Part II: Picturing Even More Quantum Processes

Aleks Kissinger

Spring School on Quantum Structures in Physics and CS

August 9, 2014

Outline

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4. Specialise to qubits and define the **ZX-calculus**

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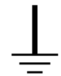
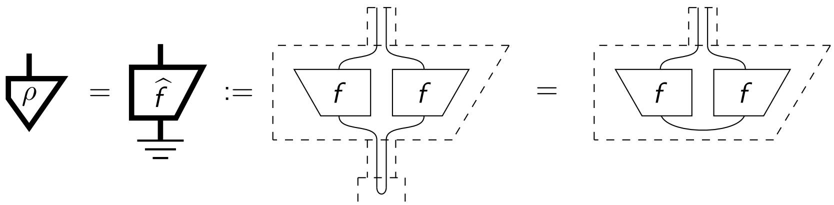
- They can always be written in terms of a **pure state** +  :



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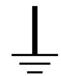
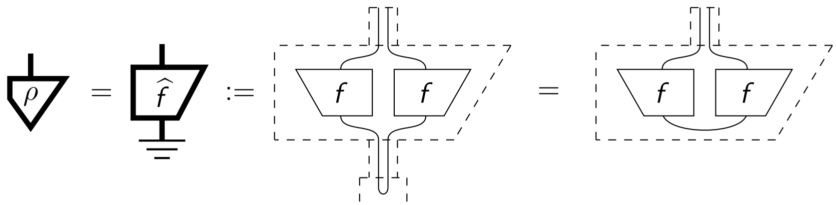
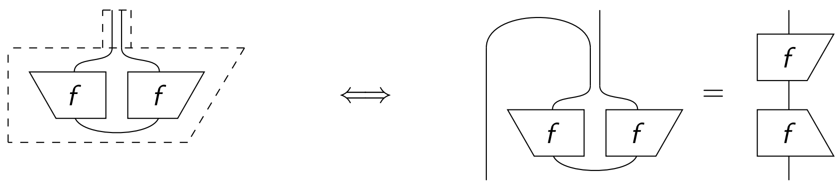
- They can always be written in terms of a **pure state** +  :



- So 'up to bending', a.k.a. partial transpose:



quantum state ρ **positive map** $f^\dagger f$

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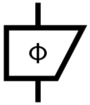
- **Quantum maps** look like this: 

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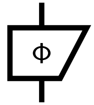
- **Quantum maps** look like this: 

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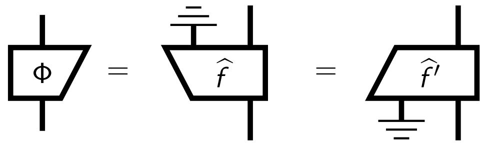

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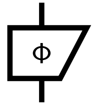
- **Quantum maps** look like this: 
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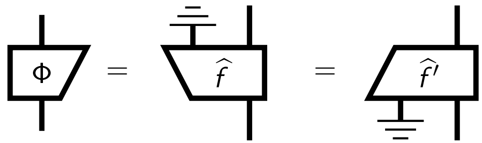
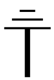


-  $= \sum_i$  for any ONB, so Φ has a **Kraus form**:

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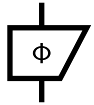
- **Quantum maps** look like this: 

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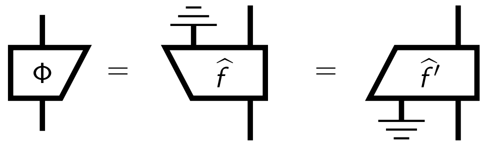
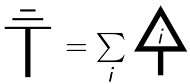
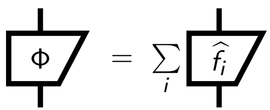


-  for any ONB, so Φ has a **Kraus form**:



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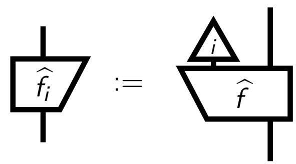

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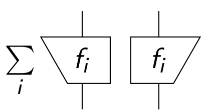
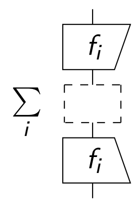


$\Longleftrightarrow$



quantum map Φ

CP-map $\sum_i f_i(-)f_i^\dagger$

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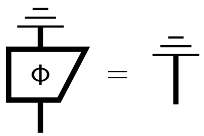

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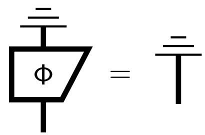

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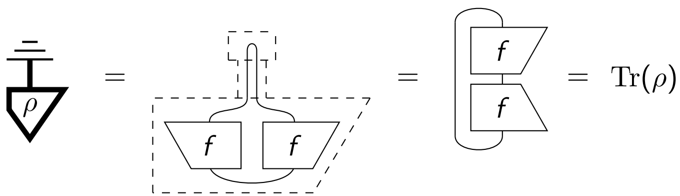

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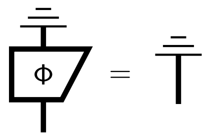

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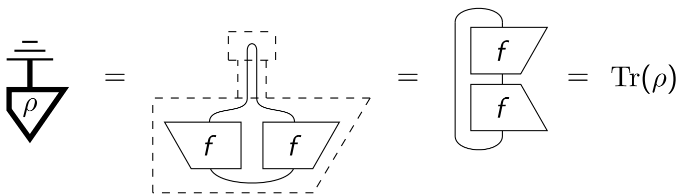

- **Causal states ↔ positive operators with trace 1**
  Causal maps ↔ trace-preserving CP-maps (CPTPs)

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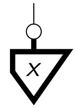
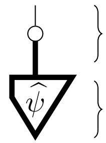
- **Classical states** look like this: 

- They can always be written as: 

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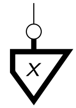

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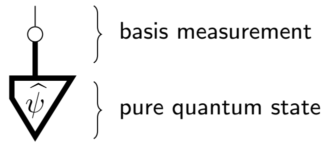

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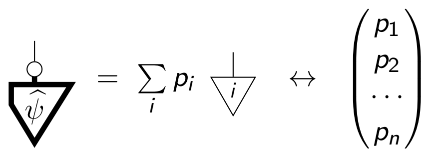

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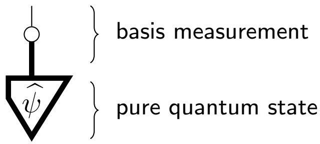
- They can always be written as: 

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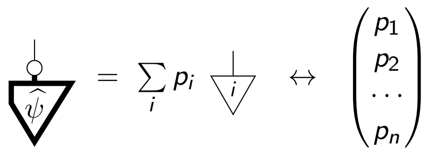

- Causality forces these numbers to sum to 1:

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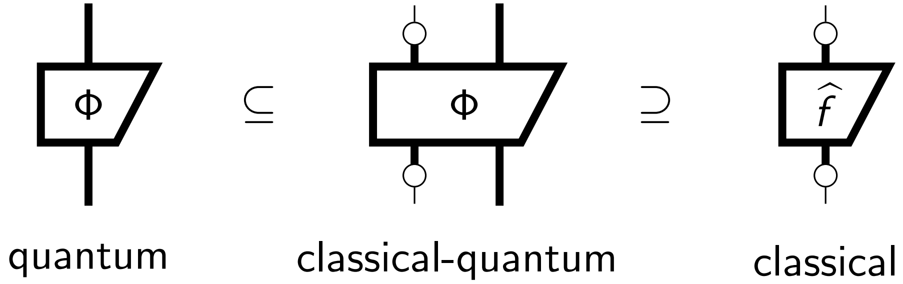
- **Quantum/classical maps** generalise both **CP-maps** and **stochastic maps**.



