Quantum teleportation, diagrams, and the one-time pad

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Outline

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Processes

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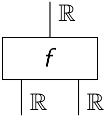

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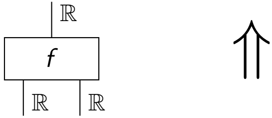

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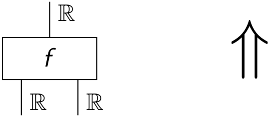

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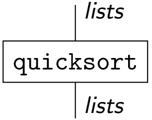

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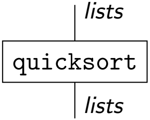
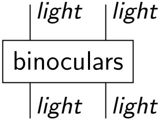
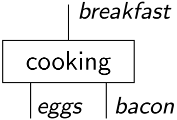
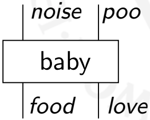


- These are also perfectly good processes:







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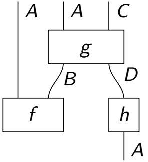

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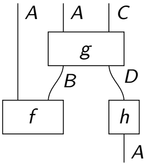
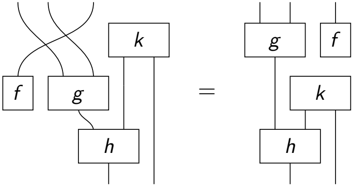
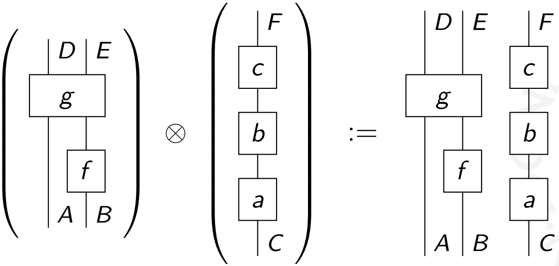


- The golden rule: only connectivity matters!



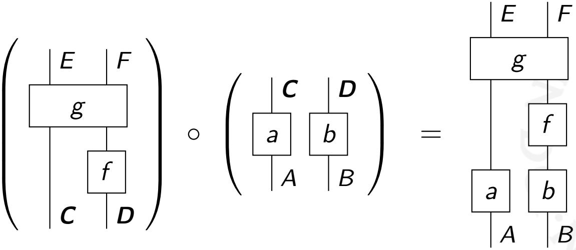
Diagrams

- Special cases are parallel composition:



Diagrams

- ...and sequential composition:



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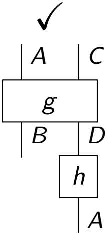

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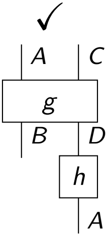

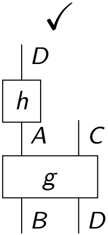

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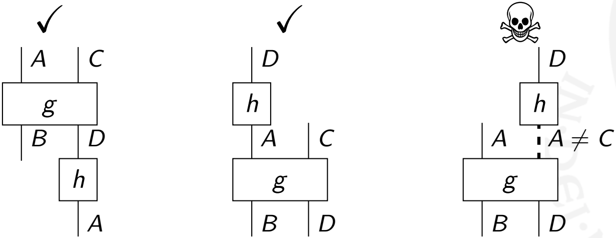

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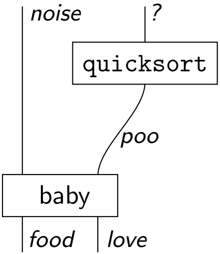

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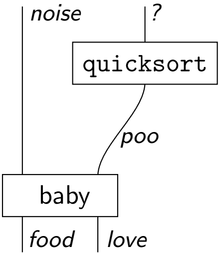

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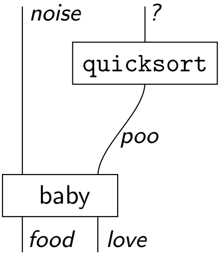

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- A family of processes which *do* make sense together is called a process theory

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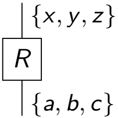

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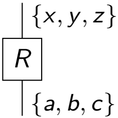

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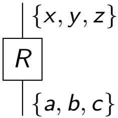
- ...which we can think of as non-deterministic computations:



$$= \begin{cases} a \mapsto \{x, y\} \\ b \mapsto z \\ c \mapsto \emptyset \end{cases}$$

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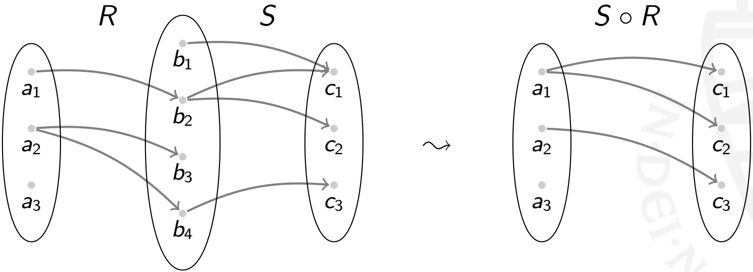
Relations compose in sequentially just like you learned in school:



