Picturing Quantum Processes

Aleks Kissinger

QTFT, Växjö 2015

June 10, 2015

Quantum Picturalism: what it **is**, what it **isn't**

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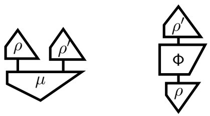
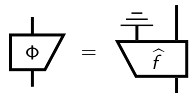


local/process tomography



purification

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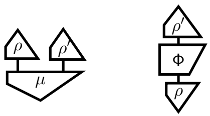

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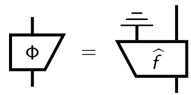
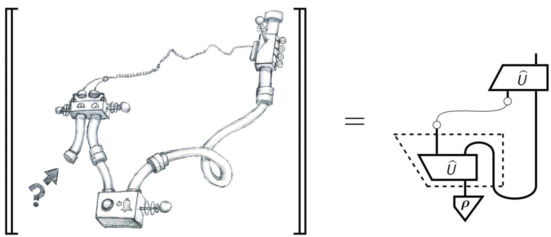


purification

- ...and relationship between *operational setups* and *theoretical models*:

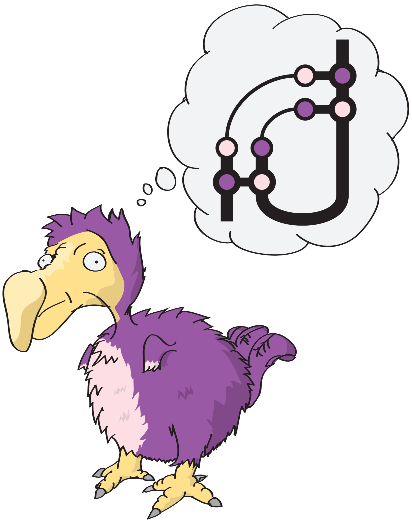


Picturing Quantum Processes

A first course in quantum theory and diagrammatic reasoning

Bob Coecke & Aleks Kissinger

CUP 2015



Outline

Picturing Quantum Processes

chapters 4-9 (roughly)

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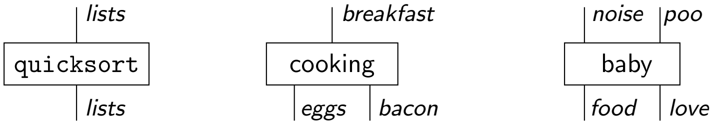
4. Classical/quantum interaction
5. Complementarity

Recap

- Wires represent *systems*, boxes represent *processes*



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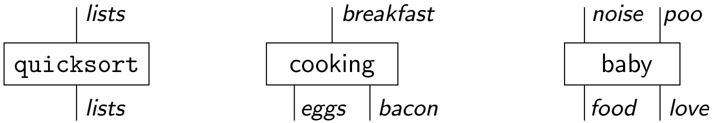
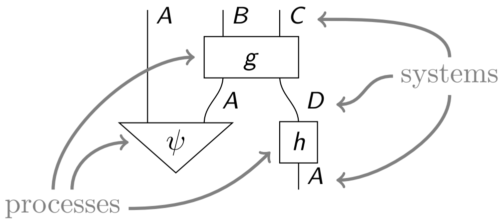
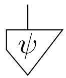


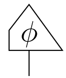
- The world is organised into *process theories*, collections of processes that make sense to combine into *diagrams*



Recap

- Certain processes play a special role:

states: 

effects: 

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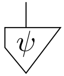
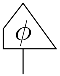
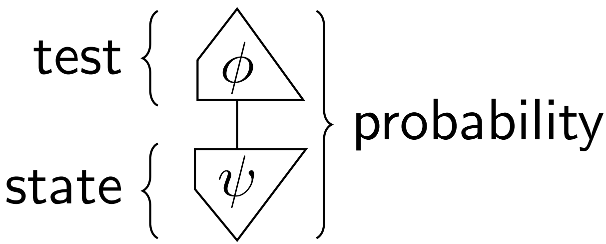
states:  *effects:*  *numbers:* 

- State + effect = number, interpreted as:



this is called the *Born rule*.

linear maps

In the process theory of **linear maps**:

linear maps

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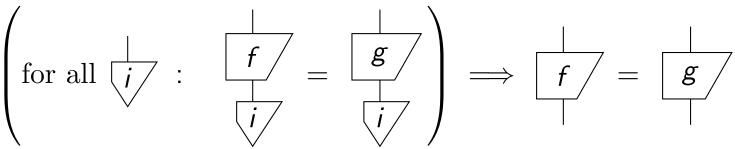
(L1) Every type has a (finite) *basis*:



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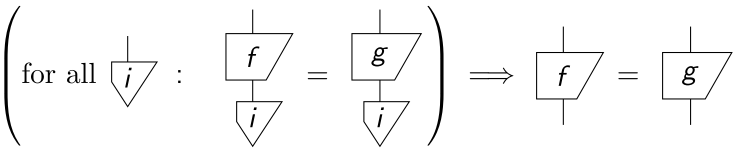
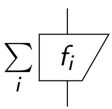


