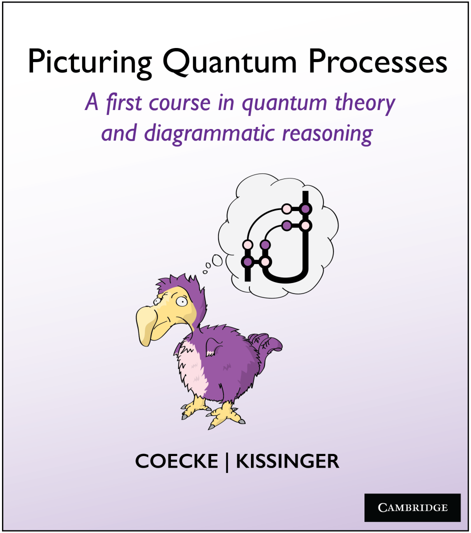
Process Theories and Graphical Language

Aleks Kissinger

Institute for Computing and Information Sciences
Radboud University Nijmegen

28th June 2016


Picturing Quantum Processes
A first course in quantum theory and diagrammatic reasoning
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Picturing Quantum Processes

> When two systems [...] enter into temporary physical interaction due to known forces between them, [...] then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. **I would not call that one but rather the characteristic trait of quantum mechanics**, the one that enforces its entire departure from classical lines of thought.
>
> — *Erwin Schrödinger, 1935.*

In quantum theory, *interaction* of systems is everything. **Diagrams** are the language of interaction.

Picturing Quantum Processes

Q: How much of quantum theory can be understood just using diagrams and diagram transformation?

A: Pretty much everything!

Outline

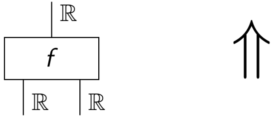
Processes

- A process is anything with zero or more *inputs* and zero or more *outputs*
- For example, this function:

$$f(x, y) = x^2 + y$$

...is a process when takes two real numbers as input, and produces a real number as output.

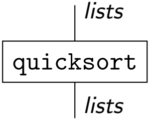
- We could also write it like this:



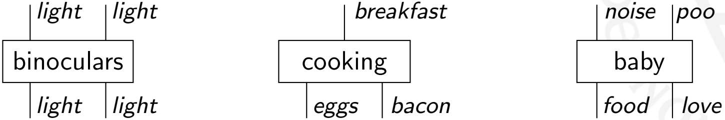
- The labels on wires are called system-types or just types

More processes

- Similarly, a computer programs are processes
- For example, a program that sorts lists might look like this:



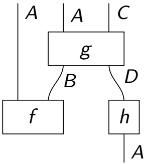
- These are also perfectly good processes:



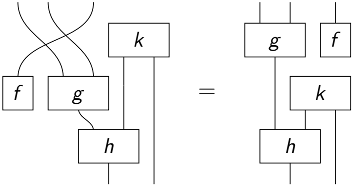
- We always think of a process as something that *happens*
- E.g. ‘binoculars’ represents one use of binoculars

Diagrams

- We can combine simple processes to make more complicted ones, described by diagrams:

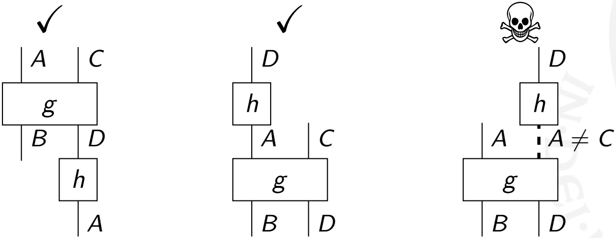


- The golden rule: only connectivity matters!



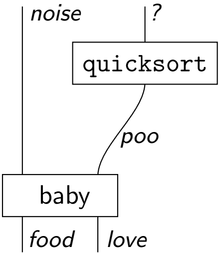
Types

- Connections are only allowed where the types match, e.g.:



Types and Process Theories

- Types tell us when it makes sense to plug processes together
- Ill-typed diagrams are undefined:



- In fact, these processes don't ever sense to plug together
- A family of processes which *do* make sense together is called a process theory

Process Theory: Definition

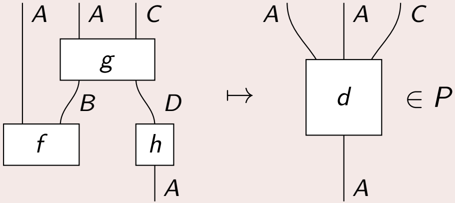
Definition

A *process theory* consists of:

(i) a collection T of *system-types* represented by wires,

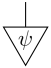
(ii) a collection P of *processes* represented by boxes, with inputs/outputs in T, and

(iii) a means of interpreting diagrams of processes as processes:



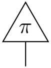
Special processes: states and effects

- Processes with no inputs are called states:



Interpret as: preparing a system in a particular configuration, where we don't care what came before.

- Processes with no outputs are called effects:

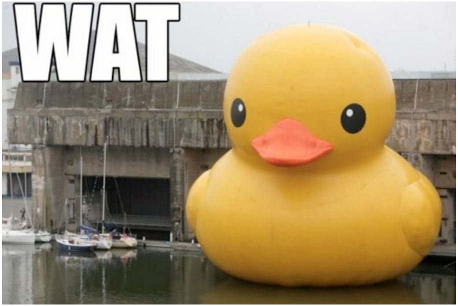


Interpret as: testing for a property π, where we don't care what happens after.

Special processes: numbers

- A number is a process with no inputs or outputs, written as:

$$\diamond\lambda \qquad \text{or just:} \qquad \lambda$$



Why are “numbers” called numbers?