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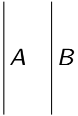

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- so relations compose like this:

$$\boxed{R}\;\boxed{S} \;::\; (a,b) \mapsto (c,d) \iff \left(\boxed{R} \;::\; a \mapsto c \text{ and } \boxed{S} \;::\; b \mapsto d \right)$$

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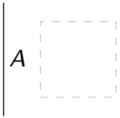

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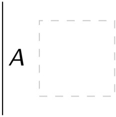

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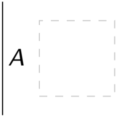 $= \{(a, \bullet) \mid a \in A\} \cong A =$ 

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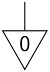

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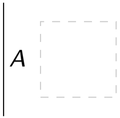

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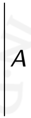

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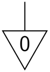

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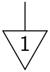

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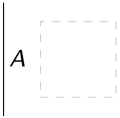

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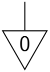

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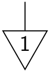

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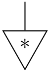

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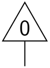

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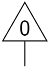
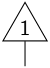
 $= \{0 \mapsto \bullet$  $= \{1 \mapsto \bullet$

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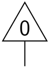

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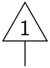

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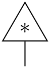

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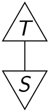

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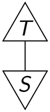

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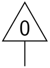

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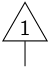

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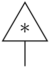

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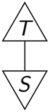

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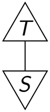

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These stand for **true** and **false**.

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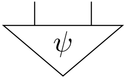

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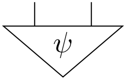

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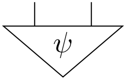

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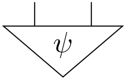

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- States of the two systems no longer have their own, separate identities
- Hence we get...

Outline

Process theories

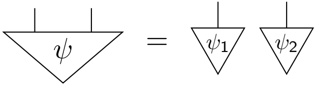
Non-separability

One-time pad

Quantum teleportation

Separable states

- A state ψ on two systems is *separable* if there exist ψ_1, ψ_2 such that:



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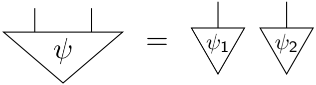

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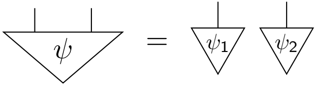

- **Intuitively:** the properties of the system on the left are *independent* from those on the right
- In the deterministic-land, *all states* to separate...

Characterising non-separability

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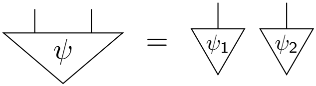

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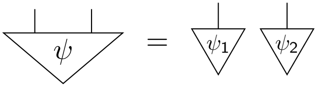

- **Problem:** Showing that something doesn't exist is hard.

Characterising non-separability

Solution: Replace a negative property with a postive one:

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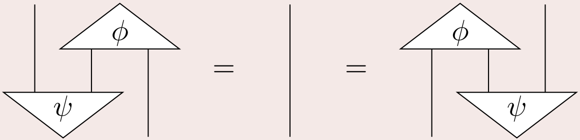
Definition

A state ψ is called *cup-state* if there exists an effect ϕ, called a *cap-effect*, such that:



Cup-states

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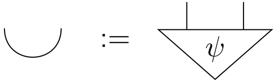

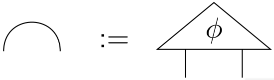

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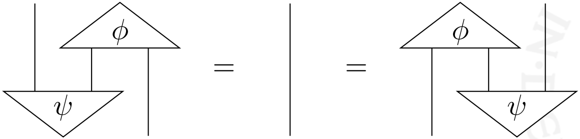

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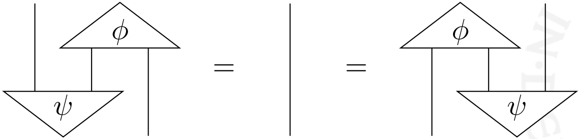
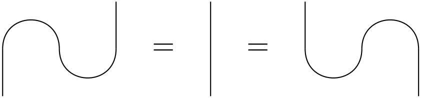
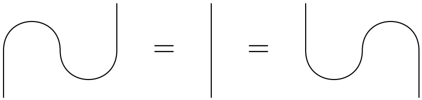


- ...look like this:



Yank the wire!



