

Stability and attractivity of absorbing sets for discrete-time Markov processes

Ilya Tkachev and Alessandro Abate

Abstract—The presence of absorbing sets within the continuous state space of a Markov process plays a crucial role in the verification of properties of the process over infinite time horizons. Specifications such as probabilistic invariance, reachability, and reach-avoid have been characterized and computed based on this structural property of the model. This paper extends these results by investigating stability properties of the model over absorbing sets. Theoretical results are developed to study attractivity properties of such sets within the state space of a Markov process. The outcomes are applied over a case study.

I. INTRODUCTION

Markov processes provide a powerful framework to model phenomena in diverse areas such as biology, finance, engineering etc. Such a framework allows one to incorporate the structured randomness affecting the system under study, but the price to pay for this modeling capability is the increased complexity related to its analysis [6].

An important topic in the analysis of dynamical systems is that of stability and of attractivity [7], which deals with the limiting behavior of trajectories over infinite horizons. For probabilistic models one can either think of stability of the limiting (invariant) distribution of the state [6], or consider the convergence of the process to an attractor set [4]. Usually, results in this arena are given with binary probability, thus restricting the class of models under consideration.

In this work, we are interested in path properties of the process, rather than in stability over an invariant distribution. In particular, we look at the problem of computing the actual value of the probability that the process converges to a set, with an explicit bound on the computation error.

The contribution puts forward a definition of stochastic attractivity over subsets of the state space, and studies conditions under which sets are attractive. Over such sets, the work provides methods to compute the associated probability of convergence, which further leads to the characterization of domains of attraction over the whole state space.

Technically, the work leverages the concept of absorbing set [6], [12], the use of PCTL as a modal logic to express infinite-horizon properties [2] for discrete-time Markov processes, and the use of operator theory and of Bellman equations to characterize such properties. The theoretical results are elucidated through a case study.

The authors are with the Delft Center for Systems & Control, Technische Universiteit Delft - Delft University of Technology, The Netherlands. Emails: {i.tkachev, a.abate}@tudelft.nl

II. PRELIMINARIES

A. Notation

Let (\mathcal{X}, ρ) be a Polish space, namely a complete and separable metric space [5], and \mathcal{B} its Borel σ -algebra. A Markov kernel T is such that for all $B \in \mathcal{B}$, $x \mapsto T(B|x)$ is a measurable function on \mathcal{X} ; and $T(\cdot|x)$ is a probability measure on $(\mathcal{X}, \mathcal{B})$ for any $x \in \mathcal{X}$. The event space is given by $\Omega = \mathcal{X}^{\mathbb{N}_0}$ with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and \mathcal{F} is a product σ -algebra on Ω .

The kernel T yields the unique family of probability measures $(P_x)_{x \in \mathcal{X}}$, which defines the discrete-time homogeneous Markov process X (see [9, Chapter 1.2]) such that

$$P_x\{X_1 \in B\} = T(B|x)$$

for any $x \in \mathcal{X}$ and $B \in \mathcal{B}$. We denote by \mathbb{B} the Banach space of all bounded measurable functions $f: \mathcal{X} \rightarrow \mathbb{R}$, endowed with a sup-norm $\|f\| = \sup_{x \in \mathcal{X}} |f(x)|$. On this space we define the Markov operator given for any $f \in \mathbb{B}$ by

$$\mathcal{P}f(x) = \int_{\mathcal{X}} f(y)T(dy|x).$$

For $f, g \in \mathbb{B}$ we write $f \leq g$ if $f(x) \leq g(x)$ for all $x \in \mathcal{X}$.

For the subset of \mathbb{B} of upper (lower) semi-continuous functions we use a shorthand u.s.c. (l.s.c.) [5]. For $f \in \mathbb{B}$ we denote $f_* = \inf\{f(x) : x \in \mathcal{X}\}$ and $f^* = \sup\{f(x) : x \in \mathcal{X}\}$. In the same way:

$$m_f = \{x \in \mathcal{X} : f(x) = f_*\} \text{ and } M_f = \{x \in \mathcal{X} : f(x) = f^*\}.$$

For any $r \in \mathbb{R}$ we denote $f_{\leq r} = \{x \in \mathcal{X} : f(x) \leq r\}$.