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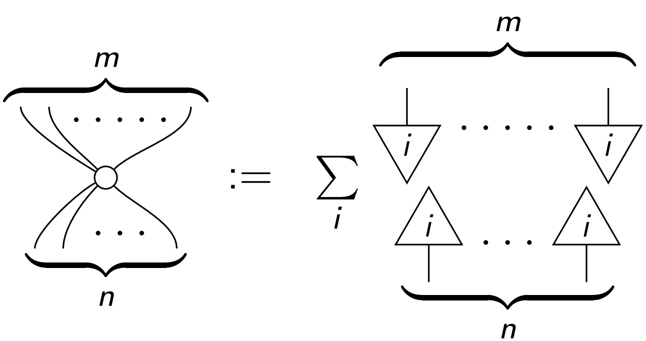

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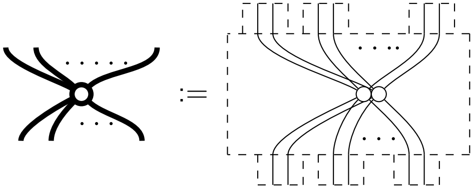

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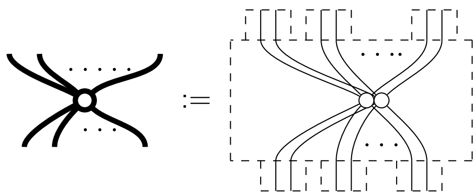
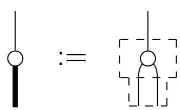
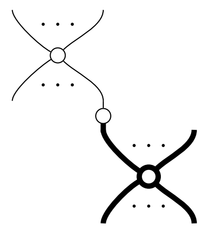


- ...and classical/quantum (a.k.a. bastard) spiders:





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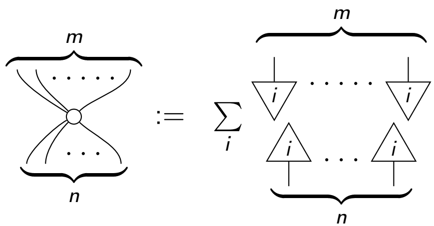
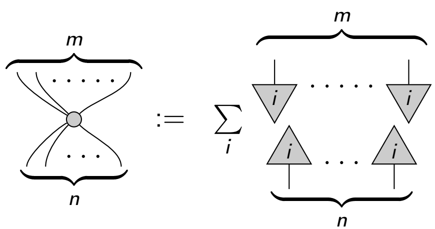
- Different bases $\rightarrow$ different coloured spiders





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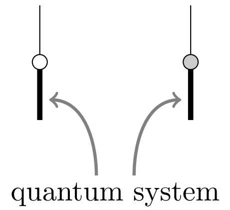

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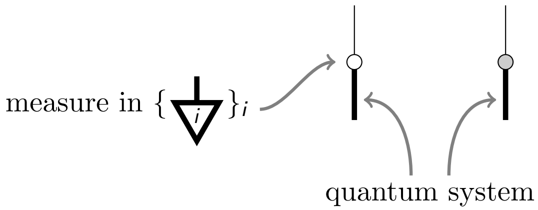

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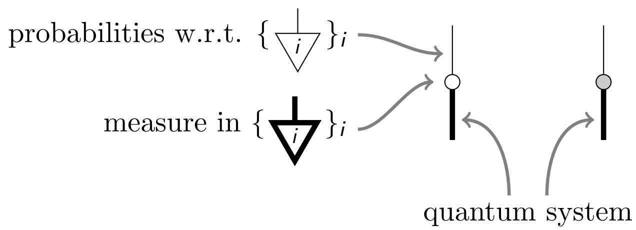

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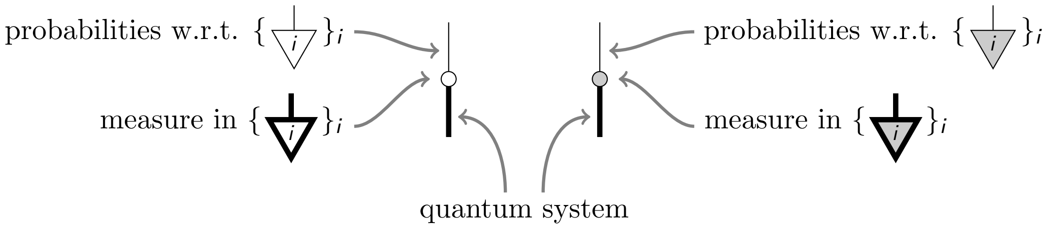

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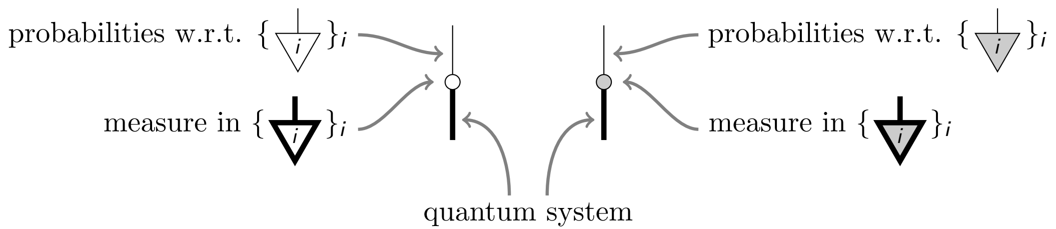
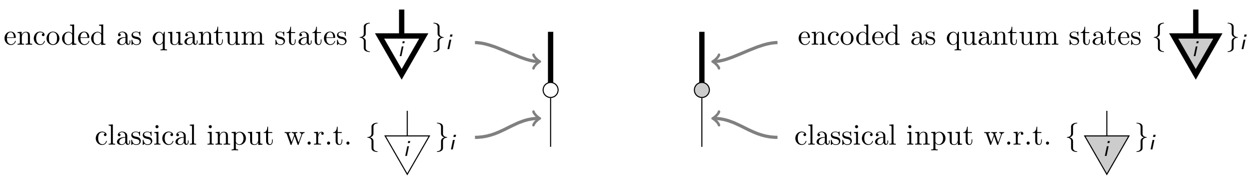


