

On Infinite Horizon Switched LQR Problems With State and Control Constraints

Maximilian Balandat^a, Wei Zhang^a, Alessandro Abate^b

^a*Department of Electrical Engineering and Computer Sciences
University of California, Berkeley, CA, USA*

^b*Delft Center for Systems and Control
Technische Universiteit Delft, Delft, The Netherlands*

Abstract

This paper studies the Discrete-Time Switched LQR problem over an infinite time horizon, subject to polyhedral constraints on state and control inputs. Specifically, we aim to find an infinite-horizon hybrid-control sequence, i.e., a sequence of continuous and discrete (switching) control inputs, that minimizes an infinite-horizon quadratic cost function, subject to polyhedral constraints on state and (continuous) control input. The overall constrained, infinite-horizon problem is split into two subproblems: (i) an *unconstrained*, infinite-horizon problem and (ii) a *constrained*, finite-horizon one. We derive a stationary suboptimal policy for problem (i) with analytical bounds on its optimality, and develop a novel formulation of problem (ii) as a Mixed-Integer Quadratic Program. By introducing the concept of a *safe set*, the solutions of the two subproblems are combined to achieve the overall control objective. Through the connection between (i) and (ii) it is shown that, by proper choice of the design parameters, the error of the overall suboptimal solution can be made arbitrarily small. The approach is tested on a numerical example.

Keywords: LQR, Discrete-Time Switched and Hybrid Systems, Infinite Horizon Constrained Optimal Control, Suboptimality

1. Introduction

Among the template problems in optimal control, the Linear Quadratic Regulator (LQR) is a fundamental one. The study of this problem in a discrete-time framework has an established and well known history, which in the recent past has witnessed enticing extensions to models subject to hard constraints on the states and control inputs [4, 9, 13, 18, 19, 20, 23, 25]. In addition, due to the recent popularity of switched and hybrid systems, an extension of the LQR problem to Discrete-Time Switched Linear Systems (DSLS), referred to as

Email addresses: balandat@eecs.berkeley.edu (Maximilian Balandat),
weizhang@eecs.berkeley.edu (Wei Zhang), a.abate@tudelft.nl (Alessandro Abate)

the Discrete-Time Switched LQR (DSLQR) problem, has also been thoroughly studied [16, 28, 29]. In particular, a numerical relaxation framework has been introduced in [28], which allows to efficiently solve a DSLQR problem with guaranteed suboptimal performance. However, one of the main restrictions of this line of work is that it deals exclusively with unconstrained DSLS.

The focus of this paper is on the Discrete-Time Constrained Switched LQR (DCSLQR) problem over an infinite horizon. Specifically, we aim to find an infinite-horizon *hybrid-control sequence*, i.e., a sequence of continuous and discrete (switching) control inputs, that minimizes an infinite-horizon quadratic cost function, subject to polyhedral constraints on state and (continuous) control input. The challenges to solve this problem include the discrete nature of the switching control sequence, the non-trivial control and state constraints, the infinite horizon length and the nonlinear performance index. To the authors' knowledge, previous studies have only considered problems exhibiting some of these challenges, but not all of them combined. The unconstrained Switched LQR problem has been studied in [27]. The constrained LQR problem for non-switched linear system has been addressed in [4, 9, 23, 25]. Under linear performance index, an algorithm has been developed in [2] to solve the constrained optimal control problem for Piecewise Affine (PWA) Systems; however, its extension to a quadratic performance index does not appear to be straightforward. Quadratic optimal control of PWA systems has also been studied extensively [3, 7, 24]; however most existing algorithms can only handle finite-horizon problems.

The main contribution of this work is the development of a framework to provide bounds on the suboptimal solution of the infinite-horizon DCSLQR problem. Motivated by the solution of the classical (non-hybrid) constrained LQR problem [23] and closely related Model Predictive Control (MPC) approaches [18, 19, 20], we decompose the overall problem into the following two related subproblems:

- (i) The first subproblem is an unconstrained, infinite-horizon DSLQR problem, whose solution can be computed efficiently using the numerical relaxation framework in [28]. The obtained suboptimal solution is a stationary hybrid-control law, characterized by a set of positive semidefinite matrices.
- (ii) The second subproblem corresponds to the solution of a Constrained, Finite-Time, Optimal Hybrid Control (CFTOHC) problem, which can be formulated as a Mixed Integer Quadratic Program (MIQP) and solved with reasonable efficiency using available optimization algorithms.

The above two subproblems are connected through a so-called *safe set*, that is a set of states for which the solution to the unconstrained DSLQR problem is guaranteed to be always feasible (with regards to the state and control constraints). We show that if the unconstrained DSLS model is stabilizable, then for reasonable constraints a non-trivial safe set always exists. Moreover, we show that in case the constrained system is stabilizable from the given initial condition, then with a sufficiently large horizon, the solution of the CFTOHC problem (ii) can always drive the state trajectory into the safe set, from where

the solution of the (unconstrained) DSLQR problem (i) is feasible and optimal. Thus, by concatenating the solutions of the first and of the second subproblems, we obtain the solution to the overall infinite-horizon DCSLQR problem. Though the idea underlying this method has previously appeared in [9, 25], the authors are not aware of its application to Switched Linear Systems.

We argue that it is generally very hard to determine the minimal length of the time horizon of the CFTOHC problem *a priori*, and instead present a decidable heuristic procedure to find a suitable horizon. This approach is similar in nature to the one presented in [20, 23, 25] and is based on the choice of a terminal cost function in combination with an iterative procedure for finding a suitable horizon length for the finite-horizon problem. In doing so, it distinguishes itself from MPC-like approaches that impose terminal state constraints on the finite-horizon problem [18, 21].

Based on the above ideas, an algorithm is developed to solve the infinite-horizon DCSLQR problem with guaranteed suboptimal performance. We will show that the suboptimality error can be made arbitrarily small through proper choice of the design parameters. The performance of the algorithm is also tested and discussed with a numerical example.

This paper is structured as follows. The infinite-horizon DCSLQR problem is stated in section 2. Section 3 deals with the (unconstrained) DSLQR problem (i). The notion of a *safe set* is introduced in section 4, and two approaches to compute such a set are discussed. Section 5 completes the solution of the overall problem by computing the optimal finite-horizon hybrid-control sequence that drives the system state into the safe set, starting from the given initial condition. Based on an MIQP formulation, an algorithm is developed to solve the associated CFTOHC problem (ii), which is tested on a case study in Section 6.

Notation: In this paper, n , p and M are some arbitrary finite positive integers; \mathbb{Z}^+ denotes the set of nonnegative integers, $\mathbb{M} \triangleq \{1, \dots, M\}$ is the set of subsystem indices, I_n is the $n \times n$ identity matrix, 0 denotes both scalar zero and the zero matrix of appropriate dimension; $\|\cdot\|$ represents the standard Euclidean norm in \mathbb{R}^n , and the induced norm over n -dim. matrices; $|\cdot|$ denotes the cardinality of a given set; \mathcal{A} denotes the set of positive semidefinite (p.s.d.) matrices; $2^{\mathcal{A}}$ is the power set of \mathcal{A} ; $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ characterize the smallest and the largest eigenvalues, respectively, of a given positive semidefinite (p.s.d.) matrix. The variable z denotes a generic initial state of system (1).