The open ball of the radius $\varepsilon > 0$ at a point $x \in \mathcal{X}$ is denoted by $\mathcal{O}_\varepsilon(x) = \{y \in \mathcal{X} : \rho(x, y) < \varepsilon\}$. For $A \subseteq \mathcal{X}$ the set B is said to be a neighborhood of A if there is an open set B' such that $A \subseteq B' \subseteq B$. The space (\mathcal{X}, ρ) is said to be locally compact if for any point $x \in \mathcal{X}$ there is a compact neighborhood of x . For $A \subseteq \mathcal{X}$ its closure is denoted by \bar{A} .

B. Convergence and attractive sets

In this section we discuss the notions of convergence and attractivity for discrete-time Markov processes. First, for $x \in \mathcal{X}$ and $A \subseteq \mathcal{X}$ we define $\rho(x; A) = \inf\{\rho(x, y) : y \in A\}$.

Definition 1. For a sequence $(x_n)_{n \geq 0}$ of elements of \mathcal{X} , we say that $x_n \rightarrow A$ with $n \rightarrow \infty$ if and only if $\lim_n \rho(x_n, A) = 0$. If for $f \in \mathbb{B}$ it holds that $f|_A = c \equiv \text{const}$, then the notation $\lim_{x \rightarrow A} f(x) = c$ denotes that $f(x_n) \rightarrow c$ for any $x_n \rightarrow A$. Equivalently, for any $\varepsilon > 0$ there is $\delta > 0$ such that $|f(x) - c| < \varepsilon$ for all $x \in \mathcal{O}_\delta(A)$.

Remark 1. We notationally distinguish between X_n , which stands for the value of the process X at moment n , and x_n or x , which denote simple points on the state space \mathcal{X} that could, as an example, represent initial conditions for the process X .

The definition of an attractive set $A \subseteq \mathcal{X}$ for a classical deterministic dynamical system [7] requires the set A 1) to be closed, 2) to be invariant under the flow of the system and 3) to have a neighborhood with the property that the system, starting within this neighborhood, never leaves it and converges to the set A . The set of all points in \mathcal{X} which are initial points for trajectories converging to A is called the domain of attraction of A . We argue that, when dealing with systems that are affected by a stochastic uncertainty, such requirements may be in general too conservative, which leads to the following.

Definition 2. We call the set $A \subseteq \mathcal{X}$ stochastically attractive for a process X if

$$\lim_{x \rightarrow A} \mathbb{P}_x \{X_n \rightarrow A\} = 1. \quad (1)$$

This definition is formal since the probability in (1) is well-defined, as we show below. The meaning of (1) is the following: selecting an initial condition x closer to A makes the probability of the event that “the process X converges to A ” closer to 1. Thus, Definition 2 is a modified version of the condition 3) above. Also, it captures the closure property: if (1) holds for the set A then it holds as well for its closure \bar{A} . However, it does not imply that within some neighborhood of set A the probability to stay always within this neighborhood is close to 1 (invariance). Moreover, the notion of being invariant required in deterministic systems and here interpreted as in Definition 3 is too restrictive for the discrete-time stochastic framework (we enlighten it below in Example 2) — this is the reason why such a requirement is not included in Definition 2.

C. Properties, specifications and related functions

The definition of stochastic attractivity leads to studying probabilities related to events, as in (1). In contrast to deterministic dynamical systems, where the study is often restricted to the attractivity of equilibria and closed orbits, in this work we embrace a more general approach which first selects potentially attractive subsets of \mathcal{X} and thereafter verifies their attractivity. A special focus is given to compact subsets of \mathcal{X} , thus excluding possible divergent behaviors of the process X . Due to this reason we introduce appropriate value functions that depend both on the point $x \in \mathcal{X}$ and on the set $A \in \mathcal{B}$. Such value functions, in general, allow one to study properties of the process X which can be expressed as events, i.e. measurable subsets of the event space Ω .

Alternatively, other approaches express dynamical properties as specifications in a modal logic, such as PCTL or probabilistic LTL [2], and proceed verifying the specification of interest over the model. In this work, we do not restrict ourselves to any particular class of logic and specify properties directly via subsets of Ω .