- Their adjoints are **preparations**:



Measuring ⇒ preparing

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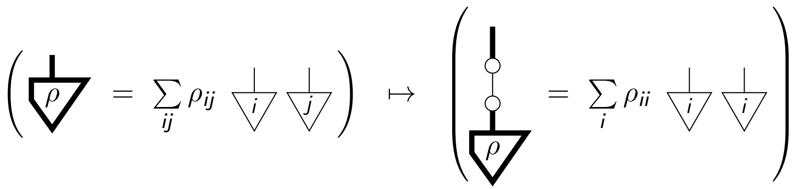

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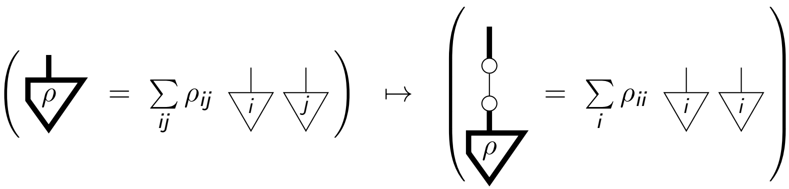

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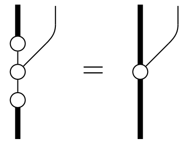

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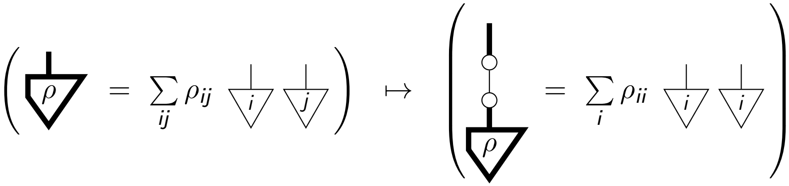

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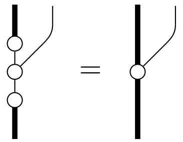
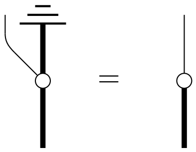


- This lets us model **non-demolition** measurement devices. The demolition measurement can be recovered just by discarding the (quantum) output:



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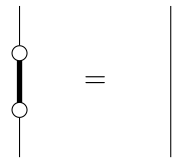

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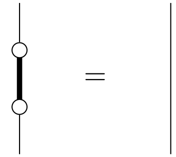

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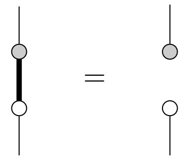

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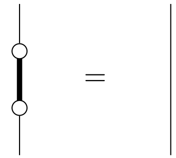

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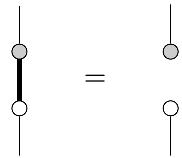

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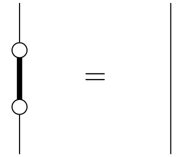

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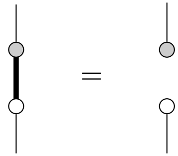

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- This is precisely what it means for two bases to be **complementary**

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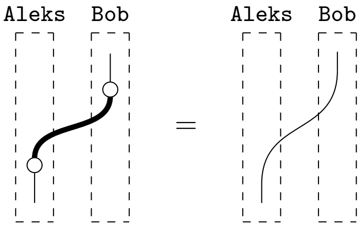

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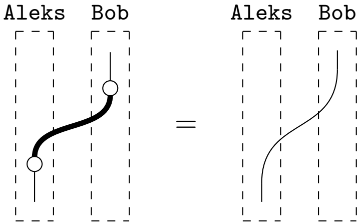
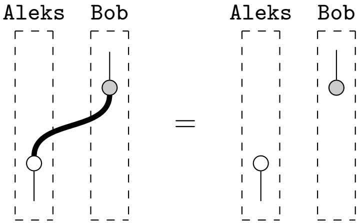


- When Bob measures in the **incorrect** basis, he gets noise:



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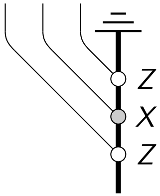

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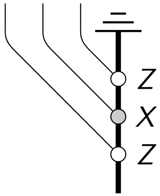

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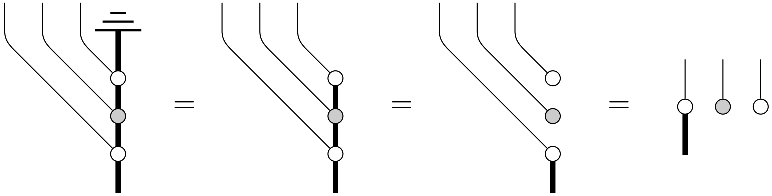

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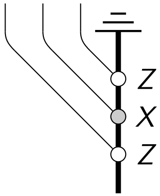

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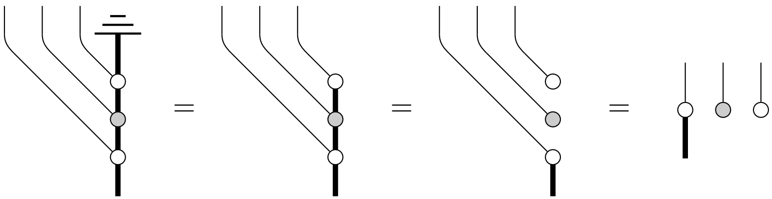
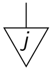
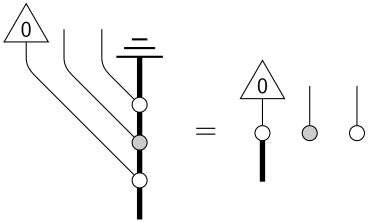


- Thus the outcome of final measurement is **uniformly random**.
(recall  = flat probability distribution w.r.t. {  }$_j$).

Complementarity – Stern-Gerlach

- Since it disconnects, the output **stays random**, even when we post-select the first measurement to be spin-up (i.e. 'block off the spin-down output'):



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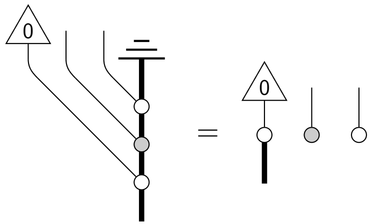
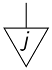
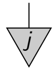
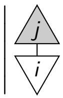


- We conclude from above that the X measurement (maximally) disturbs the system, w.r.t. the final Z measurement.