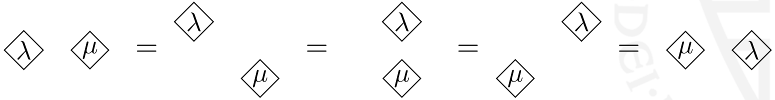
- “Numbers” can be multiplied by parallel composition:

$$\lambda \cdot \mu \quad := \quad \lambda \;\; \mu$$

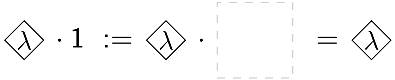
- This is associative:

$$(\lambda \cdot \mu) \cdot \nu = \lambda \;\; \mu \;\; \nu = \lambda \cdot (\mu \cdot \nu)$$

- ...commutative:



- ...and has a unit, the empty diagram:



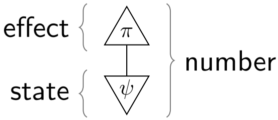
Numbers form a commutative monoid

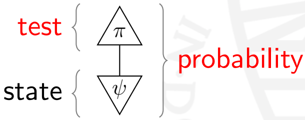
...so numbers always form a *commutative monoid*, just like most numbers we know about:

- real numbers $\mathbb{R}$
- complex numbers $\mathbb{C}$
- probabilities $[0, 1] \subset \mathbb{R}$
- booleans $\mathbb{B} = \{0, 1\}$, "$\cdot$" is AND
- ...

When a state meets and effect

- We have seen that we can to treat processes with no inputs/outputs as numbers. But why do we want to?
- **Answer:**





- state + effect = number. A probability!
- This is called the (generalised) Born rule

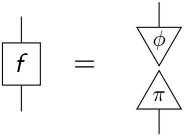
Process theories in general

Q: What kinds of behaviour can we study using just diagrams, and nothing else?

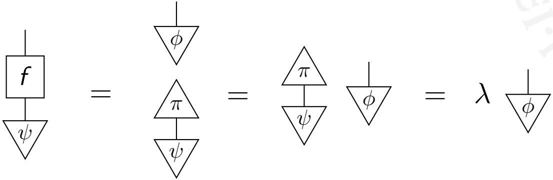
A: (Non-)separability

Separability of processes

- A process f $\circ$-*separates* if there exists a state ϕ and effect π such that:



- If we apply this process to any other state, we always (basically) get ϕ:



Trivial process theories

Hence:

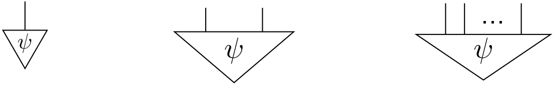
all processes $\circ$-separate $\implies$ nothing ever happens!

Definition

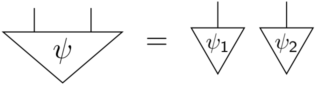
A process theory is called *trivial* if all processes $\circ$-separate.

Separable states

- States can be on a single system, two systems, or many systems:



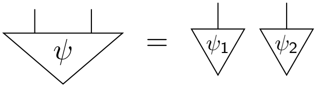
- A state ψ on two systems is $\otimes$-*separable* if there exist ψ_1, ψ_2 such that:



- **Intuitively:** the properties of the system on the left are *independent* from those on the right
- Classically, we expect *all states* to $\otimes$-separate

Characterising non-separability

- ...which is why non-separable states are way more interesting!
- But, how do we know we've found one?
- i.e. that there do not exist states ψ_1, ψ_2 such that:



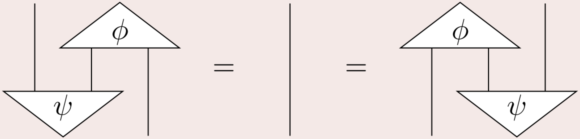
- **Problem:** Showing that something doesn't exist can be hard.

Characterising non-separability

Solution: Replace a negative property with a postive one:

Definition

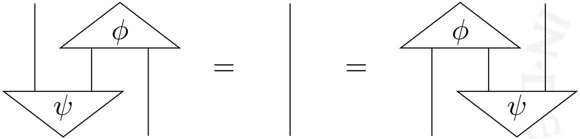
A state ψ is called *cup-state* if there exists an effect ϕ, called a *cap-effect*, such that:



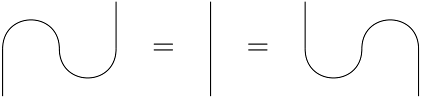
Cup-states

- By introducing some clever notation:

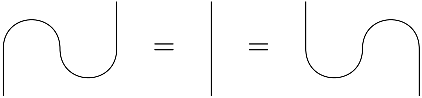
- Then these equations:



- ...look like this:



Yank the wire!

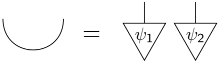


A no-go theorem for separability

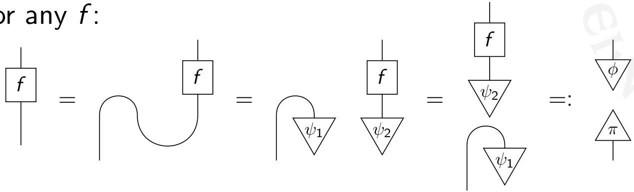
Theorem

If a process theory (i) has cup-states for every type and (ii) every state separates, then it is trivial.

Proof. Suppose a cup-state separates:

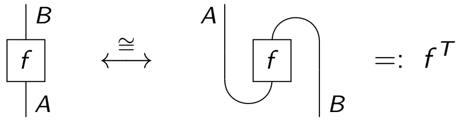


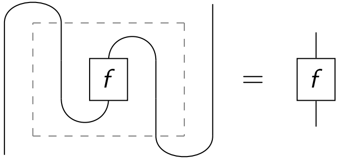
Then for any f:



□

Transpose



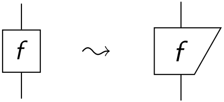


i.e.

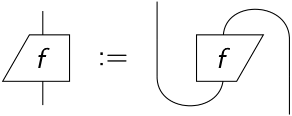
$$(f^T)^T = f$$

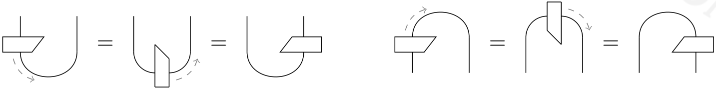
Tranpose = rotation

A bit of a deformation:



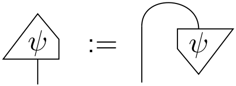
allows some clever notation:

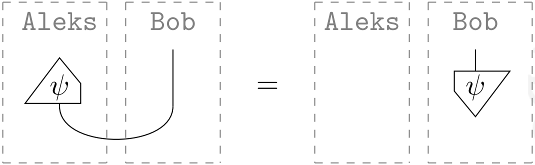


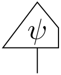


Tranpose = rotation

Specialised to states:

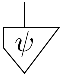


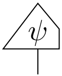


as soon as Aleks obtains  *Bob's system will be in state* 

State/effect correspondence

states of system A $\cong$ *effects for* <u>*correlated system B*</u>

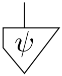


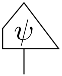


transpose

But what about...