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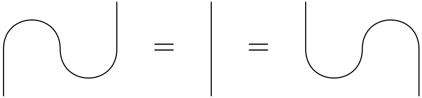

Example

- In **relations**, there is an obvious choice of cup-state:

$$:= \left\{ * \mapsto \{(0,0),(1,1)\} \right\}$$

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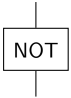
- This, plus NOT...



$$:= \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 0 \end{cases}$$

...gives us enough to start building interesting stuff.

Outline

Process theories

Non-separability

One-time pad

Quantum teleportation

An incredibly sophisticated security protocol

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    - otherwise he flips the bit.

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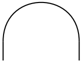
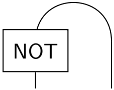
- There are two possible outcomes:

$\Bigg\{$  := "the same" , 

:= "NOT the same" $\Bigg\}$

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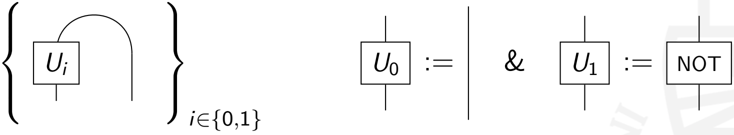
- ...which we can write as:



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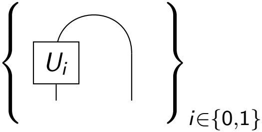
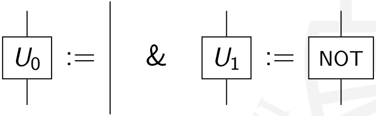
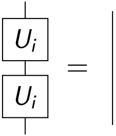




- Then, the U_i satisfy:



One-time pad diagram

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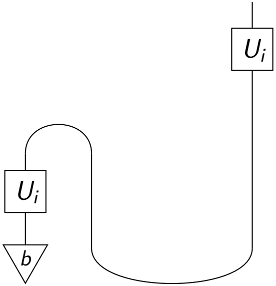

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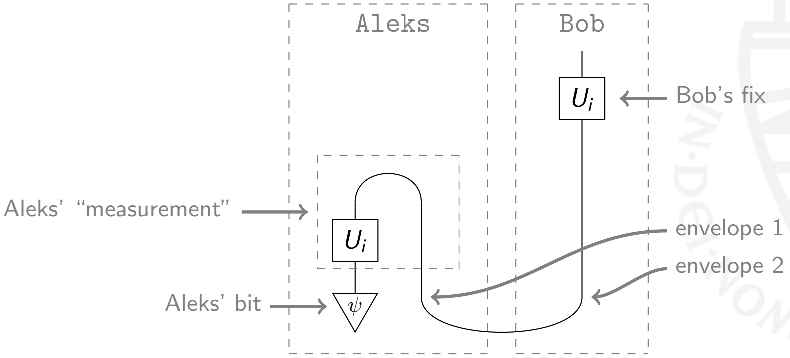

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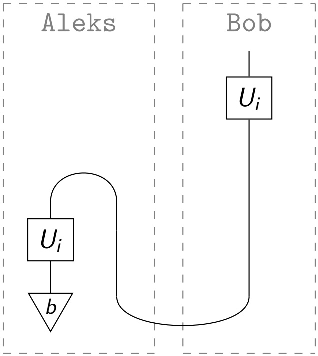

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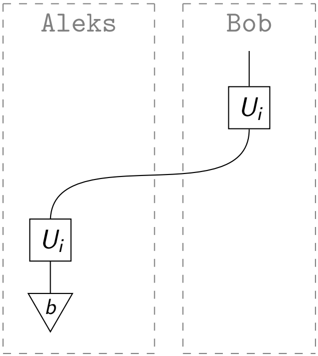

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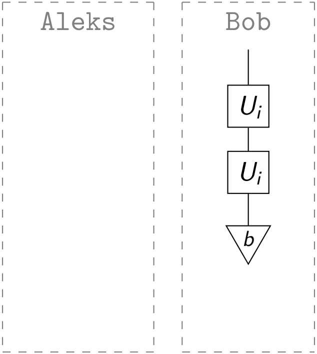

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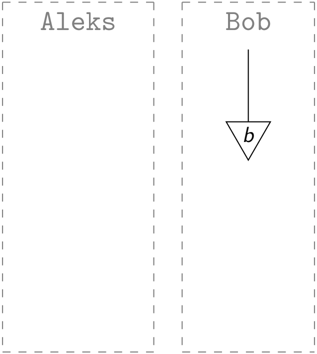

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- Example: polarization of a photon

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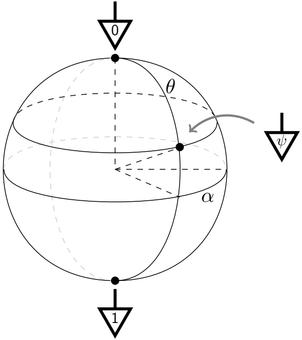

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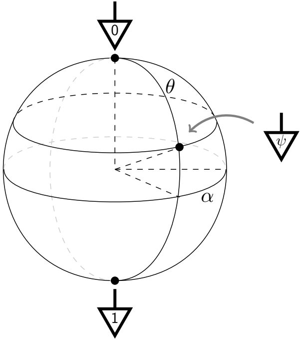

- "Plain old" bits live at the North Pole and the South Pole.