We need three basic properties, called probabilistic reachability, invariance and reach-avoid. In order to define them we use the notion of the first hitting time. Given a set $B \in \mathcal{B}$, its first hitting time is the random variable $\tau_B = \inf\{n \geq 0 : X_n \in B\}$, so that $\tau_B : \Omega \rightarrow \mathbb{N}_0 \cup \{\infty\}$. The n -horizon reachability of B is the event $\{\omega : \tau_B(\omega) \leq n\} \subseteq \Omega$, whereas the infinite-horizon reachability is $\{\tau_B(\omega) < \infty\}$ ¹. We introduce the reachability value functions $v_n(x; B) = \mathbb{P}_x\{\tau_B \leq n\}$ and $v(x; B) = \mathbb{P}_x\{\tau_B < \infty\}$, respectively. Clearly,

$$v(x; B) = \lim_{n \rightarrow \infty} v_n(x; B),$$

where the convergence is monotonically non-decreasing [11].

Invariance is the dual property of reachability, e.g. the n -horizon invariance of A is $\{\tau_{A^c} > n\}$ where $A^c = \mathcal{X} \setminus A$. We denote the associated value function by $u_n(x; A) = \mathbb{P}_x\{\tau_{A^c} > n\}$ and introduce

$$u(x; A) = \lim_{n \rightarrow \infty} u_n(x; A) = \mathbb{P}_x\{\tau_{A^c} = \infty\}.$$

Finally, the n -horizon reach-avoid event over the sets $A, B \in \mathcal{B}$ is $\{\tau_B \leq \min(n, \tau_{A^c})\}$, whereas over the infinite time horizon it is given by $\{\tau_B < \tau_{A^c}, \tau_B < \infty\}$. The corresponding value functions are denoted by $w_n(x; A, B)$ for the finite time horizon and $w(x; A, B)$ for the infinite one. The convergence $w_n \rightarrow w$ is monotonically non-decreasing [8].

For the described three problems, numerical methods based on Bellman-like dynamic programming iterations have been developed for the bounded time horizon: see [1] for reachability and invariance, and [8] for reach-avoid. The work in [1] has provided a discretization technique, which allows one to compute efficiently such value functions with explicit bounds on the error of approximation. On the other hand, infinite time horizon problems appear to be much more challenging and there is no general method to solve them. The contributions in [11], [12] deal with problems that can be solved with explicit bounds on the error – some of these methods will be also employed below.

Finally, we introduce a value function for the convergence event: $\{X_n \rightarrow A\}$. Clearly, unlike the properties above, it does not have a finite horizon version. We start by defining another, closely related event. For a set $A \subseteq \mathcal{X}$ and $\varepsilon > 0$, the ε -ball of A is $\mathcal{O}_\varepsilon(A) = \{x \in \mathcal{X} : \rho(x, A) < \varepsilon\}$. By direct application of the definition of limit, we have that $x_n \rightarrow A$ if and only if for any $m \in \mathbb{N}$ there is $N \in \mathbb{N}$, such that $x_n \in \mathcal{O}_{1/m}(A)$, for all $n \geq N$. We call the property that, for some $B \subseteq \mathcal{X}$ there is an N such that $x_n \in B$ for all $n \geq N$, “ x_n is eventually always in B ,” and denote it as $x_n \hookrightarrow B$. Now, if $x_n \hookrightarrow B$ then obviously $x_n \rightarrow B$ and $x_n \rightarrow B$ if for any $m \in \mathbb{N}$, $x_n \hookrightarrow \mathcal{O}_{1/m}(B)$.

For a set $A \in \mathcal{B}$ the event $\{X_n \hookrightarrow A\} \subseteq \Omega$ is measurable. Indeed, it is first the complement of the event “ X_n visits A^c infinitely often,” which is used to characterize properties such as transience and recurrence, and is proven to be measurable [6]. Secondly, this event is invariant, i.e. it is independent on any finite prefix of the sequence X_n . For more detailed discussion, see [6, Section 15] and [9, Section 3.3]. As a

¹Further on, we omit the argument ω in the definitions of events.

result, it is legitimate to introduce the value function defined as $h(x;A) := P_x\{X_n \in A\}$.