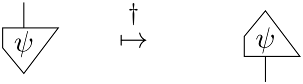
states of system A $\cong$ *effects for system A*





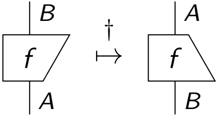
adjoint

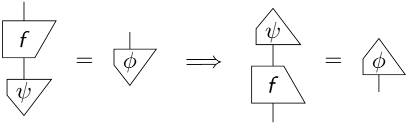
Adjoints



state ψ **testing for** ψ

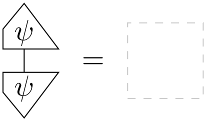
Extends from states/effects to all processes:



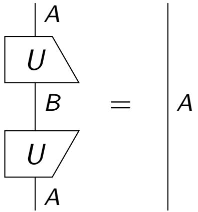


Normalised states and isometries

- Adjoints increase expressiveness, for instance can say when ψ is *normalised*:



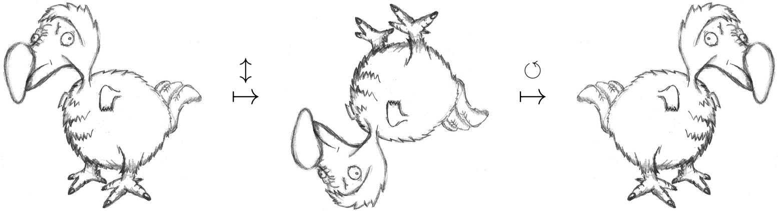
- U is an *isometry*:



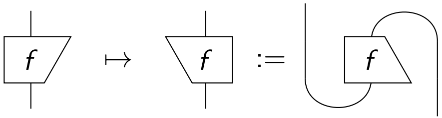
- ...and unitary, self-adjoint, positive, etc.

Conjugates

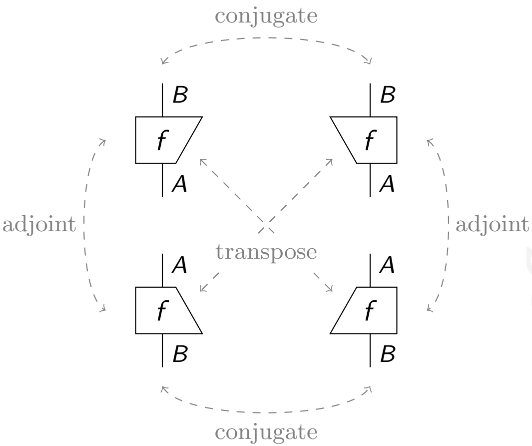
If we:



...we get horizontal reflection. The *conjugate*:

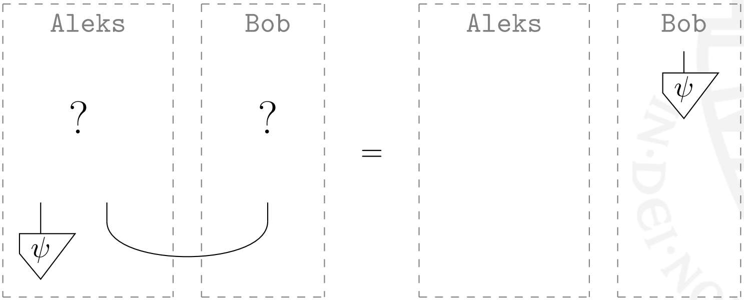


4 kinds of box



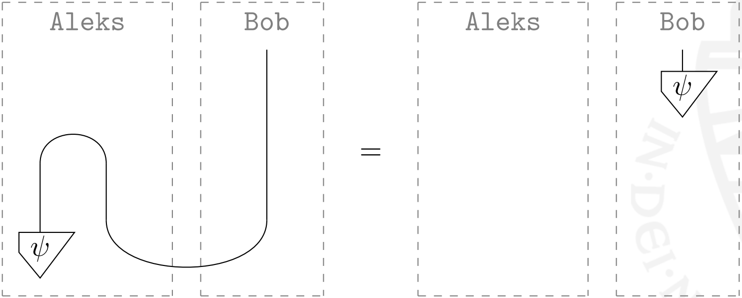
Quantum teleportation: take 1

Can we fill in '?' to get this?



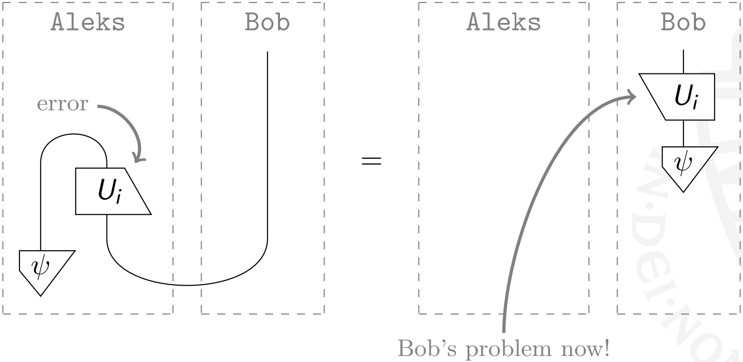
Quantum teleportation: take 1

Here's a simple solution:



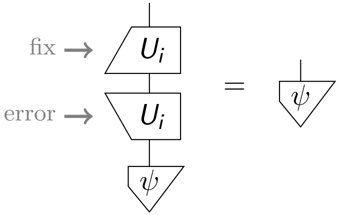
Problem: 'cap' can't be performed deterministically

Quantum teleportation: take 1

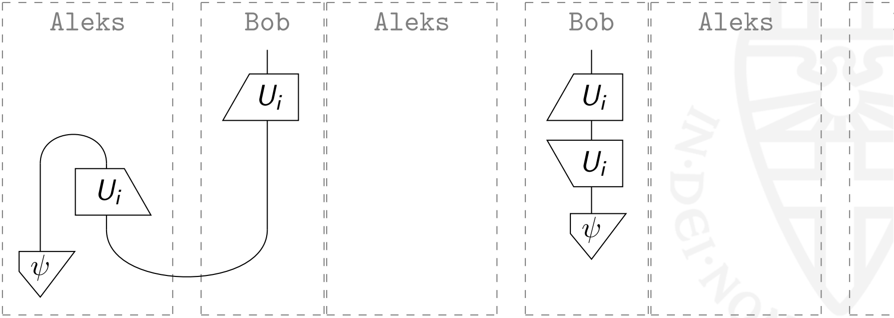


Quantum teleportation: take 1

Solution: Bob fixes the error.