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- Even though this thing is (slightly) more complicated to describe, it acts just like before

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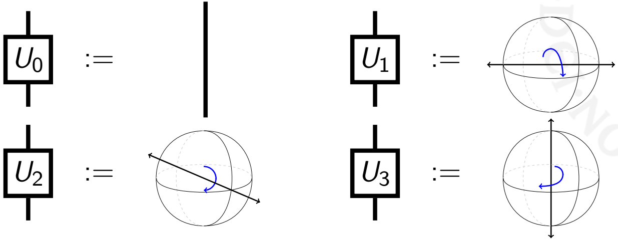
where there are now three different ways to "NOT":



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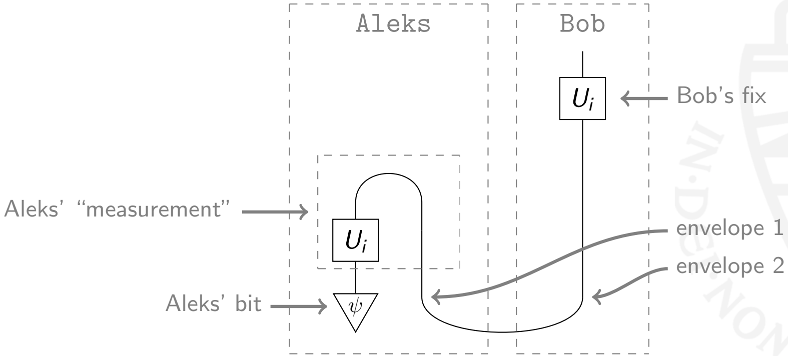

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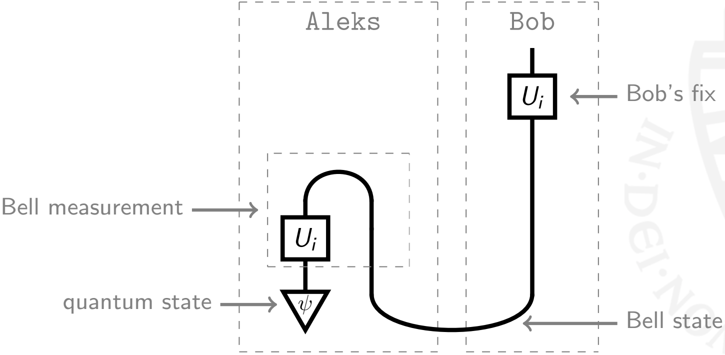

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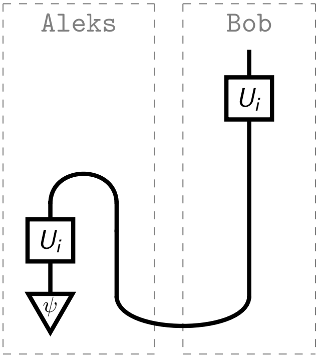

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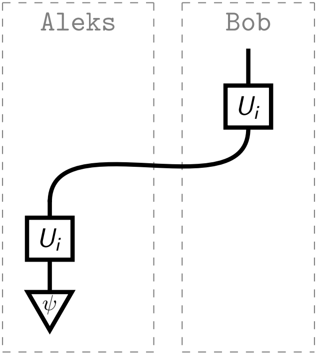

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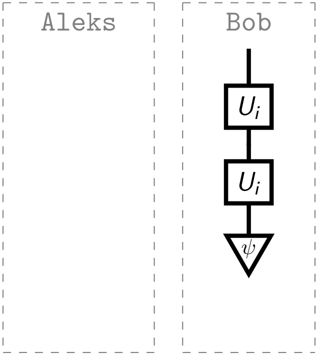

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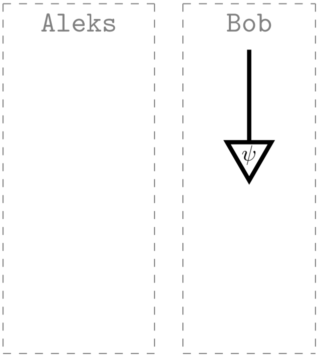

Two for the price of one

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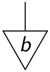

:= private data

:= quantum state

Thanks!