2. Problem Formulation

Consider the Discrete-Time Switched Linear System (DSLS) described by:

$$x_{t+1} = A_{v_t} x_t + B_{v_t} u_t \quad (1)$$

subject to the constraints

$$x_t \in \mathcal{X}, \quad u_t \in \mathcal{U}, \quad \forall t \in \mathbb{Z}^+, \quad (2)$$

where $x_t \in \mathbb{R}^n$ is the continuous state, $u_t \in \mathbb{R}^p$ is the continuous control input and $v_t \in \mathbb{M}$ is the discrete control input that determines the discrete mode at time t . The sets \mathcal{X} and \mathcal{U} are polyhedra that contain the origin in their interiors. The sequence of pairs $\psi_\infty = \{(u_t, v_t)\}_{t=0}^\infty$ is called the *hybrid-control sequence*. For each $i \in \mathbb{M}$, A_i and B_i are constant matrices of appropriate dimensions and the pair (A_i, B_i) denotes a subsystem. The switched system is time invariant in the sense that the set of available subsystems $\{(A_i, B_i)\}_{i=1}^M$ is independent of time. We assume no constraints on the switchings, i.e. at any time instant the system can switch to any mode.

For each $t \in \mathbb{Z}^+$, denote by $\xi_t \triangleq (\mu_t, \nu_t) : \mathbb{R}^n \mapsto \mathbb{R}^p \times \mathbb{M}$ the *(state-feedback) hybrid-control law* of system (1), where $\mu_t : \mathbb{R}^n \mapsto \mathbb{R}^p$ is called the *(state-feedback) continuous-control law* and $\nu_t : \mathbb{R}^n \mapsto \mathbb{M}$ is called the *(state-feedback) switching-control law*. A sequence of hybrid-control laws constitutes an *infinite horizon hybrid-control policy* $\pi_\infty = \{\xi_0, \xi_1, \xi_2, \dots\}$. A policy is called *stationary* if it consists of the same control law at all time, i.e., $\xi_t = \xi$ for all $t \in \mathbb{Z}^+$. When a policy π_∞ is applied to system (1), the closed-loop dynamics of the controlled system are given by:

$$x_{t+1} = A_{\nu_t(x_t)}x_t + B_{\nu_t(x_t)}\mu_t(x_t). \quad (3)$$

Denote by $\pi_\infty(z)$ the hybrid-control sequence generated by the policy π_∞ with initial condition $x_0 = z$, i.e., $\pi_\infty(z) = \{(\mu_t(x_t), \nu_t(x_t))\}_{t \in \mathbb{Z}^+}$ where x_t is the solution of (3) driven by π_∞ with $x_0 = z$. Define the running cost function as:

$$L(x, u, v) = x^T Q_v x + u^T R_v u, \quad \forall x \in \mathcal{X}, u \in \mathcal{U}, v \in \mathbb{M},$$

where $Q_v = Q_v^T \succ 0$ and $R_v = R_v^T \succ 0$ be the weighting matrices for the state and the continuous-control input in subsystem $v \in \mathbb{M}$. Each hybrid-control sequence $\psi_\infty = \{(u_t, v_t)\}_{t=0}^\infty$ is associated with a quadratic cost function:

$$J_\infty(z, \psi_\infty) = \sum_{t=0}^{\infty} L(x_t, u_t, v_t),$$

where x_t is the closed-loop trajectory controlled by ψ_∞ , with initial condition $x_0 = z \in \mathbb{R}^n$. Our objective is to find the optimal hybrid-control sequence that solves the following constrained optimal control problem:

$$J_\infty^*(z) = \begin{cases} \inf_{\substack{u_t \in \mathcal{U}, v_t \in \mathbb{M} \\ t \geq 0}} J_\infty(z, \psi_\infty) \\ \text{s.t. (1) and (2) with } x_0 = z. \end{cases} \quad (4)$$

The above problem is an extension of the classical discrete-time LQR controller synthesis to Switched Linear Systems (SLS) subject to polyhedral input and state constraints, and will thus be referred to as the *Discrete-Time Constrained Switched LQR problem* (DCSLQR). Because of the complexity of the problem at hand, we do not require to find the optimal policy π_∞^* for all initial states $z \in \mathcal{X}$, but rather the control sequence ψ_∞^* for a given initial condition.

Clearly, if the *unconstrained* system (1) is not stabilizable, $J_\infty(x_0, \psi_\infty)$ might be infinite for all possible control sequences ψ_∞ . Therefore, stabilizability of (1) is a minimal requirement for the well-posedness of Problem (4).

Definition 1 (Exponential stabilizability). *The unconstrained system (1) is called exponentially stabilizable if there exists a policy π_∞ and constants $a \geq 1$ and $0 < c < 1$ such that the closed-loop trajectory under the policy π_∞ starting from any initial state $x_0 = z$ satisfies $\|x_t\|^2 \leq ac^t \|z\|^2, \forall t \in \mathbb{Z}^+$.*

The following assumption is made throughout the paper:

(A1) The unconstrained SLS (1) is exponentially stabilizable.

Remark 1. *The above assumption trivially holds true if one of the unconstrained subsystems is stabilizable. Furthermore, even in case none of the subsystems is stabilizable, it is still possible for the overall unconstrained SLS to be exponentially stabilizable (see Section 6 and [28]). In such cases, the assumption can be efficiently verified using the approach developed in [28].*

3. The Unconstrained Hybrid Control Problem

In this section, we recall some recent results on the Discrete-Time Switched LQR (DSLQR) problem [29], which can be viewed as a special case of the DCSLQR problem with the trivial constraints $\mathcal{X} = \mathbb{R}^n, \mathcal{U} = \mathbb{R}^p$. For each $k \in \mathbb{Z}_+$, the k -horizon value function of the DSLQR problem is defined as:

$$J_{k,uc}^*(z) = \begin{cases} \inf_{\substack{u_t \in \mathbb{R}^p, v_t \in \mathbb{M} \\ 0 \leq t \leq k-1}} \sum_{t=0}^{k-1} L(x_t, u_t, v_t) \\ \text{s.t. (1) with } x_0 = z. \end{cases} \quad (5)$$

An important feature of the DSLQR problem is that its finite-horizon value function can be characterized analytically. For each $i \in \mathbb{M}$, define the *Riccati Mapping* $\rho_i : \mathcal{A} \mapsto \mathcal{A}$ as: $\rho_i(P) = Q_i + A_i^T P A_i - A_i^T P B_i (R_i + B_i^T P B_i)^{-1} B_i^T P A_i$. The mapping $\rho_{\mathbb{M}} : 2^{\mathcal{A}} \mapsto 2^{\mathcal{A}}$ defined by: $\rho_{\mathbb{M}}(\mathcal{H}) = \{\rho_i(P), i \in \mathbb{M}, P \in \mathcal{H}\}$ is called the *Switched Riccati Mapping* (SRM), and the sequence of sets $\{\mathcal{H}_k\}$ generated iteratively according to: $\mathcal{H}_{k+1} = \rho_{\mathbb{M}}(\mathcal{H}_k)$ with $\mathcal{H}_0 = \{0\}$, is called the *Switched Riccati Sets (SRS)*. It was shown in [29] that the k -horizon value function $J_{k,uc}^*$ can be characterized exactly by the SRS \mathcal{H}_k :

$$J_{k,uc}^*(z) = \min_{P \in \mathcal{H}_k} z^T P z, \quad \forall z \in \mathbb{R}^n. \quad (6)$$

