$$\sum_{i}^{|\Sigma_n|} P\left(\sup_{t\in[t_i,t_i+h]}(X(t)-X(t_i)\right) > \epsilon_n)$$

$$\leq |\Sigma_n| \exp^{-\frac{\left(2^{-\frac{n}{2}}-2^{-n}\right)^2}{2^{-2n}}} \leq |\Sigma_n| \exp^{-\left(2^n-2^{\frac{n}{2}}+1\right)}$$

At this point, the last, and non-trivial, step in order to derive our convergence results and relative error bounds is to show that

$$\lim_{n\to\infty} |\Sigma_n| \exp^{-\left(2^n-2^{\frac{n}{2}}+1\right)} = 0.$$

In fact, as

$$0 \leq P(\mathcal{B}|A^n) \leq |\Sigma_n| \exp^{-\left(2^n-2^{\frac{n}{2}}+1\right)},$$

this would guarantee that

$$P_{\text{safe}}(X,S,I) = \lim_{n\to\infty} P(\mathcal{A}^n \wedge \mathcal{B}^c) = \lim_{n\to\infty} P(\mathcal{A}^n).$$

To do that, it is sufficient to show that $h = \frac{2^{-n}}{C}$, for some constant $C$. In fact, this implies $|\Sigma_n| = \frac{I\cdot C}{2^{-n}}$.

Recall that we chose $h$ such that for all $t_i \in \Sigma_n$,

$$E\left[\sup_{t\in[t_i,t_i+h]}(X(t)-X(t_i))\right] \leq 2^{-n}.$$

As a consequence, it is enough to take $h$ as the greatest interval smaller than $2^{-n}$ such that this condition is verified. For $t_i \in \Sigma_n$ call $\bar{X}_i = X(t) - X(t_i)$. We can now make use of the *Dudley integral (or entropy integral)* [4], which guarantees that for $t_i \in \Sigma_n$,

$$E[\sup_{t\in[t_i,t_i+h]}(\bar{X}_i)] \leq$$

$$K\int_0^{\frac{diam([t_i,t_i+h])}{2}} \sqrt{\ln(N([t_i,t_i+h],d,\epsilon))}d\epsilon,$$

where $K \geq 12$ is a constant and $d$ is a pseudo-metric defined as

$$d(t,t+dt) = \sqrt{E[(X(t+dt)-X(t))^2]}.$$

$N([t_i,t_i+h],d,\epsilon)$ represents the smallest number of balls of radius $\epsilon$, which covers $[t_i,t_i+h]$, under metric $d$, where $diam([t_i,t_i+h])$ is defined as

$$diam([t_i,t_i+h]) = sup_{s',s\in[t_i,t_i+h]} d(s',s)$$

and with our assumptions, it is possible to show that there exists a constant $K_d$ such that

$$d(t,t+h) \leq K_d \cdot h$$

Moreover, for $\bar{T}_i = [t_i,t_i+h] \subseteq \mathbb{R}_{\geq 0}$ we have

$$N(\bar{T}_i,d,\epsilon) \leq \frac{K_d h}{2\epsilon} + 1,$$

This can be easily understood thinking at the geometry of the problem. As a consequence, we have

$$E\left[\sup_{t\in\bar{T}_i}(\bar{X}_i(t))\right] \leq K\int_0^{\sqrt{2}\cdot 2^{-n-1}} \sqrt{ln\left(\frac{K_d h}{2\epsilon}+1\right)}\, d\epsilon$$

Now, our property is satisfied if we chose $h$ such that

$$K\int_0^{\sqrt{2}\cdot 2^{-n}} \sqrt{\ln\left(\frac{K_d h}{2\epsilon}+1\right)}\, d\epsilon \leq 2^{-n}.$$

The integral inequality we need to solve cannot be solved analytically. However, as $K_d h > 0$, we can write

$$K\int_0^{\sqrt{2}\cdot 2^{-n}} \sqrt{\ln\left(\frac{K_d h}{2\epsilon}+1\right)}\, d\epsilon \leq K\int_0^{\sqrt{2}\cdot 2^{-n}} \sqrt{\frac{K_d h}{2\epsilon}}\, d\epsilon$$

$$= K\sqrt{\frac{K_d h}{2}}\int_0^{\sqrt{2}\cdot 2^{-n}} \sqrt{\frac{1}{\epsilon}}\, d\epsilon = K\sqrt{\frac{K_d h}{2}} 2\sqrt{\sqrt{2}2^{-n}}.$$