Complementarity ↔ Mutually unbiased bases

Definition

Two bases $\{$  $\}_j$ and $\{$  $\}_j$ are called *mutually unbiased* if:

$\forall i, j.$  $= \frac{1}{D}$ or equivalently, $\forall i, j.$ $\left|\right.$  $\left.\right| = \frac{1}{\sqrt{D}}$

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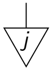
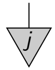
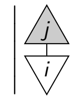
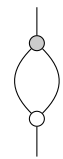
Two bases $\{$  $\}_j$ and $\{$  $\}_j$ are called *mutually unbiased* if:

$\forall i, j.$  $= \frac{1}{D}$ or equivalently, $\forall i, j.$  $= \frac{1}{\sqrt{D}}$

Theorem

Two bases are mutually unbiased iff they satisfy the *complementarity equation*:

 $= \frac{1}{D}$  or equivalently,  $= \frac{1}{D}$ 

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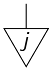
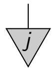
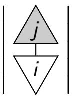
Two bases $\{$  $\}_j$ and $\{$  $\}_j$ are called *mutually unbiased* if:

$\forall i, j.$  $= \frac{1}{D}$ or equivalently, $\forall i, j.$  $= \frac{1}{\sqrt{D}}$

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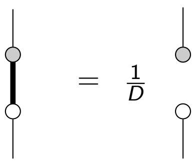

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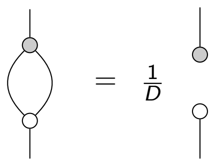
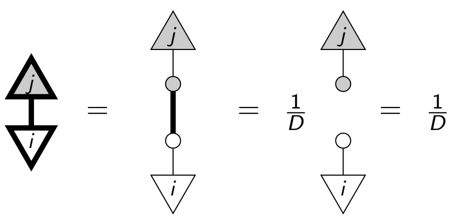


Proof.

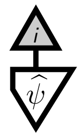
(Compl. $\Rightarrow$ MUB)



(MUB $\Rightarrow$ Compl.) follows similarly by comparing matrix entries. □

General unbiased points

- Any pure state $\widehat{\psi}$ is called *unbiased* w.r.t. to a basis if

$\forall i.$ 

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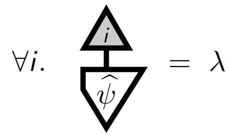

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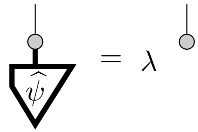

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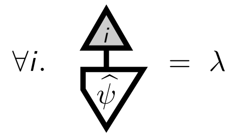

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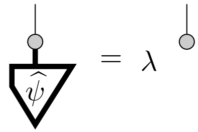
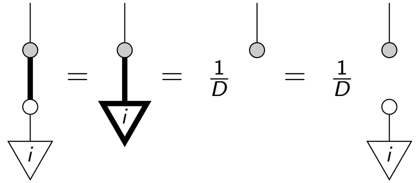
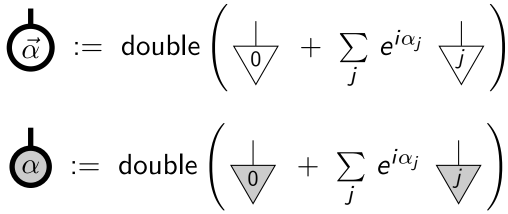


- We could just as easily use this definition of unbiasedness for MUBs. Then, the complementarity equation follows just by evaluating on basis elements:



Phase-states

- Killing the global phase, unbiased states can be parametrised by $D - 1$ complex phase factors:



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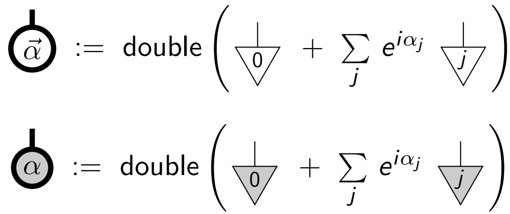

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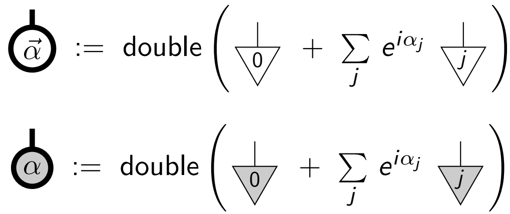

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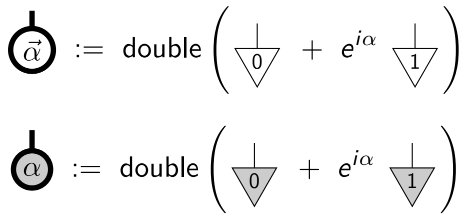
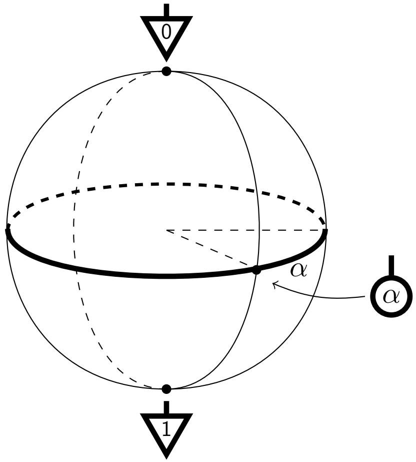
- Specialising to the 2D case:



Phase-states

- The phase states for the computational basis in 2D are just the equator of the Bloch sphere.



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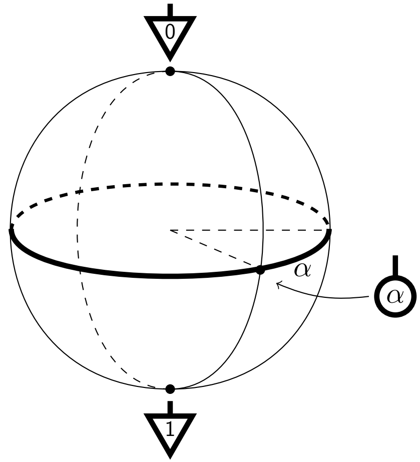

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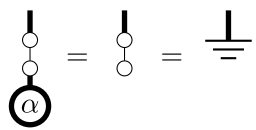

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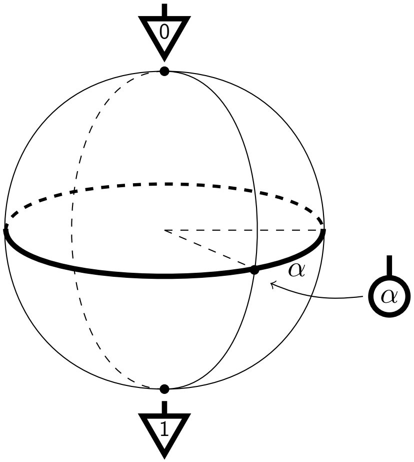

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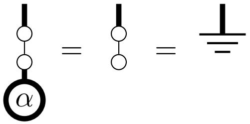

- So, phases get clobbered in the quantum/classical passage

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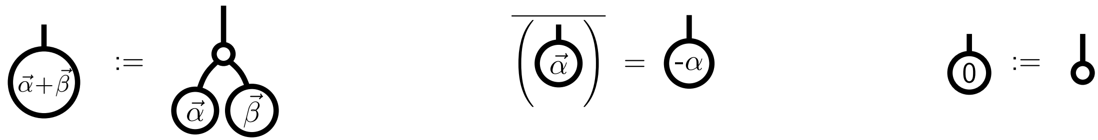