The goal of the DSLQR problem is to find an infinite-horizon policy whose closed-loop performance achieves the infinite-horizon optimal cost $J_{\infty,uc}^*$, which is the limit of $J_{k,uc}^*$ as $k \rightarrow \infty$. To deal with the exponential growth of $|\mathcal{H}_k|$, we recall a numerical relaxation scheme developed in [27]. A subset $\mathcal{H}_k^\epsilon \subseteq \mathcal{H}_k$ is called ϵ -equivalent to \mathcal{H}_k if $\min_{P \in \mathcal{H}_k^\epsilon} z^T P z \leq \min_{P \in \mathcal{H}_k} z^T P z + \epsilon \|z\|^2$. A sufficient

condition for this to hold is that for any $G \in \mathcal{H}_k \setminus \mathcal{H}_k^\epsilon$, there exists nonnegative constants $\{\alpha_P\}_{P \in \mathcal{H}_k^\epsilon}$, such that $\sum_{P \in \mathcal{H}_k^\epsilon} \alpha_P = 1$ and $\sum_{P \in \mathcal{H}_k^\epsilon} \alpha_P P \leq G + \epsilon I_n$. For given \mathcal{H}_k , \mathcal{H}_k^ϵ and $G \in \mathcal{H}_k \setminus \mathcal{H}_k^\epsilon$, the above condition can be verified through convex optimization. This leads to an efficient way to compute an ϵ -equivalent subset of \mathcal{H}_k by starting from $\mathcal{H}_k^\epsilon = \emptyset$ and gradually adding matrices from \mathcal{H}_k until the convex condition is verified. We denote by $ES_\epsilon(\cdot)$ an algorithm implementing this procedure. The algorithm can be applied on the output of each SRM, resulting in the so-called ϵ -relaxed SRS:

$$\mathcal{H}_{k+1}^\epsilon = ES_\epsilon(\rho_{\mathbb{M}}(\mathcal{H}_k^\epsilon)), \quad \text{with } \mathcal{H}_0^\epsilon = \mathcal{H}_0. \quad (7)$$

As in (6), one can define the *approximate value function* based on the SRS as

$$J_{k,uc}^\epsilon(z) = \min_{P \in \mathcal{H}_k^\epsilon} z^T P z, \quad \forall z \in \mathbb{R}^n. \quad (8)$$

The relaxed SRS \mathcal{H}_k^ϵ defines a hybrid-control law $\xi_{k,uc}^\epsilon$ given by:

$$\begin{aligned} \xi_{k,uc}^\epsilon(z) &\triangleq (\mu_{k,uc}^\epsilon(z), \nu_{k,uc}^\epsilon(z)) = \left(-K_{i_k^\epsilon(z)} (P_k^\epsilon(z)) z, i_k^\epsilon(z) \right), \\ \text{with } (P_k^\epsilon(z), i_k^\epsilon(z)) &= \arg \min_{P \in \mathcal{H}_k^\epsilon, i \in \mathbb{M}} z^T \rho_i(P) z, \end{aligned} \quad (9)$$

where $K(\cdot)$ is the Kalman gain defined by:

$$K_i(P) \triangleq (R_i + B_i^T P B_i)^{-1} B_i^T P A_i, \quad i \in \mathbb{M}, P \in \mathcal{A}. \quad (10)$$

Applying $\xi_{k,uc}^\epsilon$ at each time step yields a stationary control policy:

$$\pi_{\infty,uc}^{\epsilon,k} := \{ \xi_{k,uc}^\epsilon, \xi_{k,uc}^\epsilon, \dots \}. \quad (11)$$

A sufficient condition for the closed-loop stability under $\pi_{\infty,uc}^{k,\epsilon}$ is that: $\forall P \in \mathcal{H}_k^\epsilon$, there exists nonnegative constants $\{\alpha_G\}_{G \in \rho_{\mathbb{M}}(\mathcal{H}_k^\epsilon)}$, such that

$$\sum_G \alpha_G = 1, \quad \text{and } P \succ \sum_{G \in \rho_{\mathbb{M}}(\mathcal{H}_k^\epsilon)} \alpha_G (G + (\kappa_1 - \kappa_*) I_n), \quad (12)$$

where $\kappa_* = \min_{i \in \mathbb{M}, P \in \mathcal{H}_k^\epsilon} \lambda_{\min}(K_i(P)^T R_i K_i(P) + Q_i)$. Checking condition (12) can be formulated as a LMI feasibility problem [8]. It was proved in [27] that under Assumption (A1), there always exist constants $k < \infty$ and $\epsilon > 0$ such that $\pi_{\infty,uc}^{k,\epsilon}$ is exponentially stabilizing. In that case, the cost associated with $\pi_{\infty,uc}^{k,\epsilon}$ is also bounded. These results allow us to construct a suboptimal policy in a systematic way as described in Algorithm 1. The constants ϵ_{min} and k_{max} in the algorithm are used to provide a finite upper bound on the time dedicated to the computation of such a policy. Algorithm 1 returns a relaxed SRS that characterizes a suboptimal policy independent of the initial state z of the system. Both the relaxation algorithm $ES_\epsilon(\cdot)$ and the algorithm for checking the condition in (12) involve only simple convex optimization programs. Experience

Algorithm 1 (Unconstrained Suboptimal Policy)

Input: ϵ , ϵ_{\min} and k_{\max}

- 1: set $\mathcal{H}_0^\epsilon = \{0\}$.
 - 2: **while** $\epsilon > \epsilon_{\min}$ **do**
 - 3: **for** $k = 1$ to k_{\max} **do**
 - 4: $\mathcal{H}_k^\epsilon = ES_\epsilon(\rho_{\mathbb{M}}(\mathcal{H}_k^\epsilon))$
 - 5: **if** \mathcal{H}_k^ϵ satisfies the condition in (12) **then**
 - 6: stop and return the set \mathcal{H}_k^ϵ which characterizes the policy $\pi_{\infty,uc}^{\epsilon,k}$
 - 7: **end if**
 - 8: **end for**
 - 9: reduce ϵ
 - 10: **end while**
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shows that, although the size of the exact SRS grows exponentially fast, the size of the relaxed SRS \mathcal{H}_k^ϵ usually grows slowly and saturates at a small number, even in high-dimensional state spaces [29]. Therefore, Algorithm 1 can often be carried out efficiently.

For the remainder of this paper, we shall denote by \mathcal{H}_k^ϵ the relaxed SRS returned by Algorithm 1, whose corresponding policy $\pi_{\infty,uc}^{\epsilon,k}$ is exponentially stabilizing with a performance upper bound given in Theorem 1.