(L2) Processes can be *summed*:



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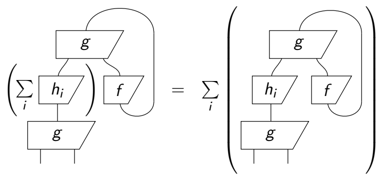

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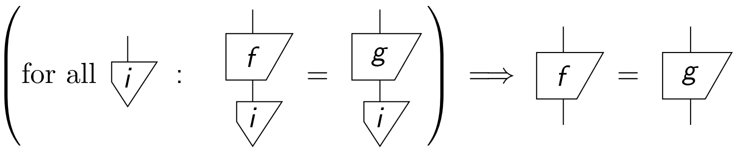

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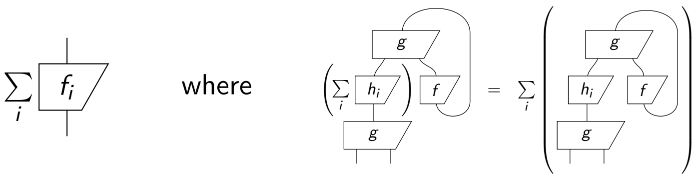
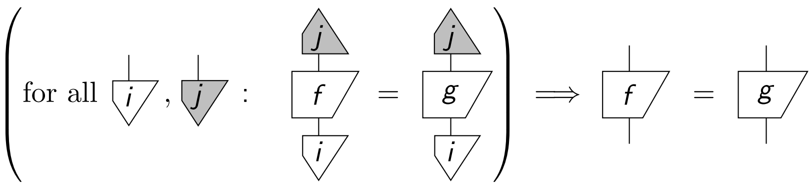


(L3) Numbers are the *complex numbers*: $\diamond\lambda \in \mathbb{C}$

Bases ⇔ process tomography

Theorem



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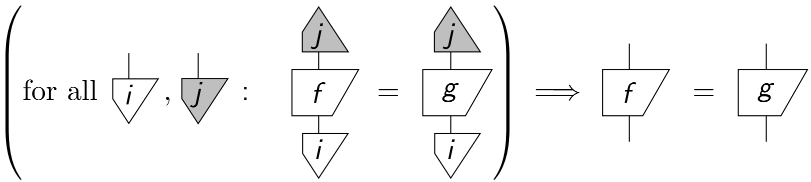
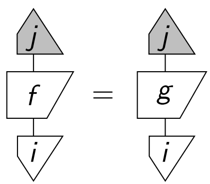


Proof.



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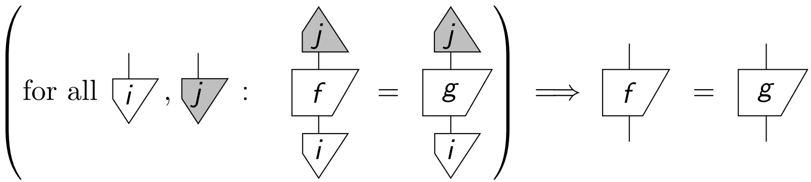

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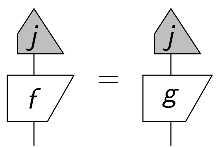

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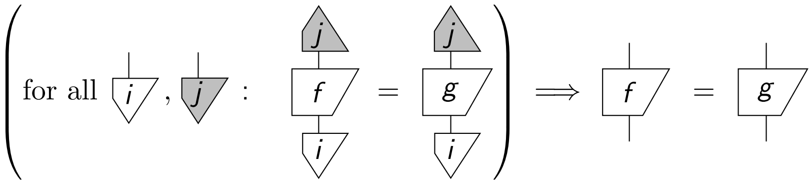

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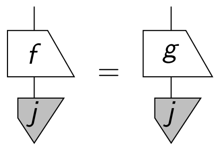

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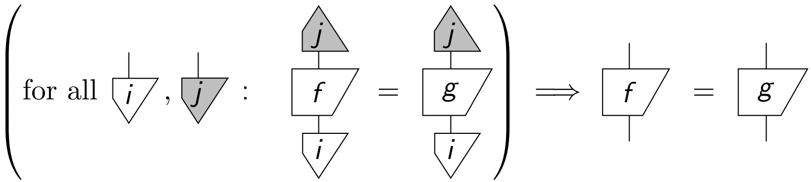

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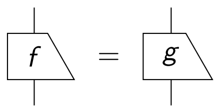

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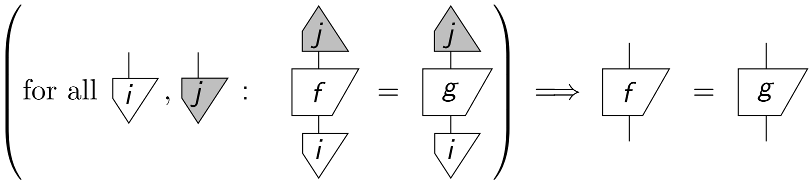