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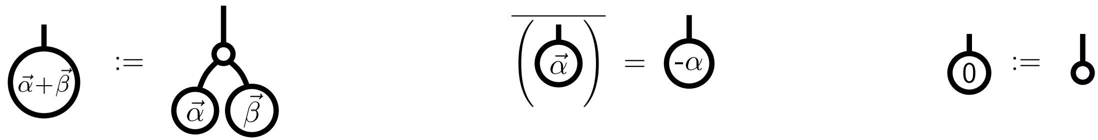
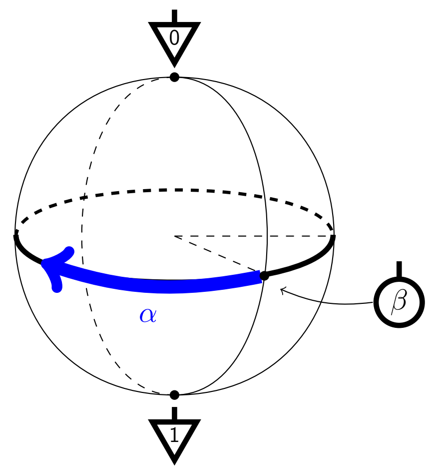


- If we multiply on the left (or the right) with a phase-state α, it performs an α rotation:



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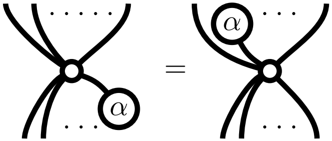
- Note that is doesn't matter where we attach a phase-state to a spider:



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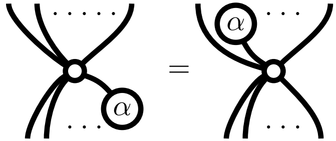
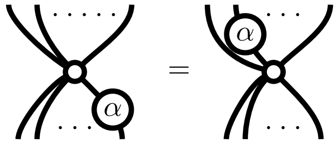


- A consequence is that **phase maps** commute through spiders:



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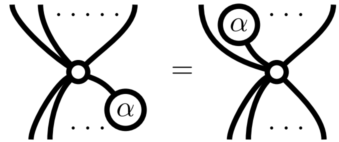

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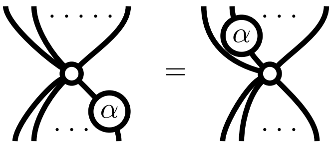
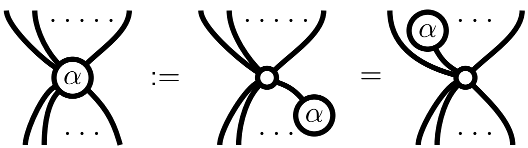
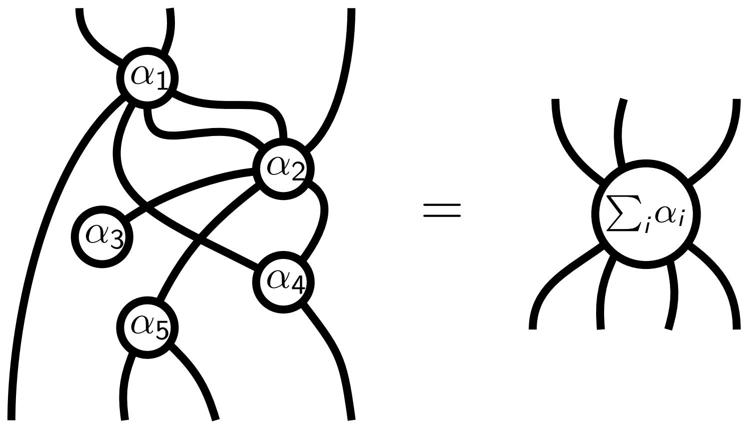


- We simplify our notation by letting spiders **eat connected phases**:



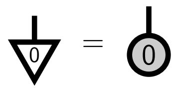
Generalised spider law

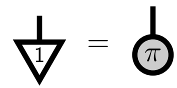
(phase group) + (spider fusion) = **(phase-spider fusion)**



Basis elements as phase states

- For a complementary pair ○/● the **basis states** of ○ are unbiased w.r.t. ●, so we could also write them as **phase states**. For ○ $:= Z$ and ● $:= X$,





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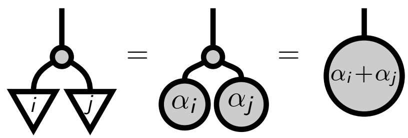

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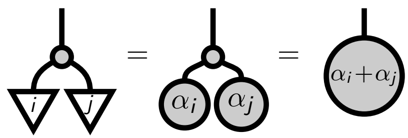
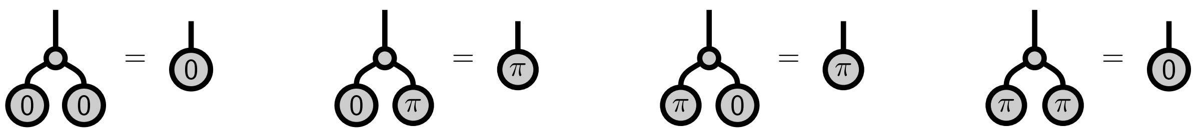


- While in general, $\alpha_i + \alpha_j$ won't be another basis element, this *is* the case for Z/X:



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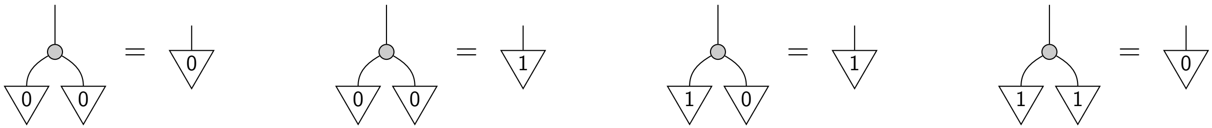
- So,  lives a double life. On the one hand, it's single version can be seen as an operation on classical data:



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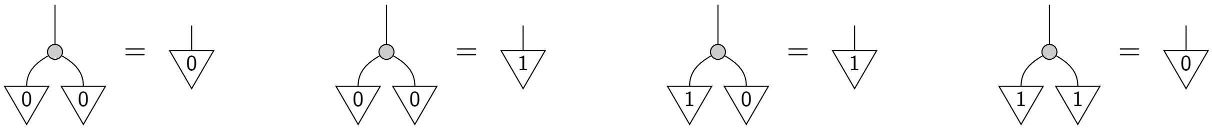

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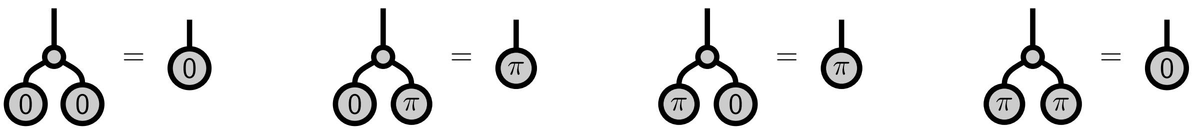