Theorem 1 ([27]). *Let \mathcal{H}_k^ϵ be the relaxed SRS returned by Algorithm 1, and let $\pi_{\infty,uc}^{\epsilon,k}$ be the corresponding control policy. Then the closed-loop system driven by $\pi_{\infty,uc}^{\epsilon,k}$ is exponentially stable and there exist constants $\eta_1 < \infty$, $\eta_2 < \infty$ and $\gamma < 1$ such that $J_{k,uc}^\epsilon(z) \leq J_{\infty,uc}^*(z) + \eta_1 \gamma^k \epsilon \|z\|^2$, and $J_\infty(z, \pi_{\infty,uc}^{\epsilon,k}(z)) \leq J_{\infty,uc}^*(z) + \eta_2 \gamma^k \epsilon \|z\|^2$, $\forall z \in \mathbb{R}^n$.*

4. Safe Sets and Their Computation

To handle non-trivial constraints, we will in this section introduce and study the concept of *safe sets*. Sets of this kind play a crucial role in solving the DCSLQR problem, as will be further discussed in Section 5.2.

4.1. Safe Sets

We define a safe set of an unconstrained, infinite-horizon, hybrid-control policy as a set of initial states for which the closed-loop system driven by this policy satisfies the constraints (2) for all $t \geq 0$. The largest of these safe sets is the *maximal invariant set* of the closed-loop system.

Definition 2 (Maximal Invariant Set). *For an arbitrary infinite-horizon policy $\pi_\infty = \{(\mu_t, \nu_t)\}$, $t \in \mathbb{Z}^+$, the associated maximal invariant set $\mathcal{X}(\pi_\infty)$ is:*

$$\begin{aligned} \mathcal{X}(\pi_\infty) = \left\{ x_0 \in \mathbb{R}^n \mid x_t \in \mathcal{X}, \mu_t(x_t) \in \mathcal{U}, \right. \\ \left. x_{t+1} = A_{\nu_t(x_t)} x_t + B_{\nu_t(x_t)} \mu_t(x_t), \forall t \in \mathbb{Z}^+ \right\}. \end{aligned} \quad (13)$$

With the above definition, given a policy π , if $x_{t_0} \in \mathcal{X}(\pi)$ for some $t_0 \in \mathbb{Z}^+$, then $x_t \in \mathcal{X}(\pi)$ for all $t \geq t_0$. Other authors have referred to (13) as the *maximal output admissible set* [11]. In the following, denote $\mathcal{X}_{\infty,uc} = \mathcal{X}(\pi_{\infty,uc}^{\epsilon,k})$. Note that any arbitrarily shaped subset $\underline{\mathcal{X}}_{\infty,uc} \subset \mathcal{X}_{\infty,uc}$ is a safe set, i.e. the closed-loop trajectory starting from any point in $\underline{\mathcal{X}}_{\infty,uc}$ will stay inside $\mathcal{X}_{\infty,uc}$ for all time and thus never violate the constraints. Consequently, while safe sets must be “strongly returnable” [10], they need not be invariant.

Since an exact characterization of $\mathcal{X}_{\infty,uc}$ is generally much harder to obtain than in the non-switched case, the rest of this subsection is devoted to the computation of a subset $\underline{\mathcal{X}}_{\infty,uc} \subset \mathcal{X}_{\infty,uc}$. As will be shown in Section 5.2, this subset will allow to solve a general DCSLQR problem by concatenating the solution of a finite-horizon constrained optimization problem and the infinite-horizon unconstrained suboptimal policy $\pi_{\infty,uc}^{\epsilon,k}$.

4.2. Analytical Characterization of a Safe Set

Proposition 1. *Under Assumption (A1), there exists a constant $r^* > 0$ such that the set*

$$\underline{\mathcal{X}}_{\infty,uc} = \{z \in \mathbb{R}^n \mid \|z\| \leq r^*\} \quad (14)$$

is a subset of $\mathcal{X}_{\infty,uc}$.

PROOF. Proposition 1 can be viewed as an application of the more general result proved in [10]. We still give a proof here, since it directly leads to a possible way of determining r^* of a safe ball $B(r^*)$. To this end, let r_0 be the radius of the largest Euclidean ball $B(r_0)$ centered at the origin that is contained in \mathcal{X} , i.e. $r_0 = \max\{r : \|z\| \leq r \Rightarrow z \in \mathcal{X}\}$. Since we assume $0 \in \text{int}(\mathcal{X})$, we have $r_0 > 0$. By Theorem 1, $\pi_{\infty,uc}^{\epsilon,k}$ is exponentially stabilizing, thus the associated closed-loop trajectory satisfies $\|x_t\| \leq c_1 \|x_0\|$, $\forall t \geq 0$, for some finite positive constant c_1 . From (9), we know $\|u_t\| = \|K_i(P)x_t\| \leq c_2 \|x_t\|$, $\forall t \in \mathbb{Z}^+$, where $c_2 = \max_{i \in \mathbb{M}, P \in \mathcal{H}_k^\epsilon} \{\|(K_i(P))\|\}$. Let $\underline{\mathcal{X}}_{\infty,uc}$ be the Euclidean ball $B(r^*)$, with $r^* = \min\{\frac{r_0}{c_1}, \frac{r_0}{c_1 c_2}\}$, centered at the origin. The values of r_0 , c_1 and c_2 are all finite and thus is r^* . It can be easily seen that for any initial state in $\underline{\mathcal{X}}_{\infty,uc}$, the closed-loop trajectory and the corresponding continuous-control sequence will always satisfy constraints (2). \square

Following the above proof, the characterization of the safe subset $\underline{\mathcal{X}}_{\infty,uc}$ requires estimating three constants r_0 , c_1 and c_2 . Estimating r_0 given the set \mathcal{X} is trivial. The estimation of c_1 and c_2 can be carried out after obtaining the relaxed SRS \mathcal{H}_k^ϵ using Algorithm 1. Therefore, the safe subset $\underline{\mathcal{X}}_{\infty,uc}$ in (14) can be computed in a state space of arbitrary dimension. However, this approach may be overly conservative, resulting in a set $\underline{\mathcal{X}}_{\infty,uc}$ that is much smaller than the maximal invariant set $\mathcal{X}_{\infty,uc}$. We next discuss a computational approach which can be used to under-approximate $\mathcal{X}_{\infty,uc}$ in lower-dimensional state spaces.

4.3. Computational Approach

The most straightforward way to obtain a safe set $\underline{\mathcal{X}}_{\infty,uc}$ is to approximately compute a positive invariant set as in Definition 2. The computation of invariant sets can be reframed as the dual of a reachability problem and is reminiscent of the seminal work in [5, 12]. Different approaches have been developed in the literature to compute reachable sets for dynamical systems, such as polytopic or zonotopic methods, ellipsoidal methods, level-set methods and others [15]. In [22], a technique is proposed to characterize and compute positively invariant and control invariant sets of PWA models through a terminating algorithm. In our case, there are two main issues preventing the direct use of the mentioned approaches. The first one is the implicit form of the control law (9), while most algorithms assume the closed-loop dynamics to be available in an explicit form. Furthermore, it turns out that the *decision regions* associated with the control law (9), i.e. the regions in the state space that yield the same pair $(P_k^\epsilon, i_k^\epsilon)$, are possibly non-convex second-order cones that can not be approximated by polyhedra or ellipsoids [29].