Let us now introduce a value function for the convergence probability: $c(x;A) = P_x\{X_n \rightarrow A\}$. We have

$$\{X_n \rightarrow A\} = \bigcap_{m=0}^{\infty} \{X_n \in \mathcal{O}_{1/m}(A)\},$$

hence the probability in (1) is well defined. Furthermore:

$$c(x;A) = \lim_m h(x; \mathcal{O}_{1/m}(A)), \quad (2)$$

for any $x \in \mathcal{X}$ and $A \in \mathcal{B}$. As a result, (1) is equivalent to

$$\lim_{x \rightarrow A} c(x;A) = 1. \quad (3)$$

Functions c and h play a prominent role in our paper.

III. ANALYSIS AND COMPUTATION OF VALUE FUNCTIONS FOR CONVERGENCE AND ATTRACTIVITY

A. Characterization through harmonic functions

We provide a formal way to derive h through the invariance value function u . First of all, let us define an invariance operator $\mathcal{I}_A : \mathbb{B} \rightarrow \mathbb{B}$ as

$$\mathcal{I}_A f(x) = 1_A(x) \mathcal{P}f(x),$$

where 1_A is the indicator function of A : $1_A(x) = 1$ if $x \in A$, otherwise $1_A(x) = 0$. We obtain:

$$\begin{cases} u_{n+1}(x;A) &= \mathcal{I}_A u_n(x;A), \quad \text{for } n \geq 0, \\ u_0(x;A) &= 1_A(x), \end{cases}$$

and, as discussed in [11],

$$u(x;A) = \mathcal{I}_A u(x;A). \quad (4)$$

Let us further name the function $f \in \mathbb{B}$ superharmonic if $f \geq \mathcal{P}f$, subharmonic if $f \leq \mathcal{P}f$, and harmonic if $f = \mathcal{P}f$. Clearly, the function u is subharmonic: if $x \in A$ then

$$u(x;A) = \mathcal{I}_A u(x;A) = \mathcal{P}u(x;A),$$

whereas if $x \in A^c$, then $0 = u(x;A) \leq \mathcal{P}u(x;A)$, since the right-hand side is non-negative.

Theorem 1. For all $x \in \mathcal{X}$ and $A \in \mathcal{B}$ it holds that

$$h(x;A) = \lim_n \mathcal{P}^n u(x;A), \quad (5)$$

so that $h = \mathcal{P}h$ and $c = \mathcal{P}c$. Moreover, if f is a harmonic function and $f \geq u$ then $f \geq h$, i.e. h is the smallest harmonic majorant of the function u . In particular,

$$\inf_{x \in \mathcal{X}} |h(x;A) - u(x;A)| = 0. \quad (6)$$

Let us raise some remarks. First, the convergence $\mathcal{P}^n u \rightarrow h$ is monotonically non-decreasing: $\mathcal{P}u \geq u$, because u is subharmonic, furthermore $\mathcal{P}^{n+1}u \geq \mathcal{P}^n u$, because \mathcal{P} is a monotone operator. On the other hand, the convergence may not be uniform, which makes it difficult to find the bounds on the quantity $\|\mathcal{P}^n u - h\|$. Second, the equation $h = \mathcal{P}h$ is called the *Laplace equation* and admits infinitely many solutions, e.g. any constant function. Finally, although h is

the least harmonic majorant of u and (6) holds, there are cases when $h(x;A) > u(x;A)$ for all $x \in \mathcal{X}$.

These remarks emphasize possible difficulties related to finding the function h . Moreover, recall that the problem of finding the invariance function u (and thus $\mathcal{P}^n u$) is difficult by itself [11]. On the other hand, although the function h in general cannot be computed with any accuracy, we show that there are cases when this problem has a solution. Also, this work provides techniques that allow one to find the value of h directly, without calculating u in advance.

With focus on the function c , its evaluation is even more difficult: from a computational perspective view we have that

$$c(x;A) = \lim_i \lim_j \lim_k \left(\mathcal{P}^j \mathcal{I}_{O_i}^k \right) 1_{O_i}(x), \quad (7)$$

for all $x \in \mathcal{X}$, $A \in \mathcal{B}$, and where $O_i = \mathcal{O}_{1/i}(A)$. In order to tackle this problem, we show how to eliminate the limit with respect to i , which leads to the calculation of function h .