Basis elements as phase states

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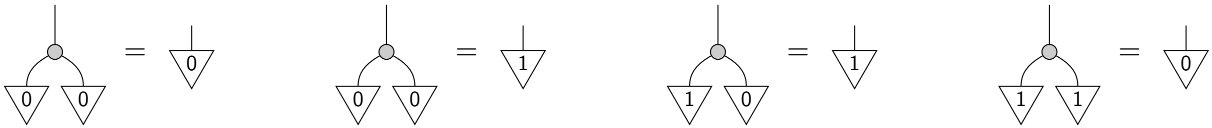

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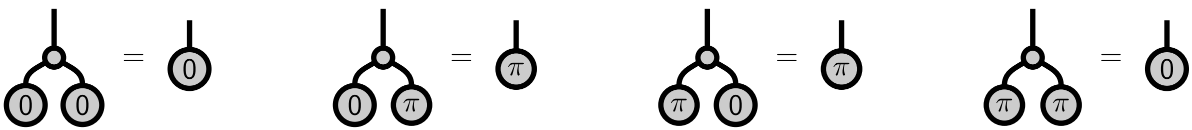
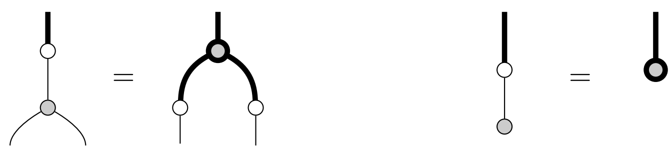


- ...and since $\{$ j $\}_j$ **encodes** the phase-states (via ○ preparation):



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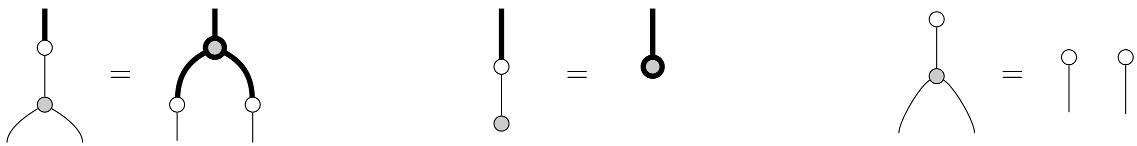
A pair of spiders is said to be *strongly complementary* if the following equations are satisfied:



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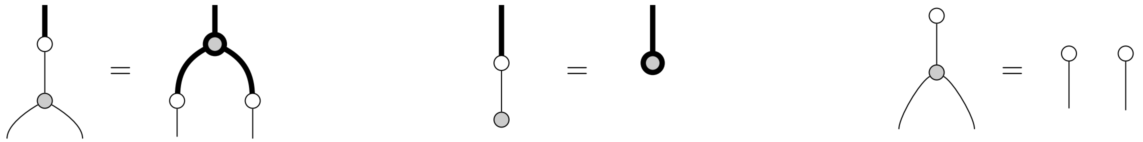

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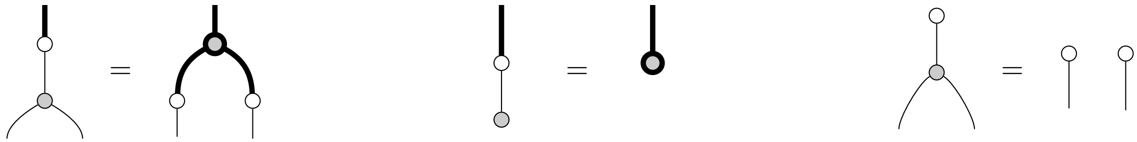

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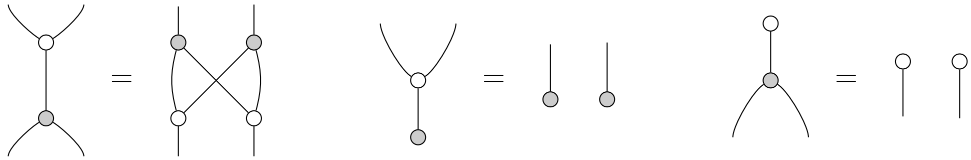

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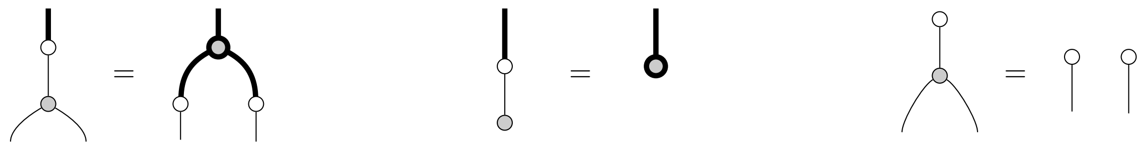

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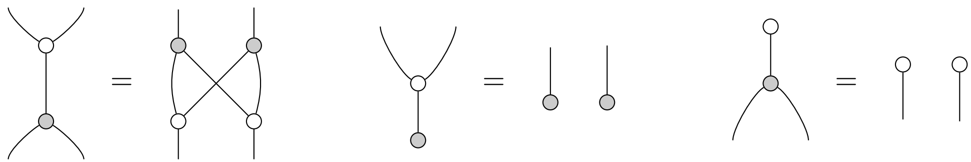

- Strongly complementary pairs of spiders form **bi-algebras**!

Strong complementarity ⇒ complementarity

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Strongly complementarity $\implies$ complementarity.

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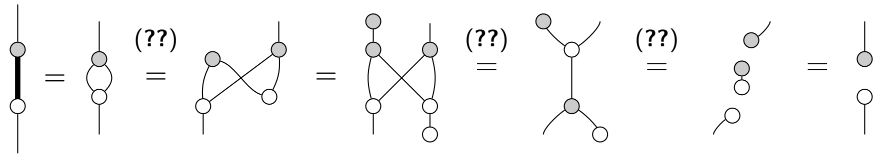

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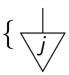
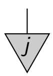
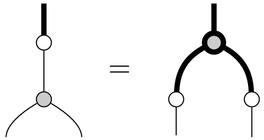
(sketch)  acts as a group operation on $\{$  $\}_j$. Fixing *which* group operation totally characterises , and hence $\{$  $\}_j$. □

Making sense of phase-multiply

- We tried to give some (pseudo-)operational interpretation of this equation:



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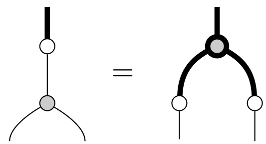
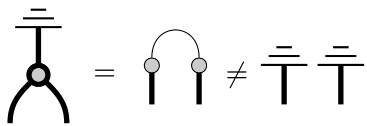


- But it falls down because, while  is a good **quantum map**, it isn't **causal**:



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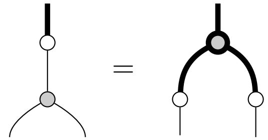

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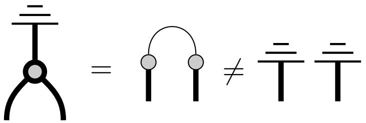

  So it isn't *physical*.