One immediate approach for computing a safe set is by approximation via gridding of the state space, as implemented in Algorithm 2 (for the proposed algorithm, \mathcal{X} is assumed to be bounded). Let $G_{\bar{\mathcal{X}}}$ be the set of all points that constitute a uniform grid with step size δ_{grid} over the smallest hyperrectangle $\bar{\mathcal{X}}$ in the state space that contains the constraint polyhedron \mathcal{X} . Let \mathcal{G} describe the region in the state space covered by the gridpoints g_i in $G_{\bar{\mathcal{X}}}$. A mapped state z is regarded as contained in \mathcal{G} if $\min_{g \in G_{\bar{\mathcal{X}}}} \|z - g\|_\infty \leq \delta_{grid}/2$.

Algorithm 2 (Grid-based computation of $\mathcal{X}_{\infty,uc}$)

Input: $\xi_{k,uc}^\epsilon = \{(\mu_{k,uc}^\epsilon, \nu_{k,uc}^\epsilon)\}$, \mathcal{X} , \mathcal{U} , $G_{\bar{\mathcal{X}}}$

- 1: set $G_0 = \{g_i \in G_{\bar{\mathcal{X}}} \mid g_i \in \mathcal{X}, \mu_{k,uc}^\epsilon(g_i) \in \mathcal{U}\}$
- 2: $G_{k+1} = \{g_i \in G_k \mid A_{\nu_{k,uc}^\epsilon(g_i)} g_i + B_{\nu_{k,uc}^\epsilon(g_i)} \mu_{k,uc}^\epsilon(g_i) \in G_k\}$
- 3: **if** $G_{k+1} = G_k$ **then**
- 4: return $G_{\mathcal{X}_{\infty,uc}} = G_{k+1}$
- 5: **else**
- 6: go to 2
- 7: **end if**

Algorithm 2 was implemented in MATLAB and tested for state dimensions $n \leq 4$. In principle, it also works in higher dimensions. However, the computational complexity grows exponentially with the state dimension. This ‘‘curse of dimensionality’’ prohibits dense gridding for higher dimensional problems.

Remark 2. *We do not necessarily have to use the obtained (possibly complex) approximation of $\mathcal{X}_{\infty,uc}$. The gridded approximation $G_{\mathcal{X}_{\infty,uc}}$ can again be under-approximated by a safe set of simpler shape, e.g. a set of polytopes or ellipsoids. This minimizes the effort of checking whether a given state z is contained in $\underline{\mathcal{X}}_{\infty,uc}$. Also, it would allow to explicitly invoke a terminal constraint*

(see Remark 3 for more on this). However, a smaller safe set $\underline{\mathcal{X}}_{\infty,uc}$ will at the same time negatively impact the minimal finite time horizon of the optimal control problem introduced in Section 5.2.

5. Solution to a General DCSLQR Problem

In this section, we propose a solution procedure for the general DCSLQR problem with nontrivial constraints. The goal is to find an infinite-horizon hybrid-control sequence for a given initial state z to achieve at least suboptimal performance with respect to the cost function $J_{\infty}^*(z)$.

5.1. Stabilizable Set

The DCSLQR problem is meaningful only when the given initial state z results in a finite cost $J_{\infty}^*(z)$. To characterize the set of such initial states, we introduce the following:

Definition 3 (Stabilizable Set). *The set defined by*

$$\mathcal{S}_{\infty} = \{z \in \mathbb{R}^n \mid \exists \psi_{\infty} = \{(u_t, v_t)\}_{t \in \mathbb{Z}^+} \text{ such that} \\ x_t \in \mathcal{X}, u_t \in \mathcal{U} \text{ and } x_t \rightarrow 0 \text{ exponentially fast}\}$$

is called the stabilizable set of system (1) subject to constraints (2), where x_t is the closed-loop trajectory driven by ψ_{∞} with initial state $x_0 = z$.

As proved in [14], asymptotic stability is equivalent to exponential stability for switched linear systems. Therefore, the stabilizable set defined in terms of exponential stability actually characterizes all initial states that yield bounded infinite-horizon costs.

For constrained LQR of linear systems ($M=1$), it is possible to compute the stabilizable set \mathcal{S}_{∞} for compact sets \mathcal{X} and \mathcal{U} [13]. This is achieved by combining multiparametric quadratic programming [4] with reachability analysis. The obtained set \mathcal{S}_{∞} is then given as a convex polyhedron. In principle, the same idea also applies to SLS ($M>1$). However, if we wanted to generalize the algorithmic approach from [13], in each step we would have to solve multiparametric mixed-integer quadratic programs, for which the resulting state-space partitions are no longer polyhedral partitions. See [6] for further details on this problem.

Despite the challenge mentioned above, an under-approximation of \mathcal{S}_{∞} can still be obtained using the algorithm presented in [13]. For each $i \in \mathbb{M}$, let \mathcal{S}_{∞}^i be the stabilizable set associated with subsystem (A_i, B_i) subject to constraints (2). Since one can stay in one mode throughout the entire horizon, the following proposition holds true.

Proposition 2. *The set $\underline{\mathcal{S}}_{\infty} \triangleq \bigcup_{i \in \mathbb{M}} \mathcal{S}_{\infty}^i$ is a subset of \mathcal{S}_{∞} .*

For each mode $i \in \mathbb{M}$, the set \mathcal{S}_{∞}^i is nontrivial ($\mathcal{S}_{\infty}^i \neq \{0\}$) if and only if the unconstrained subsystem (A_i, B_i) is stabilizable. Thus, $\underline{\mathcal{S}}_{\infty}$ is nontrivial if at least one unconstrained subsystem is stabilizable.

5.2. DCSLQR Formulation as an Mixed-Integer Quadratic Program

Following [9, 23], our strategy in solving the DCSLQR problem is to first drive the system state into a safe set $\underline{\mathcal{X}}_{\infty,uc}$, and then use the stationary sub-optimal infinite-horizon policy $\pi_{\infty,uc}^{\epsilon,k}$ to further regulate the state towards the origin. To this end, we introduce the following *constrained finite-time optimal hybrid control (CFTOHC)* problem:

$$J_N^*(z; \phi) = \begin{cases} \min_{(u_t, v_t)} \left\{ \phi(x_N) + \sum_{t=0}^{N-1} L(x_t, u_t, v_t) \right\}, \\ \mathbf{s.t} \text{ (1) and (2) with } x_0 = z \in \mathcal{S}_{\infty}, \end{cases} \quad (15)$$

with the terminal cost function $\phi : \mathcal{X} \rightarrow \mathbb{R}^+$. Denote by $x_{N|0}$ the state at time $t = N$ when system (1) is controlled by the solution of (15). The reason for introducing the above optimization problem is that with a properly chosen terminal cost function $\phi(\cdot)$, the optimal cost J_N^* will coincide with the value function J_{∞}^* of the DCSLQR problem.