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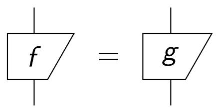

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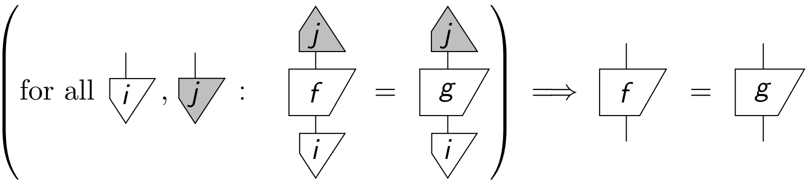
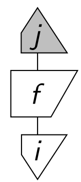
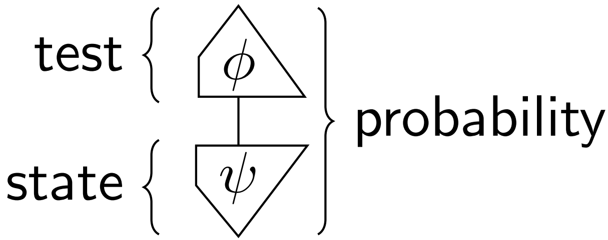


- In other words, f is uniquely fixed by its *matrix*:

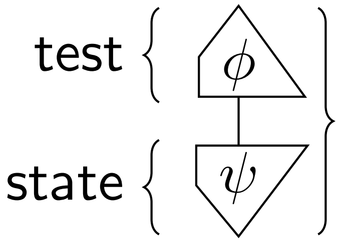
$$\begin{pmatrix} f_1^1 & f_2^1 & \cdots & f_m^1 \\ f_1^2 & f_2^2 & \cdots & f_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ f_1^n & f_2^n & \cdots & f_m^n \end{pmatrix} \quad \text{where} \quad f_i^j :=$$



What about the Born rule?



The Born rule for **relations**



$$\mathbb{B} := \{0, 1\}$$

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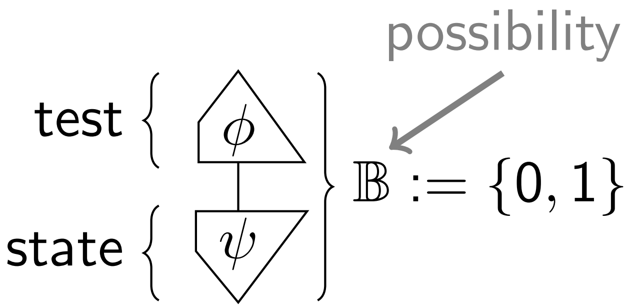

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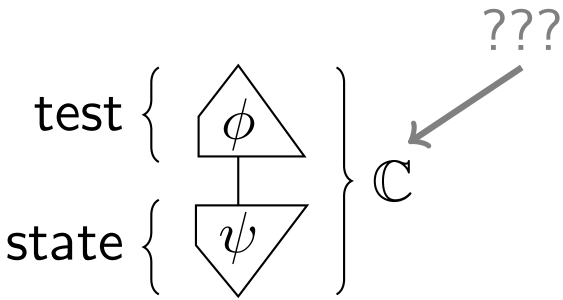
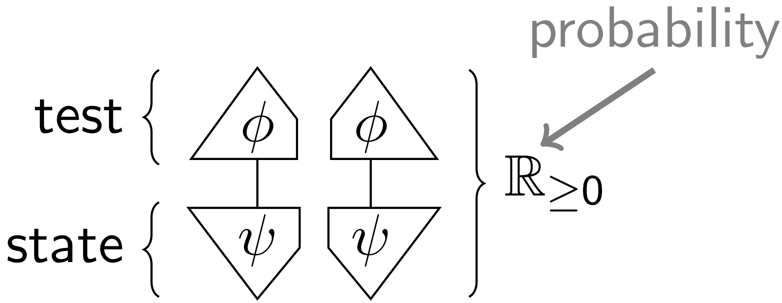


Fixing the problem



Doubled states and effects

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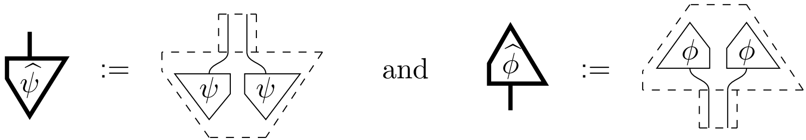

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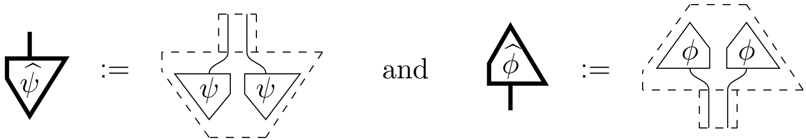
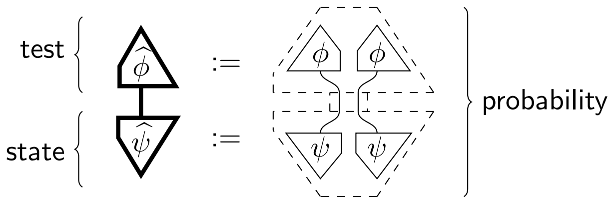
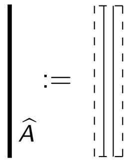


yields...