Quantum teleportation: take 1



Hilbert space

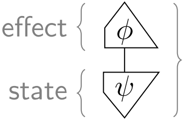
The starting point for quantum theory is the process theory of **linear maps**, which has:

1. **systems:** Hilbert spaces
2. **processes:** complex linear maps

...in particular, numbers are *complex numbers*.

Hilbert space

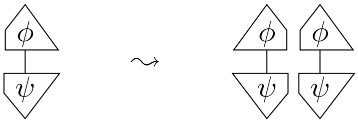
Looking at the ‘Born rule’ for **linear maps**, we have a problem:



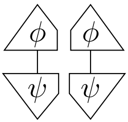
complex number $\neq$ probability!

Doubling

Solution: multiply by the conjugate:



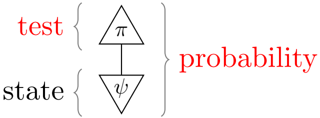
Then, for normalised ψ, ϕ:

$0 \leq$  ≤ 1

(i.e. the 'usual' Born rule: $\overline{\langle\phi|\psi\rangle}\langle\phi|\psi\rangle = |\langle\phi|\psi\rangle|^2$)

Doubling

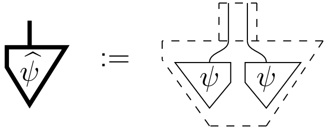
New problem: We lost this:

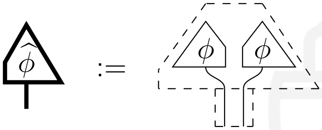


...which was the basis of our interpretation for states, effects, and numbers.

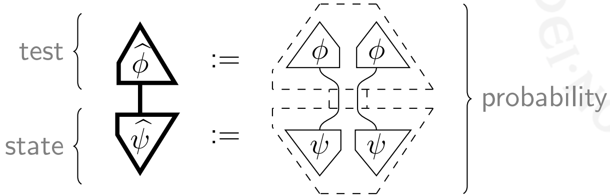
Doubling

Solution: Make a new process theory with doubling 'baked in':



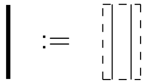


Then:

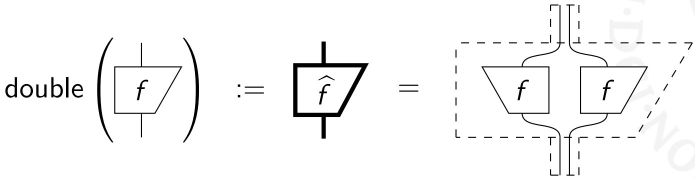


Doubling

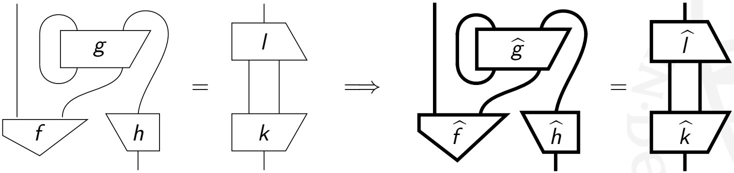
The new process theory has doubled systems $\widehat{H} := H \otimes H$:



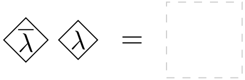
and processes:



Doubling preserves diagrams

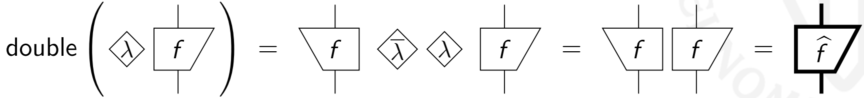


...but kills global phases



(i.e. $\lambda = e^{i\alpha}$)

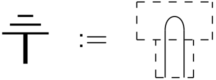
$$\Longrightarrow$$



Discarding

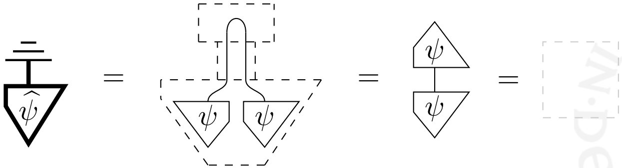
Doubling also lets us do something we couldn't do before: throw stuff away!

How? Like this:



Discarding

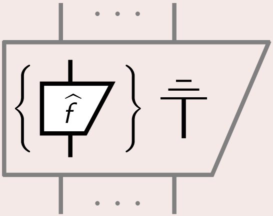
For normalised ψ, the two copies annihilate:



Quantum maps

Definition

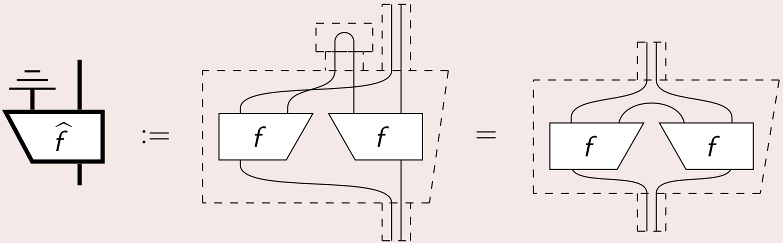
The process theory of **quantum maps** has as types (doubled) Hilbert spaces $\widehat{H}$ and as processes:



Purification

Theorem

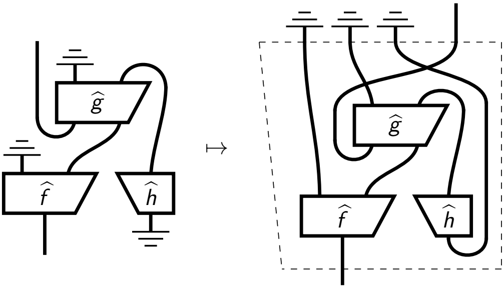
All quantum maps are of the form:



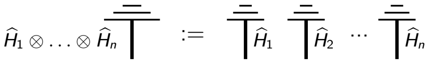
for some linear map f.