Theorem 2. *If $\phi(\cdot) = J_{\infty,uc}^*(\cdot)$ and $x_{N|0} \in \mathcal{X}_{\infty,uc}$, then $J_N^*(z; \phi) = J_{\infty}^*(z)$.*

PROOF. For each $z \in \mathcal{X}$, define $\Gamma(z) = \{(u, v) \in \mathcal{U} \times \mathbb{M} : A_v z + B_v u \in \mathcal{X}\}$. By a standard result of dynamic programming, J_{∞}^* must satisfy the Bellman equation, namely, $J_{\infty}^*(z) = \min_{(u,v) \in \Gamma(z)} \{L(z, u, v) + J_{\infty}^*(A_v z + B_v u)\}$. The result of Theorem 2 follows from applying the iteration N times and noticing that $J_{\infty,uc}^*(x_{N|0}) = J_{\infty}^*(x_{N|0})$ for all $x_{N|0} \in \mathcal{X}_{\infty,uc}$. \square

The value of the setup of Problem (15), which is based on a terminal cost function, is that by Theorem 1, the function $J_{\infty,uc}^*$ can be accurately approximated by $J_{k,uc}^{\epsilon}$ for large k and small ϵ . Thus, $J_{k,uc}^{\epsilon}$ serves us as a local CLF inside a safe subset $\underline{\mathcal{X}}_{\infty,uc}$. With $\phi(z) = J_{k,uc}^{\epsilon}(z) = \min_{P \in \mathcal{H}_k^{\epsilon}} z^T P z$, Problem (15) becomes:

$$J_N^*(z; J_{k,uc}^{\epsilon}) = \begin{cases} \min_{P \in \mathcal{H}_k^{\epsilon}} \left\{ \min_{(u_t, v_t)} \left[x_N^T P x_N + \sum_{t=0}^{N-1} L(x_t, u_t, v_t) \right] \right\} \\ \mathbf{s.t} \text{ (1) and (2) with } x_0 = z \in \mathcal{S}_{\infty}. \end{cases} \quad (16)$$

The above formulation is obtained by first substituting $\phi(z) = \min_{P \in \mathcal{H}_k^{\epsilon}} z^T P z$ into (15) and then changing the order of the two minimizations. The change on the order of the minimizations will not affect the solution because there are only finitely many matrices in \mathcal{H}_k^{ϵ} . By Theorem 2, $J_N^*(z; J_{k,uc}^{\epsilon})$ will be close to $J_{\infty}^*(z)$ if the controlled terminal state $x_{N|0}$ is in $\underline{\mathcal{X}}_{\infty,uc}$. This can be always guaranteed if N is chosen sufficiently large.

Theorem 3 (Existence of a finite time horizon). *For every initial condition $x_0 = z \in \mathcal{S}_{\infty}$, there exists a finite $\tilde{N}(z)$ such that for all $N \geq \tilde{N}(z)$, the terminal state $x_{N|0}$ of the closed-loop system controlled by the solution of (16) resides inside $\underline{\mathcal{X}}_{\infty,uc}$.*

PROOF. By Definition 3, there exists a hybrid-control sequence that exponentially stabilizes system (1) subject to constraints (2). This implies that the optimal cost of Problem (16) satisfies $J_N^*(z; J_{k,uc}^\epsilon) < C$ for all $N \in \mathbb{Z}^+$ and some $C < \infty$. Since $Q_i \succ 0$ for all $i \in \mathbb{M}$, the controlled terminal state $x_{N|0}$ must converge to zero and hence stay inside $\underline{\mathcal{X}}_{\infty,uc}$ for all large N . \square

For the case of linear systems ($M=1$) and compact sets \mathcal{X} and \mathcal{U} , it is possible to compute \hat{N} for all $z \in \mathcal{S}_\infty$ beforehand [13]. However, the issues mentioned in Section 5.1 for the computation of a stabilizable set for SLS again limit the application here. In [9], a method to estimate \hat{N} given an initial condition $x_0 = z \in \mathcal{S}_\infty$ is developed. Unfortunately, this approach relies on convexity of the stabilizable set, which is generally not given in the case of SLS.

In the following section, we therefore employ a straightforward approach inspired by [23, 25]. Notice that the size of the safe set $\underline{\mathcal{X}}_{\infty,uc}$ can have a significant impact on the value of \hat{N} : the smaller $\underline{\mathcal{X}}_{\infty,uc}$, the longer it will take to drive the trajectory x_t inside this set. Hence, it is desirable to compute an $\underline{\mathcal{X}}_{\infty,uc}$ as large as possible. One contribution of this work is that we are able to characterize the control policy $\pi_{\infty,uc}^{\epsilon,k}$ by the set of p.s.d. matrices \mathcal{H}_k^ϵ . This allows us to cast Problem (16) as a single augmented Mixed-Integer Quadratic Program (MIQP) for a given initial state z and prediction horizon N . The obtained MIQP can then be solved with reasonable efficiency using state-of-the-art optimization software. We introduce the following short-hand notation:

$$\begin{aligned} \bar{\mathbb{X}} &= \{x_0, x_1, \dots, x_N\} & \bar{\mathbb{X}} &= \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_N\} & \mathbb{V} &= \{v_0, v_1, \dots, v_{N-1}\} \\ \bar{\mathbb{U}} &= \{u_0, u_1, \dots, u_{N-1}\} & \bar{\mathbb{U}} &= \{\bar{u}_0, \bar{u}_1, \dots, \bar{u}_{N-1}\} \end{aligned}$$

Here, $v_t \in \{0, 1\}^M$ are binary vectors for the mode selection input, $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^p$ are the state vectors and continuous input vectors with associated slack variables $\bar{x}_t \in \mathbb{R}^n$ and $\bar{u}_t \in \mathbb{R}^p$, respectively. In addition, the binary vector $v_{(w)} \in \{0, 1\}^{|\mathcal{H}_k^\epsilon|}$ selects the p.s.d. matrices $P^{(j)}$, $j = 1, \dots, |\mathcal{H}_k^\epsilon|$, for the $|\mathcal{H}_k^\epsilon|$ different terminal weights and the slack variable $w \in \mathbb{R}^n$ represents the terminal cost. With this notation, Problem (16) is equivalent to the following MIQP:

$$\begin{aligned} J_N^*(z; J_{k,uc}^\epsilon) &= \min_{\mathbb{V}, v_{(w)}, \bar{\mathbb{X}}, \bar{\mathbb{X}}, w, \bar{\mathbb{U}}, \bar{\mathbb{U}}} w^T w + \sum_{t=0}^{N-1} (\bar{x}_t^T \bar{x}_t + \bar{u}_t^T \bar{u}_t) \\ \text{s.t.} \quad & x_t \in \mathcal{X}, \quad u_t \in \mathcal{U}, \quad x_0 = z \\ & v_t(i) = 1 \Rightarrow \begin{cases} x_{t+1} = A_i x_t + B_i u_t \\ \bar{x}_t = (Q_i)^{\frac{1}{2}} x_t \\ \bar{u}_t = (R_i)^{\frac{1}{2}} u_t \end{cases} \quad (17) \\ & v_{(w)}(j) = 1 \Rightarrow w = (P^{(j)})^{\frac{1}{2}} x_N \\ & v_t^T v_t = 1, \quad v_w^T v_w = 1 \end{aligned}$$

In addition to the primary polyhedral constraints (2) on state and input, in (17) we are also optimizing subject to logical constraints (indicated by “ \Rightarrow ”).