- This is because, it is both **pure**, and **it throws stuff away**. E.g. for the Z/X example before, it is $\mathbb{Z}_2$-multiply, a.k.a. XOR.

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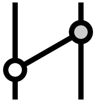

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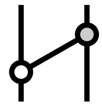

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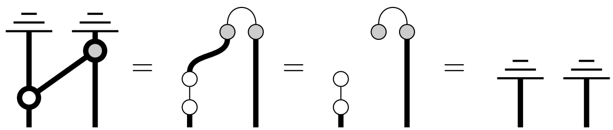

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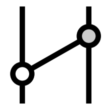

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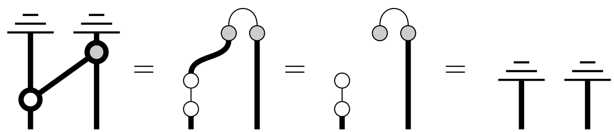
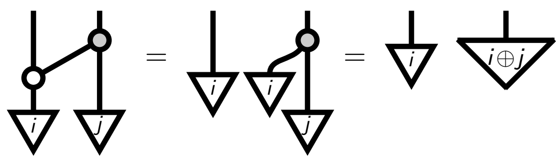


- Returning to the Z/X example, this in fact gives us a CNOT gate:



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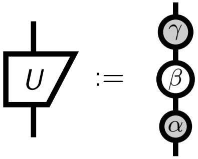

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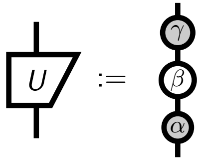

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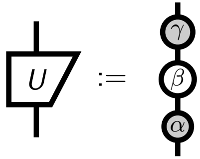

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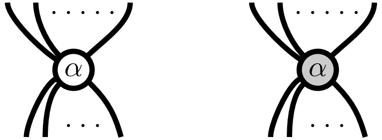
The following maps suffice to build any **quantum circuit** (i.e. unitary quantum map from qubits to qubits):



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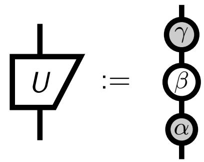

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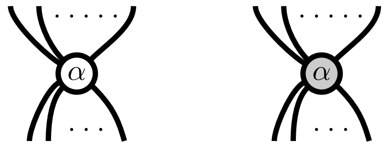

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Corollary

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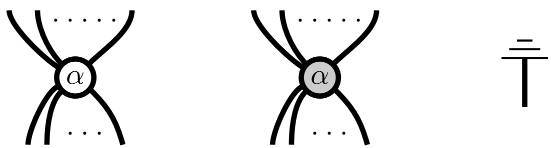

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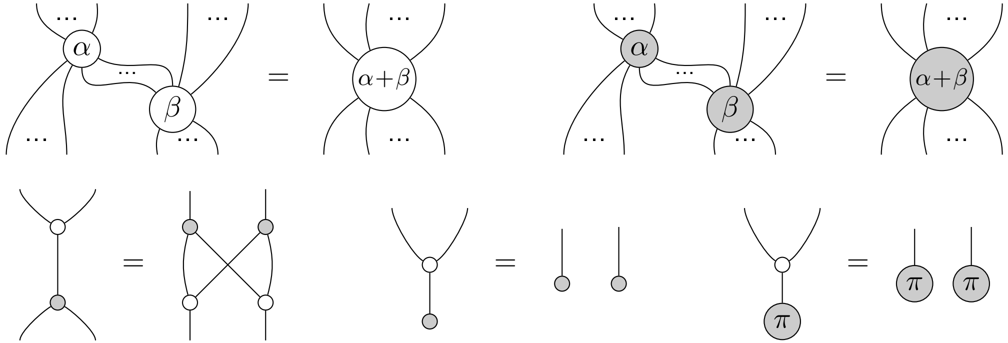
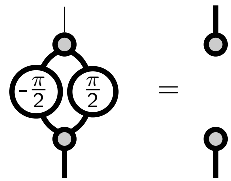


Clifford maps

- But there there are still some equations that can't be proven, e.g.



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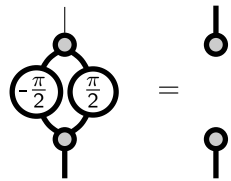

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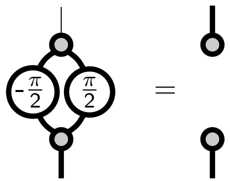

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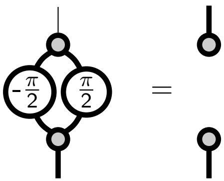

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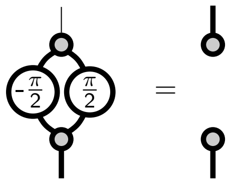

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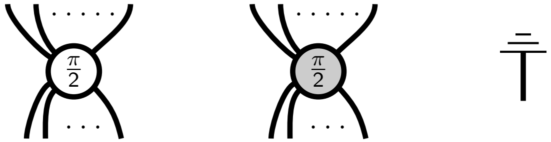
Let the family of *Clifford maps* consist of any map generated by:



(*Clifford circuit* := unitary Clifford map)

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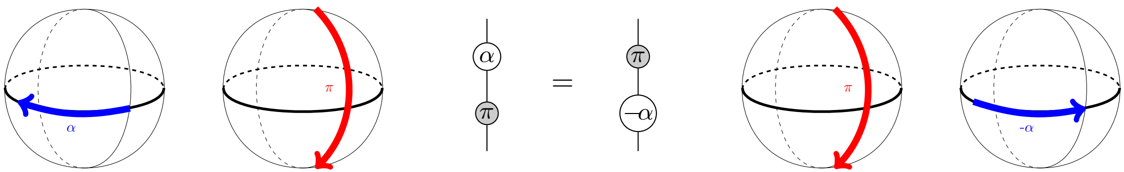

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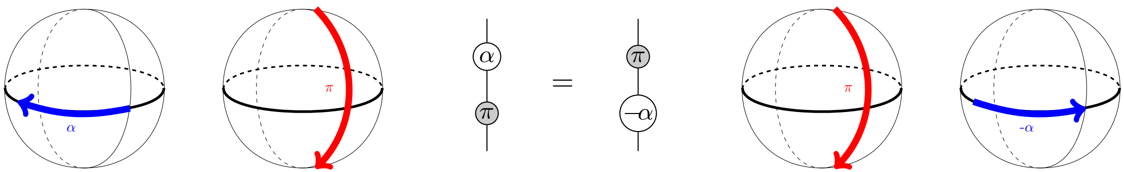

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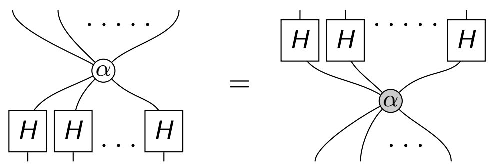