A new process theory from an old one...

- The theory of **pure quantum maps** has types:



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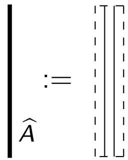
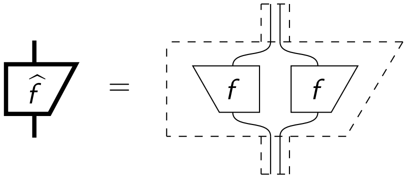


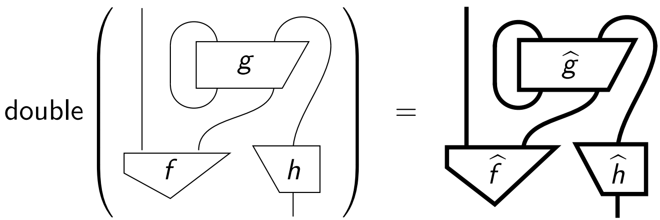
- and processes:



for all processes f from **linear maps**.

Embedding the old theory

- **linear maps** embed in **quantum maps**, and this embedding preserves diagrams:



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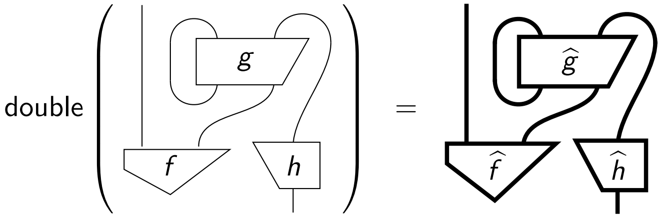

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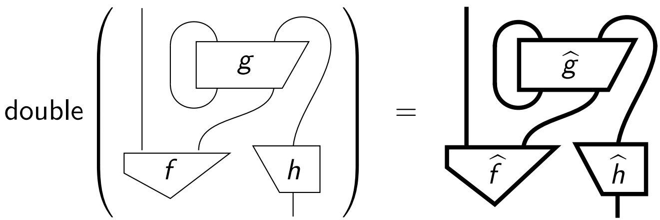
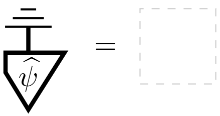


- But now we're in a bigger space, so there is room for something new, *discarding*:



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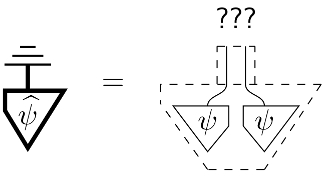

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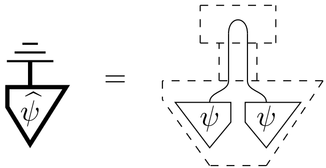

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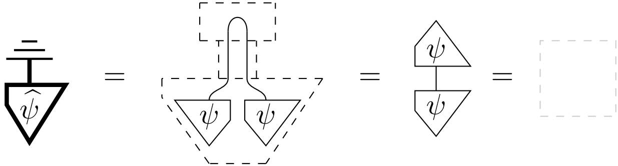

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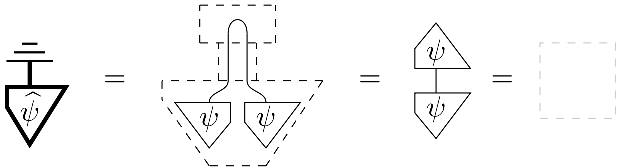

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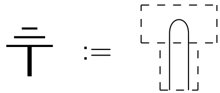

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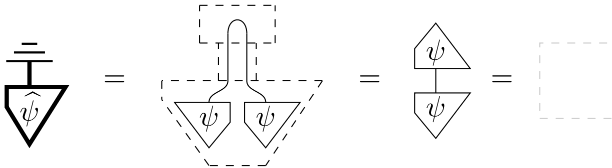

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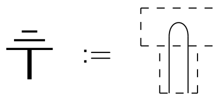
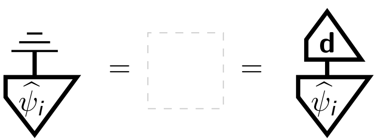


- In fact, this is the unique map with this property. Let $\{\widehat{\psi}_i\}_i$ be a basis of pure states (e.g. z^+, z^-, x^+, y^+), then:



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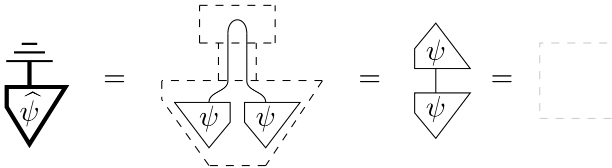