Asking for this quantity to be smaller than $2^{-n}$, we obtain the following bound for the sampling time $h$ :

$$h \leq min\left\{\frac{2^{-n}}{2\sqrt{2}K^2K_d}, 2^{-n}\right\}.$$

□

# Acknowledgments

This work has been supported by the Royal Society Professorship (L.Cardelli), the Czech Grant Agency grants GA16-24707Y (M. Češka), the EU-FET project QUANTICOL (nr. 600708) and by FRA-UniTS (L. Bortolussi).

# 10. REFERENCES

[1] A. Abate. Probabilistic bisimulations of switching and resetting diffusions. In *CDC*, pages 5918–5923. IEEE, 2010.

[2] A. Abate, J.-P. Katoen, J. Lygeros, and M. Prandini. Approximate model checking of stochastic hybrid systems. *Eur. J. Control*, 6:624–641, 2010.

[3] A. Abate, M. Prandini, J. Lygeros, and S. Sastry. Probabilistic reachability and safety for controlled discrete time stochastic hybrid systems. *Automatica*, 44(11):2724–2734, November 2008.

[4] R. J. Adler. *The geometry of random fields*, volume 62. Siam, 2010.

[5] R. J. Adler and J. E. Taylor. *Random fields and geometry*. Springer Science & Business Media, 2009.

[6] A. Arapostathis, V. Borkar, and M. Ghosh. *Ergodic Control of Diffusion Processes*. Cambridge University Press, 2012.

[7] L. Arnold. Stochastic differential equations. *New York*, 1974.

[8] A. Aziz, K. Sanwal, V. Singhal, and R. Brayton. Model-checking continuous-time Markov chains. *ACM Transactions on Computational Logic*, 1(1):162–170, 2000.

[9] C. M. Bishop. *Neural networks for pattern recognition*. Oxford university press, 1995.

[10] H. Blom and J. Lygeros (Eds.). *Stochastic Hybrid Systems: Theory and Safety Critical Applications*. Number 337 in Lecture Notes in Control and Information Sciences. Springer Verlag, Berlin Heidelberg, 2006.

[11] L. Bortolussi, L. Cardelli, M. Kwiatkowska, and L. Laurenti. Approximation of probabilistic reachability for chemical reaction networks using the

linear noise approximation. In *QEST*, pages 72–88. Springer, 2016.

[12] L. Bortolussi and R. Lanciani. Model checking markov population models by central limit approximation. In *QEST*, pages 123–138. Springer, 2013.

[13] L. Bujorianu. *Stochastic Reachability Analysis of Hybrid Systems*. Springer-Verlag, London, 2012.

[14] L. M. Bujorianu. *Stochastic reachability analysis of hybrid systems*. Springer Science & Business Media, 2012.

[15] P. Bulychev, A. David, K. Guldstrand Larsen, A. Legay, M. Mikučionis, and D. Bøgsted Poulsen. *Checking and Distributing Statistical Model Checking*, pages 449–463. Springer Berlin Heidelberg, 2012.

[16] L. Cardelli, M. Kwiatkowska, and L. Laurenti. Stochastic analysis of chemical reaction networks using linear noise approximation. In *CMSB*, pages 64–76. Springer, 2015.

[17] L. Cardelli, M. Kwiatkowska, and L. Laurenti. A stochastic hybrid approximation for chemical kinetics based on the linear noise approximation. In *CMSB*, pages 147–167. Springer, 2016.

[18] C. Cassandras and J. Lygeros (Eds.). *Stochastic Hybrid Systems*. Number 24 in Control Engineering. CRC Press, Boca Raton, 2006.

[19] D. J. Daley and D. Vere-Jones. *An introduction to the theory of point processes: volume II: general theory and structure*. Springer Science & Business Media, 2007.