B. Absorbing sets

Although it is in general hard to find an analytical expression for $h(x;A)$, given an $x \in \mathcal{X}$ and an $A \in \mathcal{B}$, in some cases it is possible. We characterize such instances using the notion of absorbing set.

Definition 3. The set $A \in \mathcal{B}$ is called *absorbing* if $\mathbb{T}(A|x) = 1$ for all $x \in A$. For a given set B the set $A \subseteq B$ is called its *largest absorbing subset* if A is absorbing and for any absorbing set $A' \subseteq B$ it holds that $A' \subseteq A$.

Absorbing sets of Markov processes are analogues of equilibrium points, closed orbits and more generally, of invariant manifolds for classical deterministic dynamical systems [7]. The next lemma further highlights this similarity.

Definition 4. The Markov process X is said to be *weakly continuous* if $\mathcal{P}f$ is continuous for any continuous f .

Lemma 1. If the process X is weakly continuous and A is absorbing then \bar{A} is absorbing.

We denote with $\text{l.a.s.}(A)$ the largest absorbing subset of a given set $A \in \mathcal{B}$. It is well-defined: first let us denote

$$A_n = \{x \in A : u_n(x;A) = 1\}. \quad (8)$$

In [11] it was proved that the sequence $(A_n)_{n \in \mathbb{N}_0}$ is non-increasing: $A_{n+1} \subseteq A_n$ and the for limit set we have

$$A_\infty = \bigcap_{n=0}^{\infty} A_n = \text{l.a.s.}(A).$$

This set also admits the following characterization:

$$\text{l.a.s.}(A) = \{x \in \mathcal{X} : u(x;A) = 1\}. \quad (9)$$

Moreover, the following more general statement holds.

Lemma 2. For a superharmonic f , the set m_f is absorbing.

As a corollary, we have that for a subharmonic function f the set M_f is absorbing, and furthermore that if f is harmonic both m_f and M_f are absorbing. Let us show how these results are employed to find function h .

Theorem 2. For any set $A \in \mathcal{B}$ the function $u(x;A)$ is harmonic if and only if $m_u = \{x \in \mathcal{X} : u(x;A) = 0\}$ is absorbing. In that case $h(x;A) = u(x;A)$.

Theorem 2 implies that if the set m_u is absorbing then the problem of finding h is reduced to that of finding u . Although the analytical expression for u is in general hard to obtain and thus the verification of absorbance of m_u is not a trivial problem, there exist cases with an analytical solution.

C. Non-attractive sets

We show which measurable subsets of \mathcal{X} are essentially not attractive. First, since equation (4) is linear, it always admits a trivial zero solution, though this may happen when $u(x;A)$ is not a constant zero function – a simple example being $A = \mathcal{X}$. Due to this reason, if $u(x;A) \equiv 0$ we say that the set A is *trivial*. We say that the set A is *simple* if $\text{l.a.s.}(A) = \emptyset$.

Lemma 3. [11, Theorem 3] For a weakly continuous process X , a compact set is trivial if and only if it is simple.

Lemma 4. For a weakly continuous process X and a compact set A , the function $u(x;A)$ is u.s.c. and the set $\text{l.a.s.}(A)$ is compact.

We raise criteria for $h(x;A)$ and $c(x;A)$ to be equal to zero based on the simplicity of the set A .

Theorem 3. For a set $A \in \mathcal{B}$ it holds that:

- 1) $h(x;A) \equiv 0$ if and only if A is trivial;
- 2) in particular, if X is weakly continuous and A is compact, then $h(x;A) \equiv 0$ if and only if A is simple;
- 3) if X is weakly continuous, \mathcal{X} is locally compact, and A is compact, then $c(x;A) \equiv 0$ if and only if A is simple.

Corollary 1. It follows that:

- 1) If $A \in \mathcal{B}$ and for some $\delta > 0$ the set $\mathcal{O}_\delta(A)$ is trivial, then A is not stochastically attractive.
- 2) If \mathcal{X} is locally compact, X is weakly continuous, and A is compact and simple, then A is not stochastically attractive.