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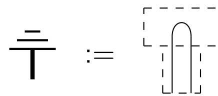

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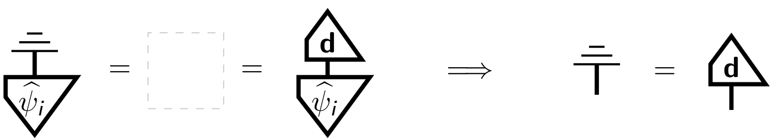
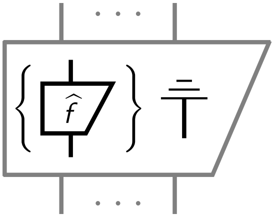


quantum maps

Definition

The process theory of **quantum maps** consists of all processes obtained from pure quantum maps and discarding:



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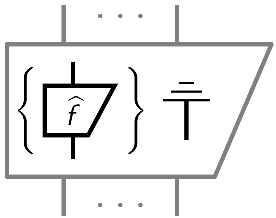
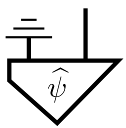
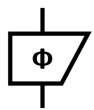
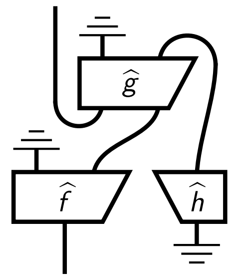


- e.g. 

:= 

and 

:= 

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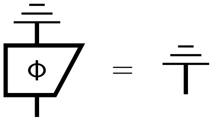

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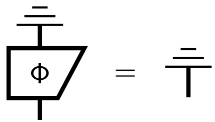

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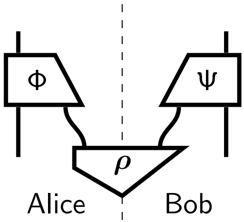

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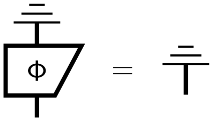

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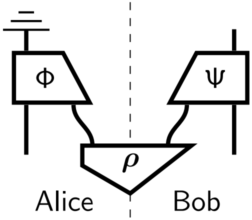

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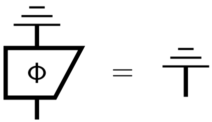

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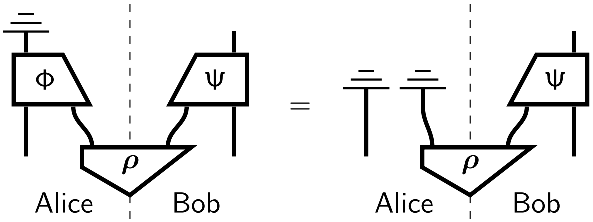

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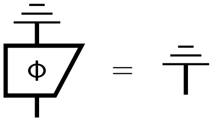

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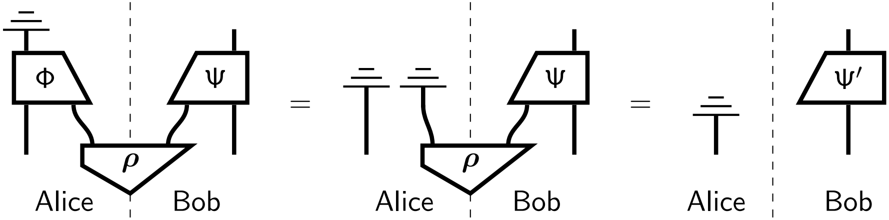
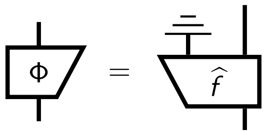


Purification

- Any quantum map extends to a pure quantum map on an extended system:



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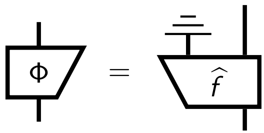

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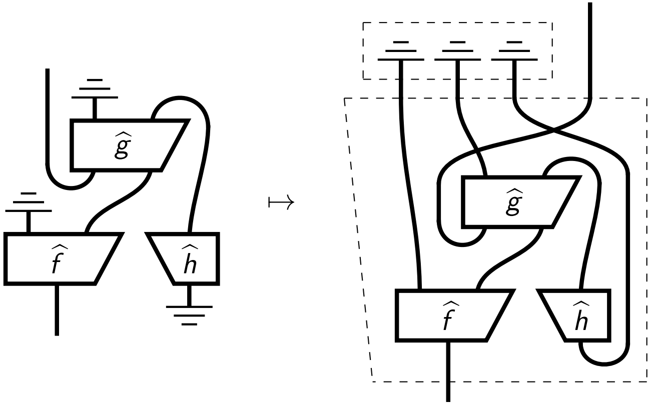

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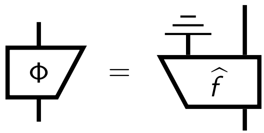