One possibility to represent those logical constraints is the so-called “big M” formulation [26]. For numerical stability, the slack variables $\bar{x}_0, \dots, \bar{x}_N, \bar{u}_0, \dots, \bar{u}_{N-1}$ and w have to be bounded explicitly. The polyhedral constraint $x_t \in \mathcal{X}$ can be expressed as $H_x x_t \leq k_x$. Since we assume positive definiteness of all Q_i , the matrix $(Q_i)^{\frac{1}{2}}$ is invertible and the computation of a constraint polytope $\bar{\mathcal{X}} : H_{\bar{x}} \bar{x}_t \leq k_{\bar{x}}$ is straightforward:

$$\underbrace{H_x(Q_i)^{-\frac{1}{2}}}_{H_{\bar{x}}} \bar{x}_t \leq \underbrace{k_x}_{k_{\bar{x}}}.$$

For the bounds on the other slack variables \bar{u}_t and w , the procedure is the same.

5.3. Overall Algorithm for DCSLQR

Theorem 3 guarantees that as N increases, the controlled terminal state $x_{N|0}$ associated with (16) eventually enters the safe subset $\underline{\mathcal{X}}_{\infty,uc}$. Then, a suboptimal infinite-horizon control sequence for Problem (4) is given by

$$\psi_{\infty} = \{(\hat{u}_0, \hat{v}_0), \dots, (\hat{u}_{N-1}, \hat{v}_{N-1}), \pi_{\infty,uc}^{\epsilon,k}(x_{N|0})\}, \quad (18)$$

where $\{(\hat{u}_t, \hat{v}_t)\}_{0 \leq t < N}$ denotes the solution to the optimization problem (16) and $\pi_{\infty,uc}^{\epsilon,k}(x_{N|0})$ denotes the infinite-horizon hybrid-control sequence generated by the policy $\pi_{\infty,uc}^{\epsilon,k}$ with initial state $x_{N|0}$.

A general procedure for solving the DCSLQR problem (4) with initial condition $x_0 = z \in \underline{\mathcal{S}}_{\infty}$ is summarized in Algorithm 3. The control sequence returned by Algorithm 3 is guaranteed to be suboptimal in the sense that by choosing k sufficiently large and ϵ sufficiently small, its performance can be made arbitrarily close to the optimal one.

Algorithm 3 (Solution of DCSLQR Problem (4))

Input: $x_0 = z \in \underline{\mathcal{S}}_{\infty}$, method for solving (17)

- 1: compute $\pi_{\infty,uc}^{\epsilon,k}$ and the corresponding relaxed SRS \mathcal{H}_k^{ϵ} using Algorithm 1
 - 2: compute $\underline{\mathcal{X}}_{\infty,uc}$ using Theorem (1) or Algorithm 2
 - 3: set $N = 1$
 - 4: solve problem (16) with time horizon N
 - 5: **if** $x_{N|0} \in \underline{\mathcal{X}}_{\infty,uc}$ **then**
 - 6: stop and return the control sequence as defined in (18)
 - 7: **else**
 - 8: set $N = N + 1$ and go to step 4
 - 9: **end if**
-

Theorem 4. For any $\delta > 0$ and $z \in \underline{\mathcal{S}}_{\infty}$, there exists a $k < \infty$ and an $\epsilon > 0$ such that the control sequence ψ_{∞} returned by Algorithm 3 satisfies

$$J_{\infty}(z, \psi_{\infty}) \leq J_{\infty}^*(z) + \delta.$$

PROOF. Due to compactness of $\underline{\mathcal{X}}_{\infty,uc}$ and by Theorem 1, there exist constants $\epsilon > 0$ and $k < \infty$ such that $|J_{k,uc}^\epsilon(x_{N|0}) - J_{\infty,uc}^*(x_{N|0})| < \delta$ for all $x_{N|0} \in \underline{\mathcal{X}}_{\infty,uc}$. For this choice of k and ϵ , since Algorithm 3 guarantees that the predicted state $x_{N|0} \in \underline{\mathcal{X}}_{\infty,uc}$ under the returned control sequence ψ_∞ , we have that

$$J_\infty(z, \psi_\infty) = J_N^*(z; J_{\infty,uc}^\epsilon) \leq J_N^*(z; J_{\infty,uc}^*) + \delta \leq J_\infty^*(z) + \delta.$$

where the last step follows from Theorem 2. \square

Theorem 4 suggests that by increasing k and decreasing ϵ , an arbitrarily small error in $J_\infty(z, \psi_\infty)$ can be achieved. This often increases the number of matrices contained in \mathcal{H}_k^ϵ and hence the complexity of Problem (16). Thus, in practical implementation, a tradeoff between performance and speed of computation needs to be found.

Remark 3. *It is possible to extend the presented result to a hybrid MPC formulation by explicitly invoking the terminal constraint $x_N \in \underline{\mathcal{X}}_{\infty,uc}$ and using a fixed prediction horizon N in (17). The computation of the hybrid-control law $\xi_{k,uc}^\epsilon$ and the computationally challenging task of obtaining an approximated safe set may be performed off-line, possibly allowing an online-implementation when using fast MIQP solvers. This is a practical way of posing the problem, but would complicate proving the degree of sub-optimality of the overall solution, which is more of a concern to this paper.*

6. A Numerical Example

A simple example with two subsystems in 2-dimensional state space has been presented in [1]. We here consider a more complex example with $M = 3$ modes, state dimension $n = 3$ and input dimension $p = 2$ defined by the following matrices:

$$\begin{aligned} A_1 &= \begin{bmatrix} -1.5 & 0.75 & 0 \\ 1 & 0.25 & -0.5 \\ -0.25 & -0.5 & 0.75 \end{bmatrix}, & A_2 &= \begin{bmatrix} -1 & 1.5 & 0.5 \\ 0.5 & -0.5 & -0.75 \\ 0.5 & -1 & -0.25 \end{bmatrix}, & A_3 &= \begin{bmatrix} -0.75 & 1 & -0.5 \\ -0.5 & 0.75 & 0 \\ -1 & 0 & 1.25 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 0.4 & -0.4 \\ -0.5 & 0.75 \\ 0.25 & -0.5 \end{bmatrix}, & B_2 &= \begin{bmatrix} 0.5 & 0.3 \\ 0.75 & -0.75 \\ -0.1 & 0.5 \end{bmatrix}, & B_3 &= \begin{bmatrix} -0.75 & 0.5 \\ -0.75 & 0 \\ 0.75 & -0.25 \end{bmatrix}, \\ Q_1 &= \begin{bmatrix} 1.5 & 0.45 & 0.45 \\ 0.45 & 1 & 0.3 \\ 0.45 & 0.3 & 1.25 \end{bmatrix}, & Q_2 &= \begin{bmatrix} 1.75 & 0.35 & 0.25 \\ 0.35 & 1 & 0.25 \\ 0.25 & 0.25 & 1.25 \end{bmatrix}, & Q_3 &= \begin{bmatrix} 1.25 & 0.25 & 0.3 \\ 0.25 & 1.25 & 0.3 \\ 0.3 & 0.3 & 1.5 \end{bmatrix}, \\ R_1 &= \begin{bmatrix} 1.2 & 0.25 \\ 0.25 & 1.1 \end{bmatrix}, & R_2 &= \begin{bmatrix} 1 & 0.3 \\ 0.3 & 1 \end{bmatrix}, & R_3 &= \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 1.1 \end{bmatrix}. \end{aligned}$$