Corollary 1 provides conditions for sets not to be stochastically contractive. Although the problem of verification of simplicity or triviality of a given set A does not have a general (respectively analytical or computational) solution, there exist sufficient conditions. The first (analytical) conditions require super- or subharmonic functions to be constants, implying that $u(x;A) = 0$ for all $A \neq \mathcal{X}$. The second (computational) conditions require that A_n , as defined in (8), is empty for some $n \in \mathbb{N}$ [11, Theorem 2].

So far we have discussed sets that do not satisfy the given definition of stochastic attractivity. Next, the attention is shifted over a class of sets that satisfies it.

D. Stochastically attractive absorbing subsets

We start with the following useful result. Notice that $\lim_n u(X_n;A)$ exists P_x -a.s. for all $x \in \mathcal{X}$, since u is a bounded

subharmonic function, hence the existence of the limit is insured by the martingale convergence theorem [3].

Lemma 5. For any $x \in \mathcal{X}$ and $A \in \mathcal{B}$ we have

$$P_x \left\{ \lim_n u(X_n;A) = \liminf_n 1_A(X_n) \right\} = 1$$

and

$$P_x \left\{ \lim_n h(X_n;A) = \liminf_n 1_A(X_n) \right\} = 1.$$

In particular,

$$h(x;A) = P_x \left\{ \lim_n u(X_n;A) = 1 \right\} = P_x \left\{ \lim_n h(X_n;A) = 1 \right\}.$$

Lemma 6. If X is weakly continuous and A is a compact set, then $\lim_n u(x_n;A) \rightarrow 1$ implies $x_n \rightarrow \text{l.a.s.}(A)$, for any sequence $(x_n)_{n \geq 0}$ of elements of \mathcal{X} .

Lemmas 5 and 6 show that, provided weak continuity of X and compactness of A , $X_n \rightarrow \text{l.a.s.}(A)$ is a necessary condition for $X_n \in A$, P_x -a.s. for all $x \in \mathcal{X}$. On the other hand, this is not a sufficient condition in general. Due to this reason we introduce the concept of *stable* absorbing set.

Definition 5. An absorbing set $A \in \mathcal{B}$ is called *stable* if there exists a compact neighborhood U_A of A such that $A = \text{l.a.s.}(U_A)$ and

$$\lim_{x \rightarrow A} u(x;U_A) = 1. \quad (10)$$

Remark 2. The stability property of absorbing sets can be related to the Lyapunov stability for classical deterministic dynamical systems [7]. Indeed, (10) means that if A is a stable absorbing subset, then for any $\varepsilon > 0$ there is a neighborhood of A starting from which the process never leaves such neighborhood with a probability at least $1 - \varepsilon$.

The compactness of set U_A plays a role in the following result.

Theorem 4. If X is weakly continuous and A is a stable absorbing set, then A is stochastically attractive and there exists a $M \in \mathbb{N}$ such that for all $x \in \mathcal{X}$ and $m \geq M$ it holds that

$$h\left(x; \overline{\mathcal{O}_{1/m}(A)}\right) = h(x;U_A) = c(x;A).$$

Let us discuss examples showing that some of the conditions we have provided are sufficient but not necessary in general. Let $\mathcal{X} = \{\pm \frac{1}{n}\}_{n \in \mathbb{N}} \cup \{0\}$ be endowed with the Euclidean metric, which makes it a complete separable compact (and locally compact) metric space. We first show that the reverse statement of Theorem 4 does not hold.

Example 1 (A stochastically attractive absorbing set is not necessary stable). Let $T(\{0\}|0) = 1$, $T(\{-1\}|1) = 1$, $T(\{\frac{1}{n-1}\}|\frac{1}{n}) = 1$ for all $n \in \mathbb{N} \setminus \{1\}$ and $T(\{-\frac{1}{n+1}\}|\frac{1}{n}) = 1$ for all $n \in \mathbb{N}$. The corresponding dynamics is clearly deterministic, still it is a weakly continuous Markov process. This process converges to an absorbing set $A = \{0\}$ starting from any initial condition, so $c(x;A) \equiv 1$ and hence A is stochastically attractive. However, there is no such neighborhood U_A of A such that $A = \text{l.a.s.}(U_A)$.

The second example shows that there may exist a set A such that $c(x;A) \equiv 1$, which in particular means that A is stochastically attractive, but it is not absorbing. This fact relates to the discussion given after Definition 2.