[20] M. Davis. *Markov Models and Optimization*. Chapman & Hall/CRC Press, London, 1993.

[21] R. M. Dudley. The sizes of compact subsets of hilbert space and continuity of gaussian processes. *Journal of Functional Analysis*, 1(3):290–330, 1967.

[22] P. M. Esfahani, D. Chatterjee, and J. Lygeros. The stochastic reach-avoid problem and set characterization for diffusions. *Automatica*, 70:43–56, 2016.

[23] S. Esmaeil Zadeh Soudjani and A. Abate. Adaptive and sequential gridding procedures for the abstraction and verification of stochastic processes. *SIAM Journal on Applied Dynamical Systems*, 12(2):921–956, 2013.

[24] M. Fränzle, E. M. Hahn, H. Hermanns, N. Wolovick, and L. Zhang. Measurability and safety verification for stochastic hybrid systems. In *HSCC*, pages 43–52, 2011.

[25] M. Fränzle, H. Hermanns, and T. Teige. Stochastic satisfiability modulo theory: A novel technique for the analysis of probabilistic hybrid systems. In M. Egerstedt and B. Misra, editors, *HSCC*, volume 4981 of *Lecture Notes in Computer Science*, pages 172–186. Springer Verlag, Berlin Heidelberg, 2008.

[26] C. W. Gardiner et al. *Handbook of stochastic methods*, volume 3. Springer-Verlag, 1985.

[27] M. K. Ghosh, A. Arapostathis, and S. I. Marcus. Optimal control of switching diffusions with application to flexible manufacturing systems. *SIAM Journal on Control and Optimization*, 31(5):1183–1204, 1993.

[28] P. E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer Science & Business Media, 2011.

[29] K. Koutsoukos and D. Riley. Computational methods for reachability analysis of stochastic hybrid systems. In J. Hespanha and A. Tiwari, editors, *HSCC*, volume 3927 of *Lecture Notes in Computer Science*, pages 377–391. Springer Verlag, Berlin Heidelberg, 2006.

[30] H. J. Kushner and P. Dupuis. *Numerical Methods for Stochastic Control Problems in Continuous Time*. Springer-Verlag, New York, 2001.

[31] M. Kwiatkowska, G. Norman, and D. Parker. Stochastic model checking. In *Formal methods for performance evaluation*, pages 220–270. Springer, 2007.

[32] P. Massart. *Concentration inequalities and model selection*, volume 6. Springer, 2007.

[33] B. Øksendal. *Stochastic differential equations*. Springer, 2003.

[34] M. Prandini and J. Hu. Stochastic reachability: Theory and numerical approximation. In C. Cassandras and J. Lygeros, editors, *Stochastic hybrid systems*, Automation and Control Engineering Series 24, pages 107–138. Taylor & Francis Group/CRC Press, 2006.

[35] S. Särkkä et al. *Recursive Bayesian inference on stochastic differential equations*. Helsinki University of Technology, 2006.

[36] I. Tkachev, A. Mereacre, J.-P. Katoen, and A. Abate. Quantitative automata-based controller synthesis for non-autonomous stochastic hybrid systems. In *HSCC*, pages 293–302, 2013.

[37] Y. Wang, N. Roohi, M. West, M. Viswanathan, and G. E. Dullerud. Statistical verification of dynamical systems using set oriented methods. In *HSCC*, pages 169–178. ACM, 2015.

[38] G. G. Yin and C. Zhu. *Hybrid switching diffusions: properties and applications*, volume 63. Springer Science & Business Media, 2009.

[39] M. Zamani and A. Abate. Symbolic models for randomly switched stochastic systems. *Systems & Control Letters*, 69:38–46, 2014.

[40] M. Zamani, P. M. Esfahani, R. Majumdar, A. Abate, and J. Lygeros. Symbolic control of stochastic systems via approximately bisimilar finite abstractions. *IEEE Transactions on Automatic Control*, 59(12):2825–2830, 2014.

[41] L. Zhang, Z. She, S. Ratschan, H. Hermanns, and E. M. Hahn. *Safety Verification for Probabilistic Hybrid Systems*, pages 196–211. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.