The constraint polytopes \mathcal{X} and \mathcal{U} are characterized by the sets of vertices $V_{\mathcal{X}}$ and $V_{\mathcal{U}}$, respectively:

$$\begin{aligned} V_{\mathcal{X}} &= \left\{ \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} \right\}, \\ V_{\mathcal{U}} &= \left\{ \begin{bmatrix} 0.8 \\ 0.8 \end{bmatrix}, \begin{bmatrix} 0.5 \\ -0.8 \end{bmatrix}, \begin{bmatrix} -0.5 \\ 0.8 \end{bmatrix}, \begin{bmatrix} -0.8 \\ -0.8 \end{bmatrix} \right\}. \end{aligned}$$

The computation of the set $\mathcal{H}_k^\varepsilon$ was performed for a horizon of $k = 25$ and a numerical relaxation parameter of $\varepsilon = 10^{-3}$. The evolution of the cardinality of $\mathcal{H}_l^\varepsilon$, $l=0, 1, \dots, k$ leads to a final number of $|\mathcal{H}_k^\varepsilon| = 28$ matrices, as opposed to $3^{25} \approx 10^{12}$ matrices in the unrelaxed SRS \mathcal{H}_k . Clearly, computing \mathcal{H}_k without numerical relaxation is computationally prohibitive even for simple systems.

For the initial state $[-0.9 \ 1.75 \ 1.0]^T$, the optimal discrete control inputs (unconstrained and constrained) for $0 \leq t \leq 5$ are given in Table 1. The compu-

Table 1: Optimal discrete switching modes

t	0	1	2	3	4	5
$v_{opt,uc}$	1	2	1	2	1	2
v_{opt}	1	3	3	3	3	3

tation time for the CFTOHC problem (17) was 1.11s using the CPLEX solver on a 3 GHz Intel Core2 CPU, where we employed YALMIP [17] to conveniently parse the optimization problem.

An approximation of the maximal positive invariant set $\mathcal{X}_{\infty,uc}$ for a gridpoint distance of $\delta_{grid} = 0.075$ is shown in Figure 1. From the shape of $\mathcal{X}_{\infty,uc}$, it is easy to see that a ball-shaped under-approximation of $\mathcal{X}_{\infty,uc}$ would be rather conservative in this case.

The optimal state trajectories are depicted in Figure 2, the associated optimal continuous control actions are shown in Figure 3. As expected, both state trajectory and control inputs converge to the origin and satisfy the constraints in this example. It is a coincidence for this specific example that at $t = 2$, the continuous control input of constrained and unconstrained controller are almost identical, and that at $t = 3$, the unconstrained controller produces a continuous control input that lies on the boundary of the feasible polyhedron in the input space. Note that although the unconstrained controller produces a state trajectory that is feasible, the associated sequence of control inputs is not.

7. Conclusion

In this paper, we presented an approach for approximately solving a discrete-time, constrained infinite horizon optimal hybrid-control problem for Switched Linear Systems with guaranteed suboptimal performance. Building on previous results, we formulated a stationary suboptimal policy for the unconstrained problem with analytical bounds on its optimality. For such a policy we showed how to obtain a conservative analytical as well as a non-conservative gridding-based characterization of a safe set. A safe set is a set of initial conditions from which the unconstrained policy is persistently feasible. We further showed how to obtain an overall, suboptimal solution by repeatedly solving a finite-horizon optimal hybrid-control problem until the terminal state is contained within the safe set.

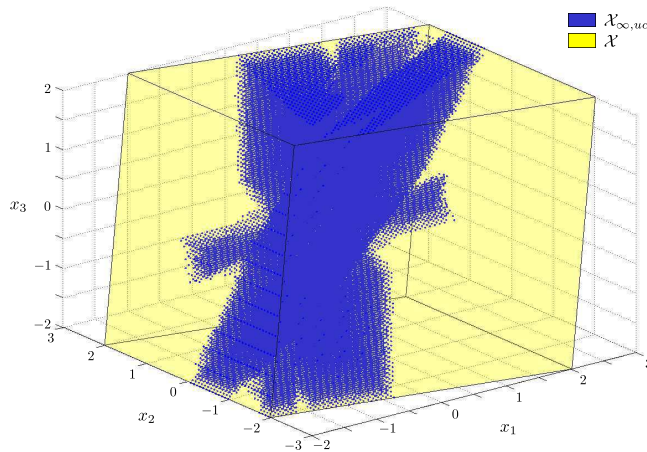


Figure 1: Feasible region \mathcal{X} and gridding-based approximation of the safe set $\mathcal{X}_{\infty,uc}$

The main contributions are the computation of a gridding-based safe set, the formulation of the finite-horizon optimal hybrid-control problem with appropriate terminal cost function as an MIQP. We have also showed that, if the system is stabilizable from a given set of initial states, our approach is able to achieve arbitrarily good suboptimal performance by proper choice of the design parameters.

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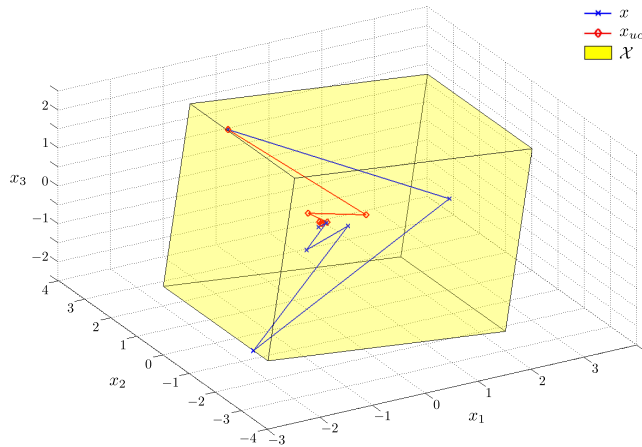


Figure 2: Feasible region \mathcal{X} and closed-loop trajectories (constrained and unconstrained case)

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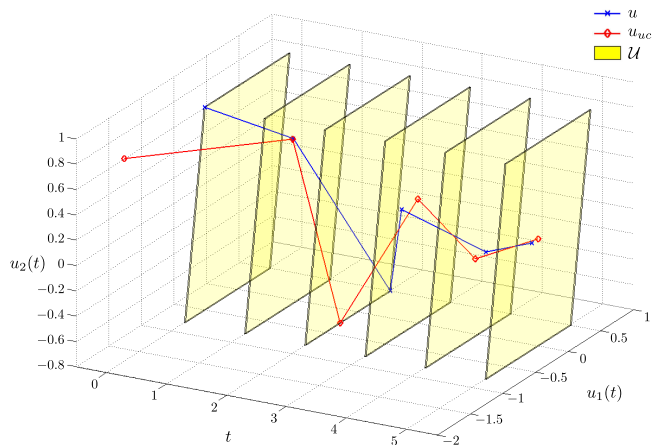


Figure 3: Continuous optimal control inputs (constrained and unconstrained case)

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