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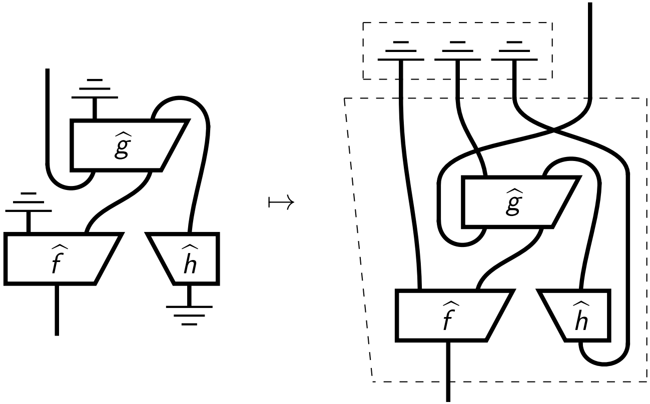
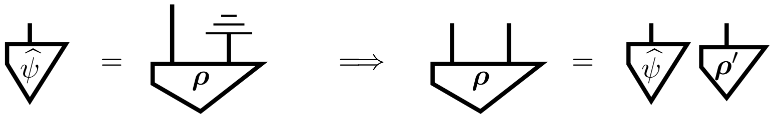


- If Ψ causal, $\widehat{f}$ must be isometry: *Stinespring dilation*.

No-broadcasting from pure extension

Theorem

A state is pure if and only if any *extension* separates:



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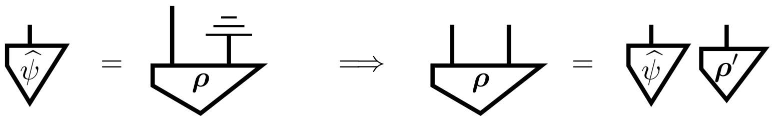
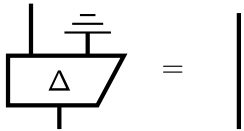


Corollary

There exists no quantum map Δ such that:

No-broadcasting from pure extension - proof

Broadcast to the left:



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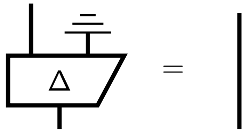
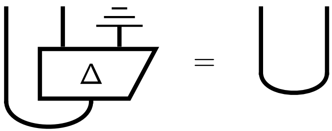


Bend the wire:



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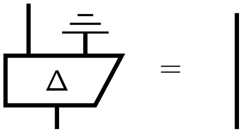

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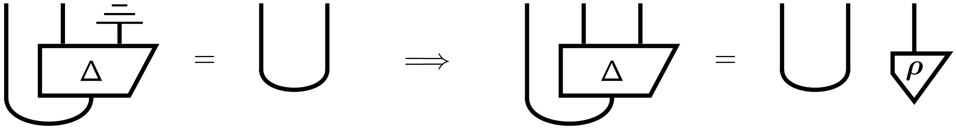

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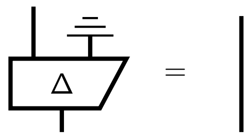

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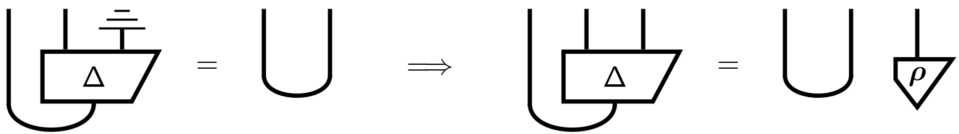
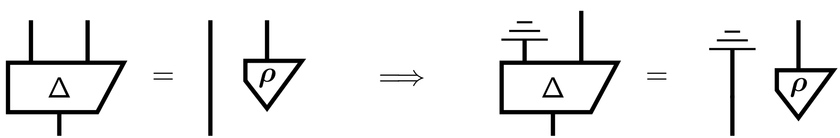


Unbend the wire and try to broadcast to the right:



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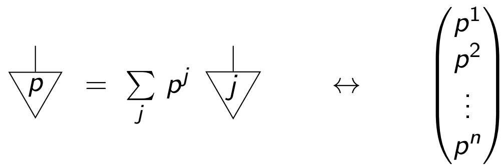

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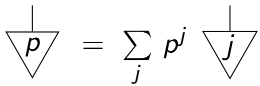

$$\leftrightarrow \begin{pmatrix} p^1 \\ p^2 \\ \vdots \\ p^n \end{pmatrix}$$

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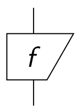

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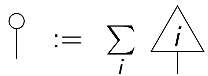
Classical operations

- Deleting is marginalisation:



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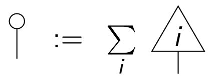

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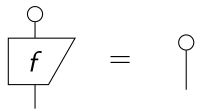

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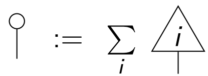

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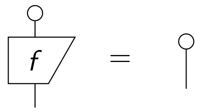
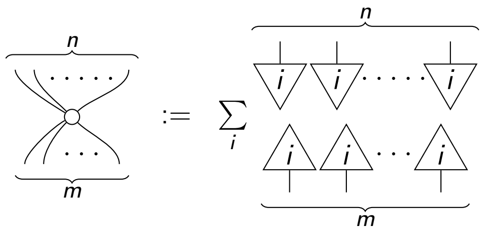