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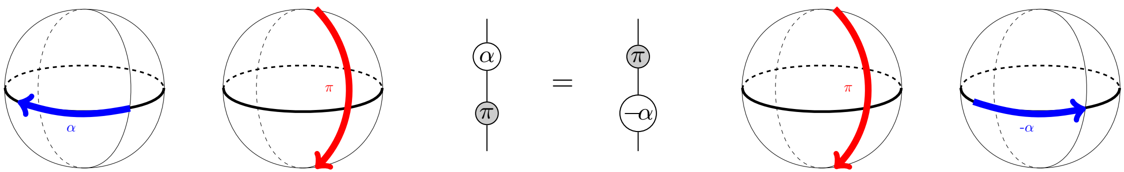

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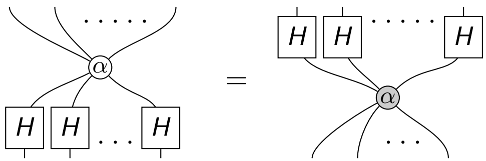
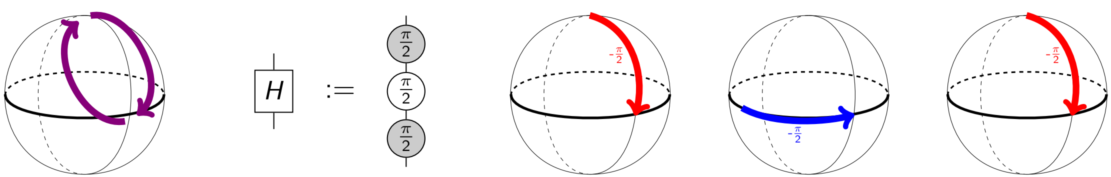


- Since it is a unitary rotation, we can give its Euler decomposition:



The ZX-Calculus

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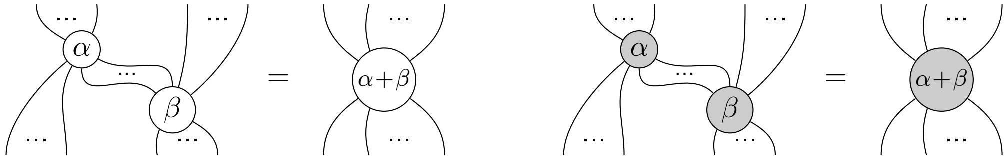
The *ZX-calculus* consists of:

- Two **spider-fusion** rules:



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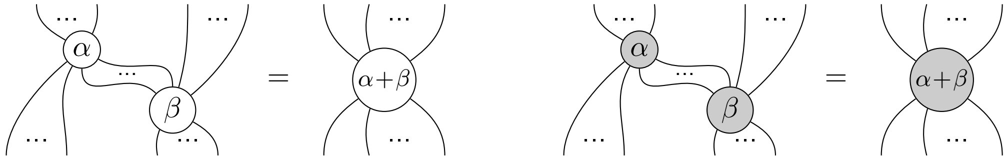

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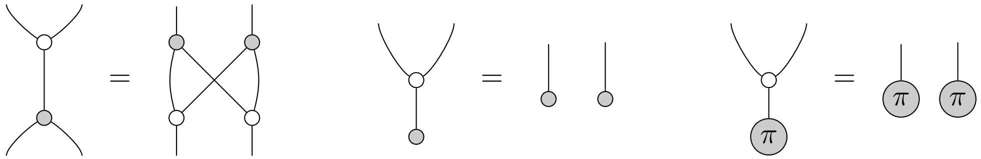

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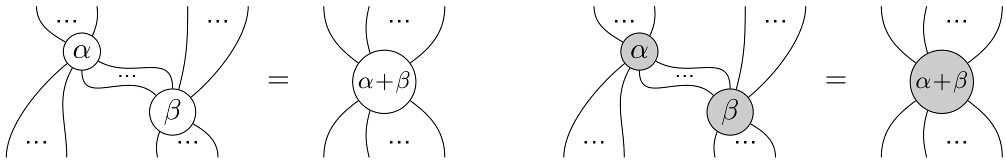

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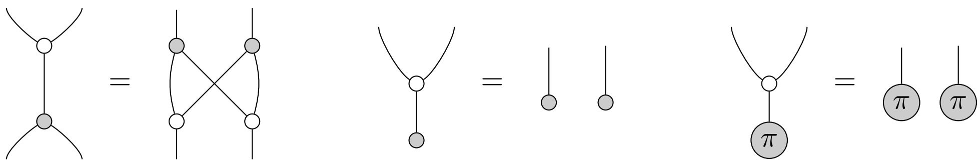

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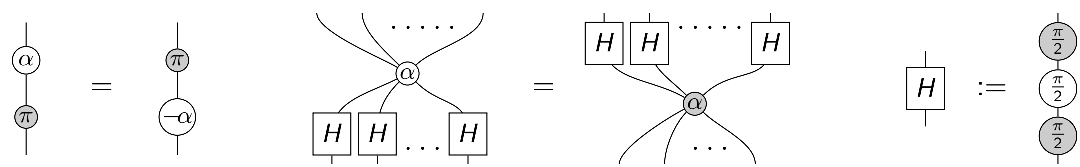

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- ...but it is complete for at least one other fragment: **single-qubit unitaries** with $\frac{\pi}{4}$ **phase maps** (a.k.a. Clifford + T).

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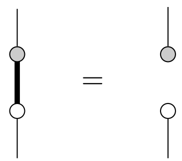

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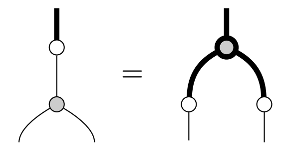

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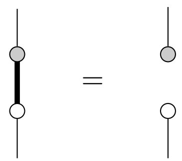
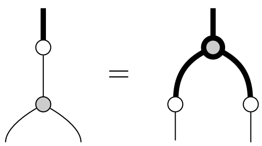


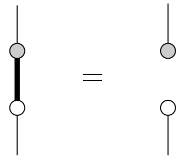
strong complementarity:

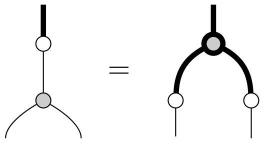


- Next time, we'll look at how to *use* the ZX-calculus in four areas:
  1. Quantum algorithms
  2. Measurement-based quantum computing
  3. Security protocols
  4. Non-locality

Summary

- We built up to the ZX-calculus, which is a graphical **swiss army knife** for calculating with **qubits**.
- Along the way, we met two important relationships between pairs of measurements:

complementarity: 

strong complementarity: 

- Next time, we'll look at how to *use* the ZX-calculus in four areas:
  1. Quantum algorithms
  2. Measurement-based quantum computing
  3. Security protocols
  4. Non-locality
- ...and demonstrate a tool for automating calculation in ZX: **QuantoDerive**