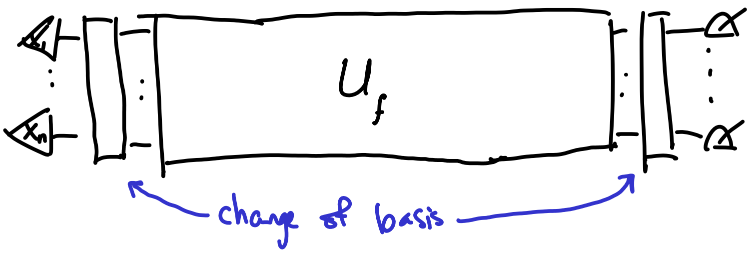
$$H|0\rangle = |+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$H|1\rangle = |-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

OR MORE GENERALLY: $\quad \mathbb{C}^N -\boxed{F}- \mathbb{C}^N \quad$ where $\quad F_j^k = \frac{1}{\sqrt{N}}\,\omega^{j\cdot k}$

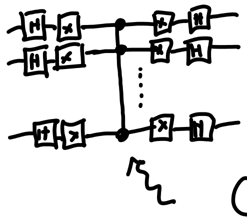
Fourier xform $\qquad \omega := e^{2\pi i/N}$

(n.b. $\; H_j^k = \frac{1}{\sqrt{2}}(-1)^{j\cdot k}, \quad \omega = e^{2\pi i/2} = e^{\pi i} = -1$)

D-J, Simon's, Shor, HSP :



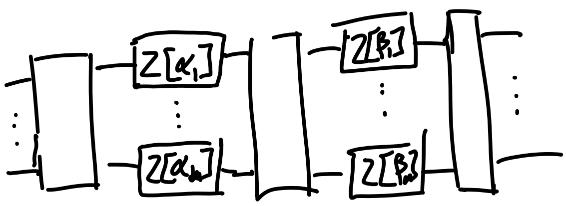
(b) Grover "diffusion"



$C^n Z :: |x_1 \dots x_n\rangle \mapsto (-1)^{x_1 \dots x_n} |x_1 \dots x_n\rangle$

(c) Hamiltonian simulation (Pauli gadgets)

(d) Variational circuits



, , , , ,

$Z[\alpha] :: \begin{array}{l} |0\rangle \mapsto |0\rangle \\ |1\rangle \mapsto e^{i\alpha}|1\rangle \end{array}$

<u>Problem</u> (synthesis): Given a (high-level) description of a computation/unitary U, build a circuit that does U.

<u>Problem</u> (optimisation) given a circuit C that does U, find a <u>smaller</u> C' that also does U.

Problem (classical simulation) given C that does U, and an input $|\psi\rangle$ either:

Strong Simulation → * compute measurement probabilities for $U|\psi\rangle$, or

weak Simulation → * Sample measurement outcomes for $U|\psi\rangle$

- ZX-diagrams are a tool for reasoning about circuits (and more!)

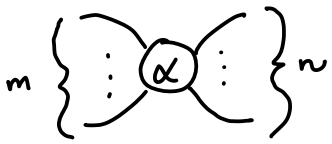
Perspective 1: ZX-diagrams are "circuits" made of spiders.

$$Z[\alpha]_m^n : \underbrace{(\mathbb{C}^2)^{\otimes m}}_{\mathbb{C}^{2^m}} \to \underbrace{(\mathbb{C}^2)^{\otimes n}}_{\mathbb{C}^{2^n}}$$

$$Z[\alpha]_m^n = |00..0\rangle\langle 00..0| + e^{i\alpha}|11..1\rangle\langle 11..1| \quad \longleftrightarrow \quad \begin{pmatrix} 1 & & & 0 \\ & 0 & & \\ & & \ddots & \\ & & 0 & \\ 0 & & & e^{i\alpha} \end{pmatrix}$$

rank 2
(usually not unitary!)

ie. $\begin{cases} |00\ldots0\rangle \longmapsto |00\ldots0\rangle \\ |11\ldots1\rangle \longmapsto e^{i\alpha}|11\ldots1\rangle \end{cases}$



$$X[\alpha]_m^n = |++\ldots+\rangle\langle ++\ldots+| + e^{i\alpha}|--\ldots-\rangle\langle --\ldots-|$$