- We *can* broadcast classically:

Generalising to *spiders*

- These generalise to a whole family of maps, called *spiders*:



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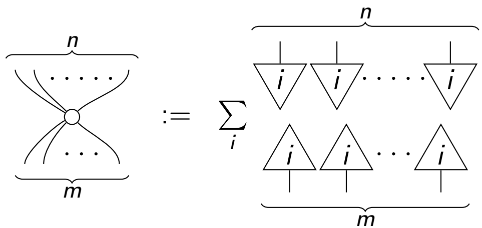
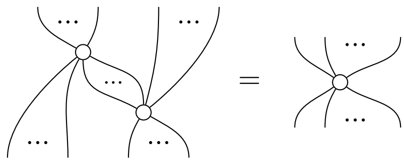


- where the only rule to remember is:



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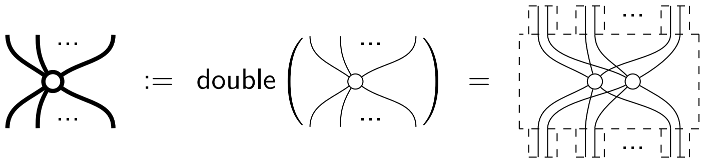

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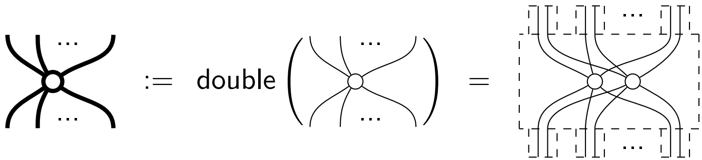

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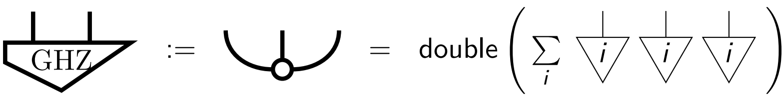

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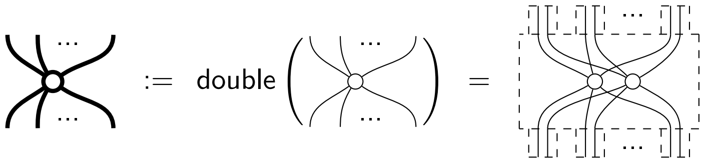

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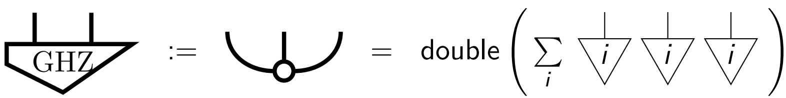
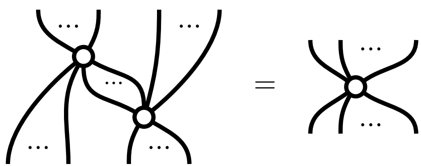


- They also fuse:



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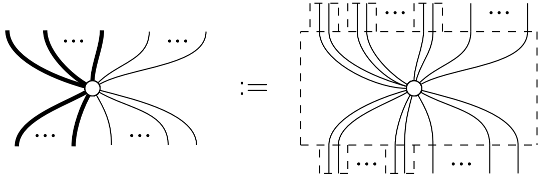

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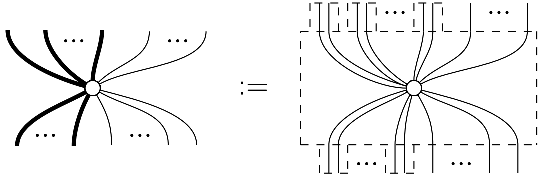

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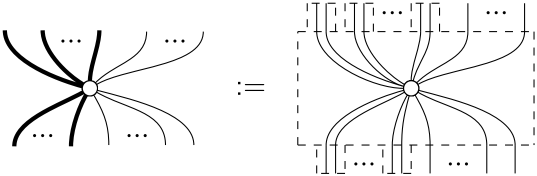

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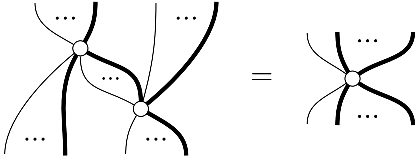

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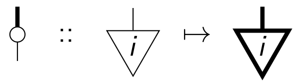

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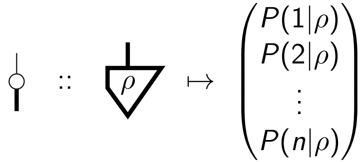