Purification

Proof. Pretty much by construction:

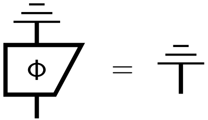


then note that:



Causality

A quantum map is called *causal* if:

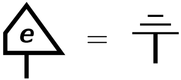


If we discard the output of a process,
it doesn't matter which process happened.

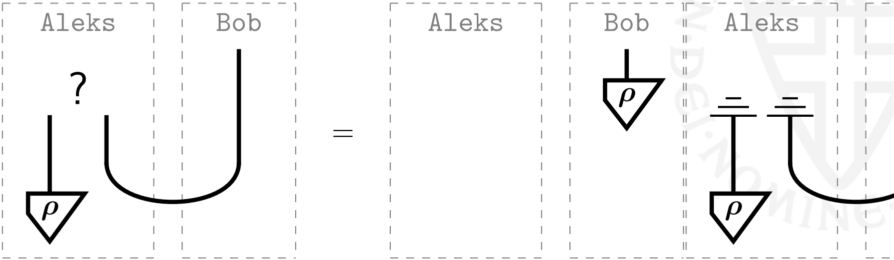
causal $\iff$ *deterministically physically realisable*

Consequence: no cap effect ☹

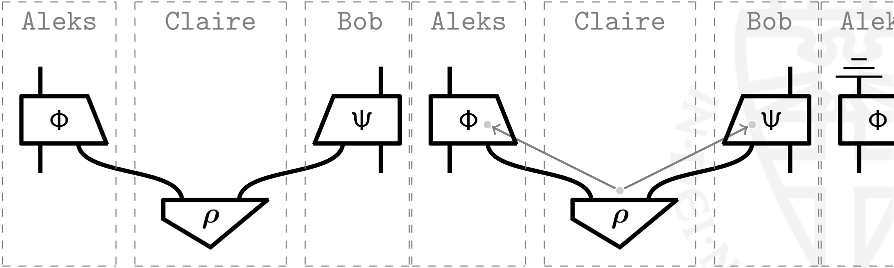
Consequence: there is a unique causal effect, discarding:



Hence 'deterministic quantum teleportation' must fail:



Consequence: no signalling ☺

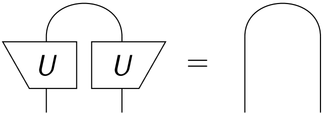


Stinespring's theorem ☺

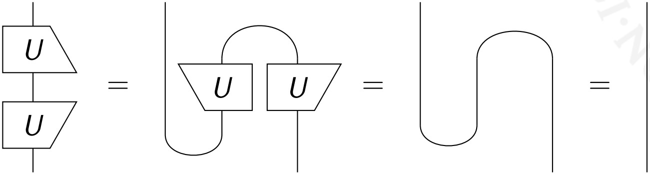
Lemma

Pure quantum maps $\widehat{U}$ are causal if and only if they are isometries.

Proof. Unfold the causality equation:



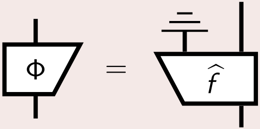
and bend the wire:



Stinespring's theorem ☺

Theorem (Stinespring)

For any causal quantum map Φ, there exists an isometry $\widehat{f}$ such that:

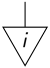


Proof. Purify Φ, then apply the lemma to $\widehat{f}$.

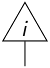
Double vs. single wires

$$\left(\text{quantum} \quad := \quad \Bigg\| \; \right) \quad \neq \quad \left(\text{classical} \quad := \quad \Bigg| \; \right)$$

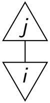
Classical values



:= ‘providing classical value i’



:= ‘testing for classical value i’

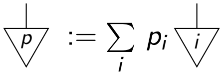


$$= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

($\Rightarrow$ ONB)

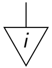
Classical states

General state of a classical system:



$\leftarrow$ probability distributions

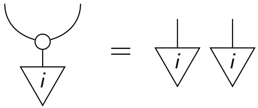
Hence:



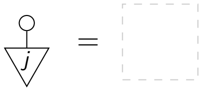
$\leftarrow$ point distributions

Copy and delete

Unlike quantum states, classical values can be *copied*:

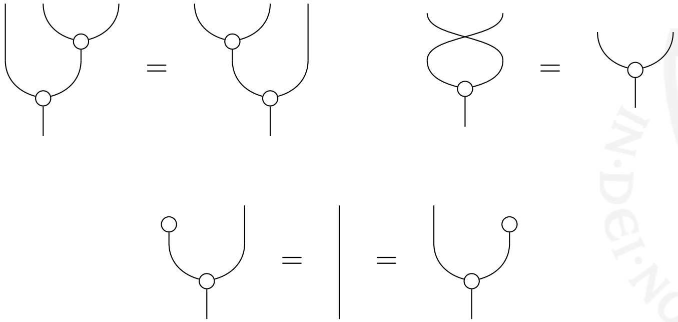


and *deleted*:

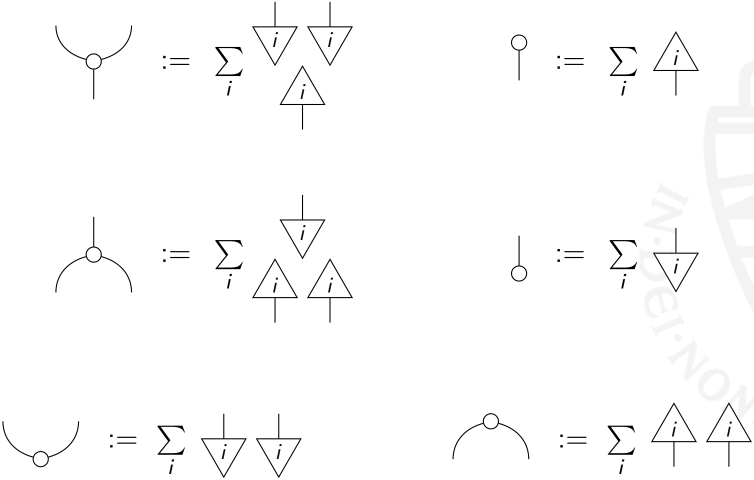


Copy and delete

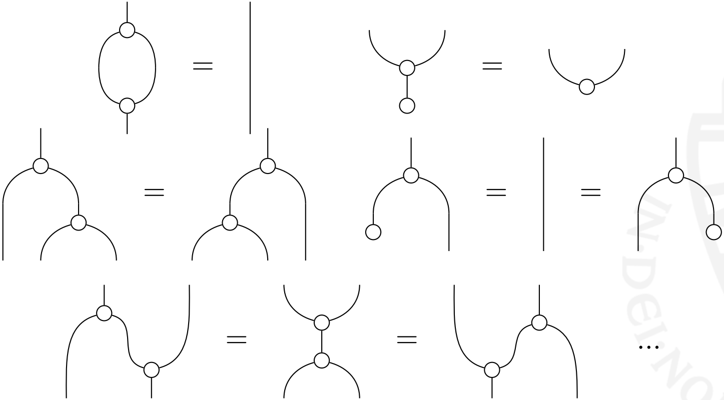
These satisfy some equations you would expect:



Other classical maps



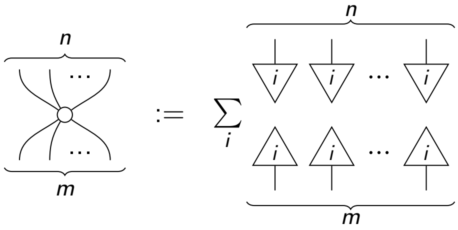
...satisfying lots of equations



When does it end???

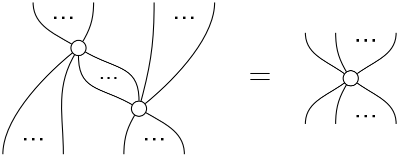
Spiders

All of these are special cases of *spiders*:



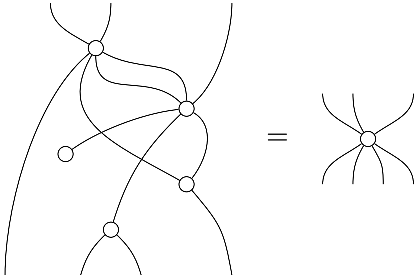
Spiders

The only equation you need to remember is this one:

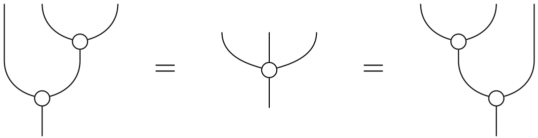


When spiders meet, they fuse together.

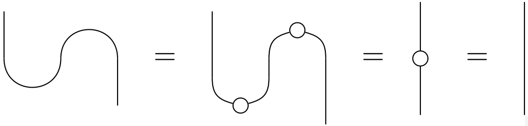
Spider reasoning



For example:



Spider reasoning ⇒ string diagram reasoning



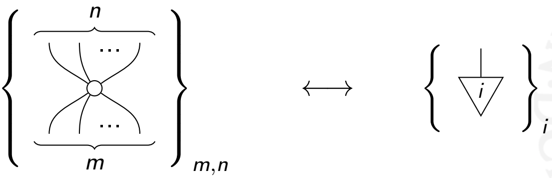
How do we recognise spiders?

Suppose we have something that ‘behaves like’ a spider:

Do we know it is one?

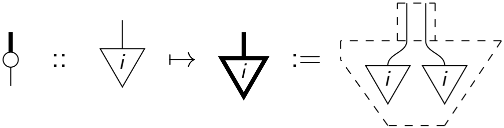
Spiders = 'diagrammatic ONBs'

Yes!

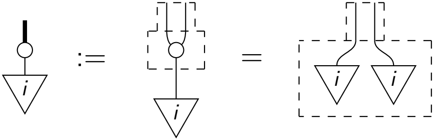


Classical and quantum interaction

Classical values can be encoded as quantum states, via doubling:

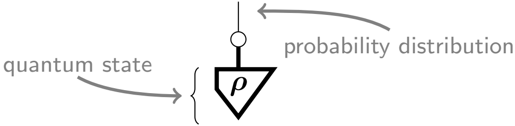


This is our first classical-quantum map, *encode*. It's a copy-spider in disguise:

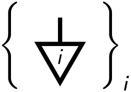


Measuring quantum states

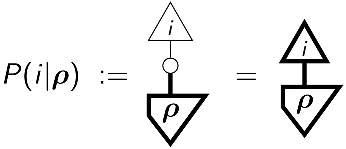
The adjoint of *encode* is *measure*:



This represents measuring w.r.t.



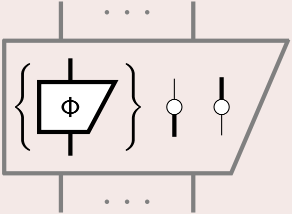
...where probabilities come from the Born rule:



Classical-quantum maps

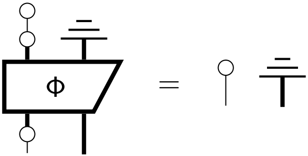
Definition

The process theory of **cq-maps** has as processes diagrams of quantum maps and encode/decode:



Quantum processes

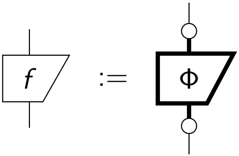
Causality generalises to cq-maps:



quantum processes := causal **cq-maps**

Special case: classical processes

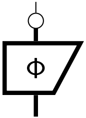
Classical processes are **quantum processes** with no quantum inputs/outputs:



These correspond exactly to stochastic maps. Positivity comes from doubling, and normalisation from causality:

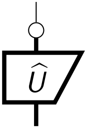
Special case: quantum measurements

A *measurement* is any **quantum process** from a quantum system to a classical one:



$\xleftrightarrow{\cong}$ POVMs

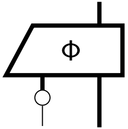
Special case:

ONB-measurement $:=$ 

$\longleftarrow$ unitary

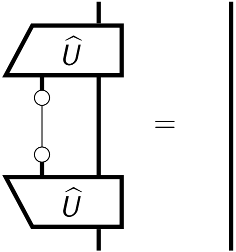
Special case: controlled-operations

A **quantum process** with a classical input is a *controlled operation*:



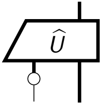
Special case: controlled-operations

A *controlled isometry* furthermore satisfies:

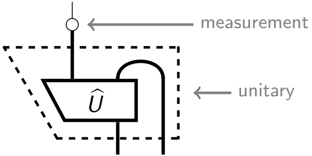


Special case: controlled-operations

Suppose we can use a single $\widehat{U}$ to build a *controlled isometry*:

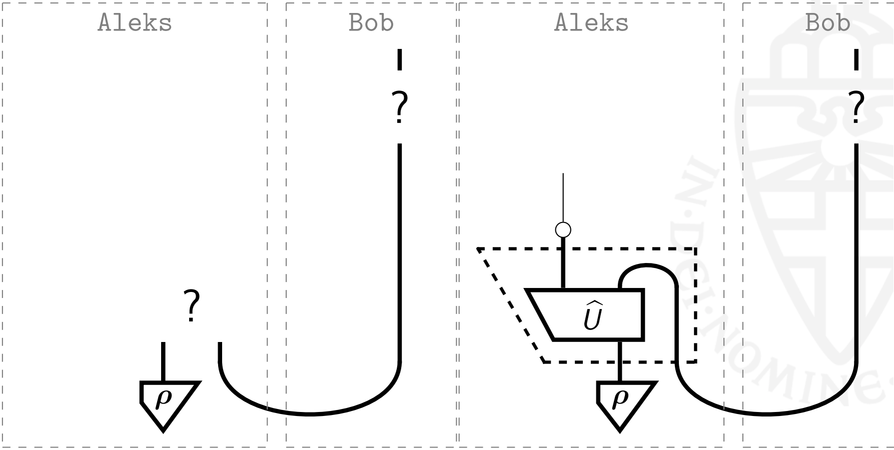


...and an ONB measurement:



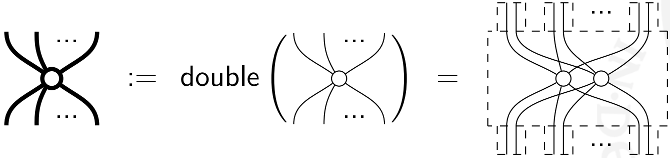
Quantum teleportation: take 2

...then teleportation is a snap!



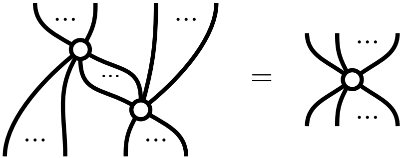
Quantum spiders

Doubling a classical spider gives a *quantum spider*:



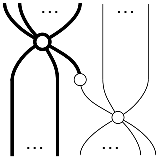
Quantum spiders

Since doubling preserves diagrams, these fuse when they meet:

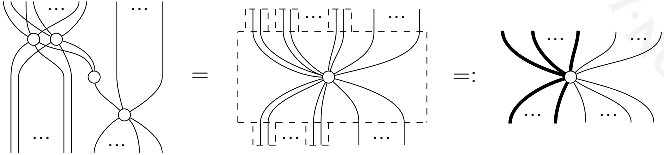


Quantum meets classical

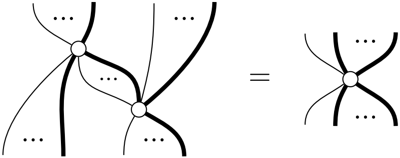
Q: What happens if a quantum spider meets a classical spider, via measure or encode?



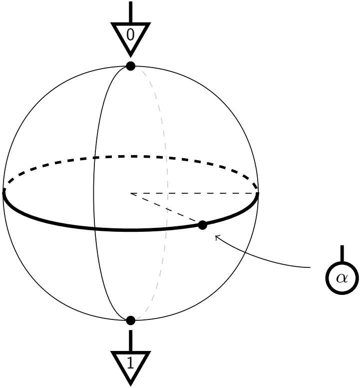
A: Bastard spiders!



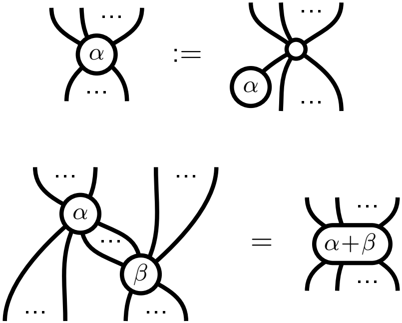
Bastard spider fusion



Phase states

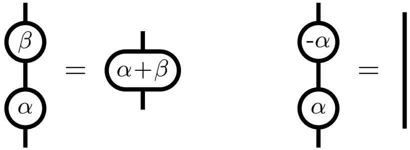


Phase spiders

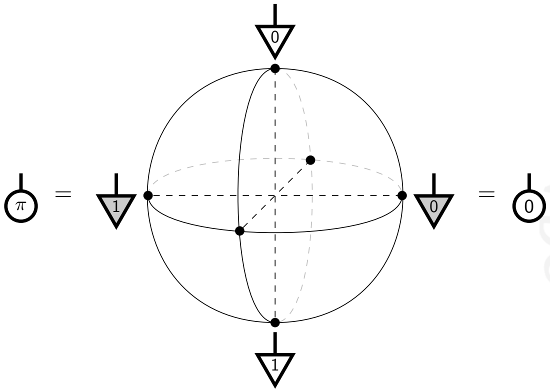


Example: phase gates

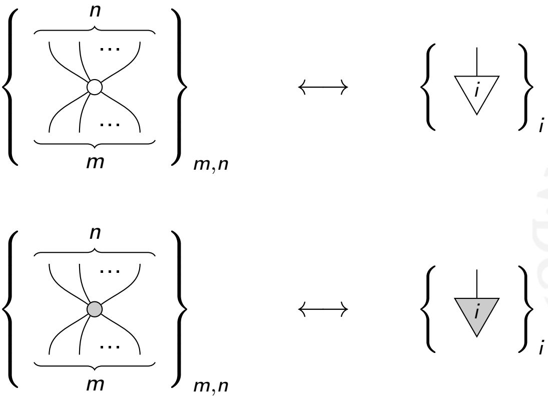
phase gate :=



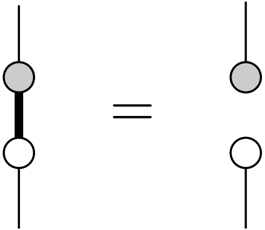
Complementary bases



Complementary bases



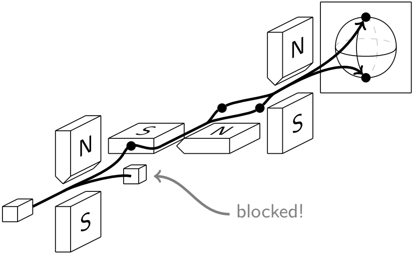
Complementarity

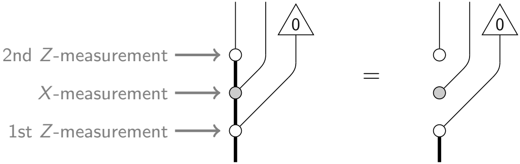


Interpretation:

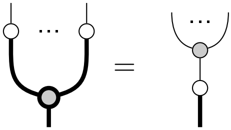
(encode in ○) THEN (measure in ●) = (no data flow)

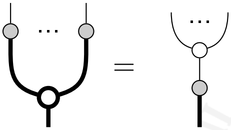
Consequence: Stern-Gerlach





Strong complementarity



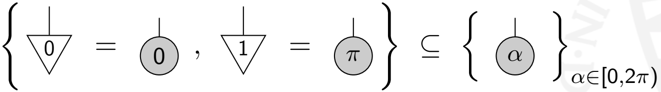


Interpretation:

Mathematically: Fourier transform. *Operationally:* ???

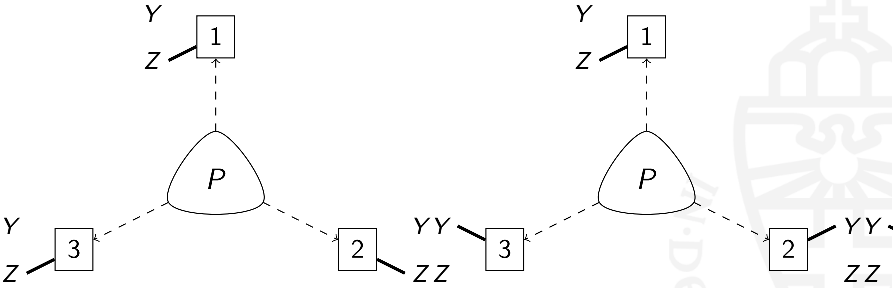
Consequences

- strong complementarity $\implies$ complementarity
- ONB of $\circ$ forms a **subgroup** of phase states, e.g.



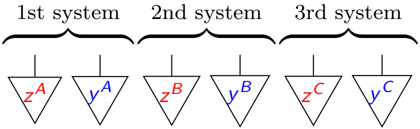
- *GHZ/Mermin non-locality*

The setup

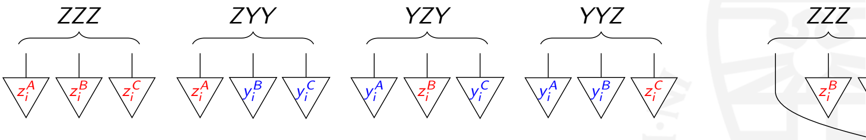


1. *ZZZ*
2. *ZYY*
3. *YZY*
4. *YYZ*

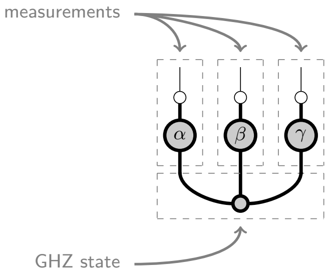
A locally realistic model



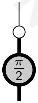
A locally realistic model



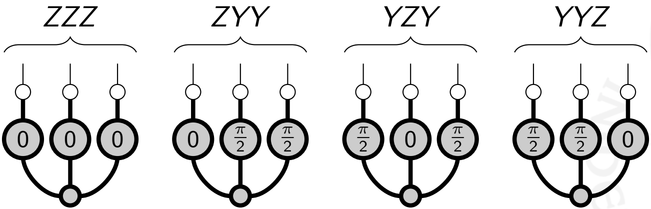
A quantum model



Z-measurement $:=$ 

Y-measurement $:=$ 

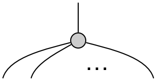
A quantum model



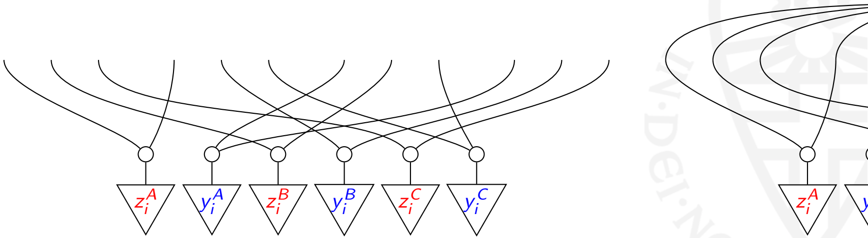
Deriving the contradiction

We prove the correlations from the quantum model are **inconsistent** with any locally realistic one, by computing:

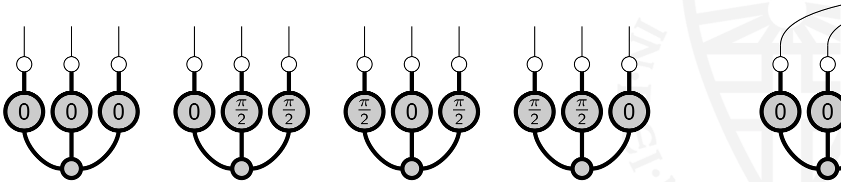
parity :=



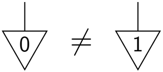
Deriving the contradiction



Deriving the contradiction



Deriving the contradiction



$\Longrightarrow$ **Quantum theory is non-local!**

Applications: the expanded menu

- **foundations**
  - strong complementarity ⇒ GHZ/Mermin non-locality
  - phase groups distinguish Spekkens' toy theory and stabilizer QM
- **quantum computation**
  - graphical calculus ⇒ circuit/MBQC transformation
  - complementarity ⇔ quantum oracles
  - strong complementarity ⇒ graphical HSP
- **quantum resource theories**
  - resource theories := 're-branded' process theories
  - graphical characterisations for convertibility relations (purity, entanglement)
  - 3 qubit SLOCC-classification ⇒ two kinds of 'spider-like arachnids'