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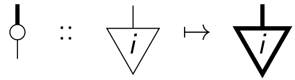
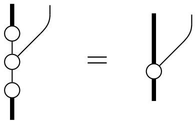
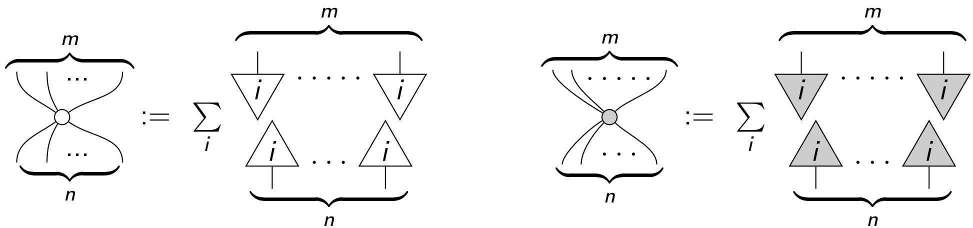


- Combining these yields more general stuff, e.g. non-demo measurements:



Multi-coloured spiders

- Different bases $\rightarrow$ different coloured spiders



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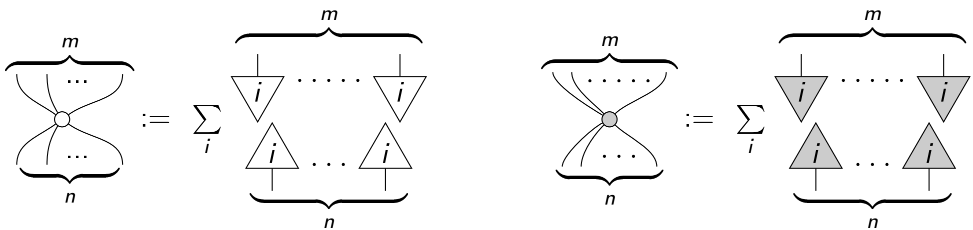
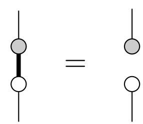


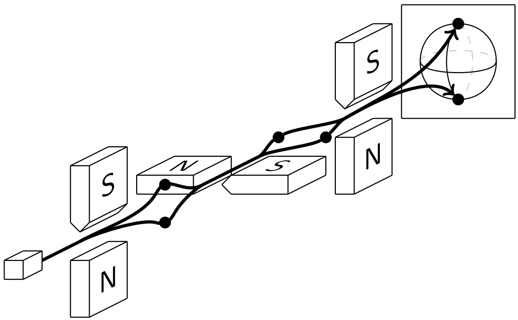
- Two spiders ○ and ● are *complementary* if:



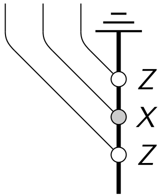
(encode in ○) + (measure in ●) = (no data transfer)

Complementarity – Stern-Gerlach

- For example, we can model Stern-Gerlach:



$\rightsquigarrow$



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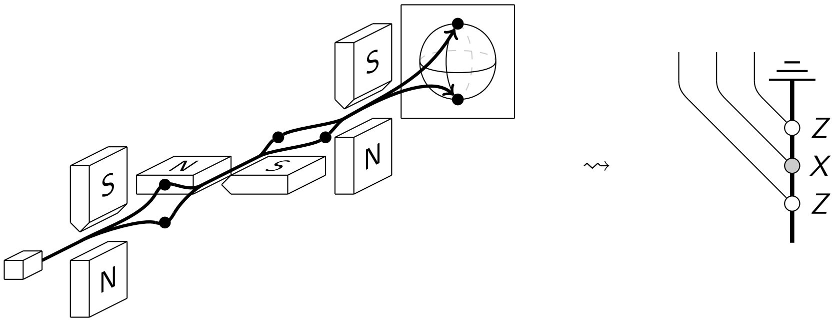
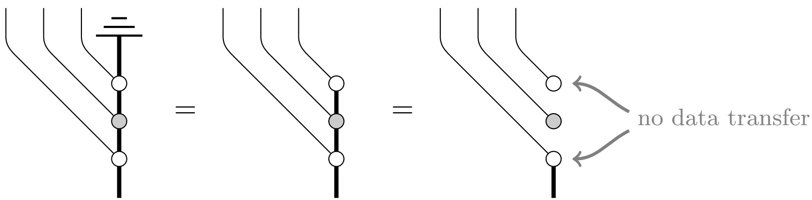


- which simplifies as:



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the rest...

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2. **Quantum computing:** Complementary spiders give a handy toolkit for building quantum circuits and MBQC
3. **Quantum resources:** Framework for resource theories, e.g. entanglement, purity
4. **Quantum foundations:** (spoiler alert!) GHZ/Mermin argument in diagrams

Applications

Picturing Quantum Processes
the rest...

1. **Quantum info:** Complementarity and cousin *strong complementary* give graphical presentations for many protocols, e.g. QKD, QSS
2. **Quantum computing:** Complementary spiders give a handy toolkit for building quantum circuits and MBQC
3. **Quantum resources:** Framework for resource theories, e.g. entanglement, purity
4. **Quantum foundations:** (spoiler alert!) GHZ/Mermin argument in diagrams

Thanks!