Example 2 (A stochastically attractive set is not necessarily absorbing). *In the previous example we only change $T(\{1\}|0) = 1$. We still have $A = \{0\}$ stochastically attractive since $c(x;A) \equiv 1$ but now A is not absorbing. Note that the update in the dynamics leads to the lack of weak continuity of the process, thus this assumption cannot be relaxed in Theorem 3, statement 3).*

E. Domains of attraction of absorbing sets

Theorem 4 shows that the stability of an absorbing set under mild conditions implies its stochastic attractivity. Moreover, it helps eliminating the outermost limit in the computation of the function c as in (7). A Lyapunov-like function can be used to prove the stability of an absorbing set.

Lemma 7. *An absorbing set A is stable if and only if the following stabilizing pair exists:*

- compact neighborhood U_A of A such that $A = \text{l.a.s.}(U_A)$;
- function $f \in \mathbb{B}$ such that $f_* = 0$, $m_f = A$, $\lim_{x \rightarrow A} f(x) = 0$ and there is a $r > 0$ such that $f_{\leq r} \subseteq U_A$ and $f(x) \geq \mathcal{P}f(x)$ for all $x \in f_{\leq r}$.

If X is weakly continuous, A is stable if and only if there is a stabilizing pair with an l.s.c. function f .

We have provided conditions for an absorbing set A to be stable, and hence stochastically attractive, under the conditions of Theorem 4. With regards to its domain of attraction, the set of points for which it holds with probability 1 is clearly given by $\{x \in \mathcal{X} : c(x;A) = 1\}$. Since c is a harmonic function, such set is itself absorbing and hence may coincide either with A or with \mathcal{X} . So, the claim that convergence must hold with probability 1 may be too conservative and instead one may consider ε -domains of attraction given by $\{x \in \mathcal{X} : c(x;A) \geq 1 - \varepsilon\}$. To characterize such domains the procedure of computing the function c with explicit bounds on the error is needed.

We show that the knowledge of a Lyapunov-like function as in Lemma 7 is not only useful to establish the stability of A , but also for such a computational procedure.

Lemma 8. *For any set $A \in \mathcal{B}$ the following trichotomy holds: either for all $x \in A$ $h(x;A) \equiv 0$ or $h(x;A) \equiv 1$, or both conditions $\inf_{x \in \mathcal{X}} h(x;A) = 0$ and $\sup_{x \in \mathcal{X}} h(x;A) = 1$ hold.*

Theorem 5. *Assume that X is weakly continuous, A is a stable absorbing set which admits a stabilizing pair (U_A, f) with an l.s.c. function f , and $r > 0$ is as in the statement of Lemma 7. Assume also that there is an open set E such that $\inf_E c(x;A) = 0$, $U_A \cap E = \emptyset$, $A = \text{l.a.s.}(E^c)$ and put $D_{r'} = (f_{\leq r'} \cup E)^c$ for all $r' \in \mathbb{R}$. Then*

$$|c(x;A) - w(x;D_{r'}, f_{\leq r})| \leq \max \left(\varepsilon, \sup_{y \in E} c(y;A) \right), \quad (11)$$

where w is the reach-avoid value function and $\varepsilon \in (0, 1)$ is arbitrary.

Let us raise some remarks on Theorem 5. First, the reach-avoid value function w in (11) can be computed with explicit bounds on the error [11]. Combining such bounds with the right-hand side in (11), we obtain an approximate value of c with a known precision. Second, weak continuity of X and stability of A ensure that a necessary stabilizing pair exists by Lemma 7. Moreover, since $c(x;A) = h(x;U_A)$, it is either a constant function equal to 1, or the set E exists by Lemma 8.

IV. CASE STUDY

Let us consider the space $\mathcal{X} = K \cup \{\partial\}$, where $K = [-2, 2]^2$, and endowed with the Euclidian metric and such that $\rho(x, \partial) = 1$ for any $x \in K$, with $\partial \notin K$ being an auxiliary ‘‘sink’’ state. The space \mathcal{X} is Polish, compact and thus locally compact. We define two functions

$$\begin{cases} f_1(x_1, x_2, \xi, \eta) &= 0.5x_2(3x_1^2 + 2x_2^2 - 0.5) \\ &+ 0.6x_1\xi + 0.6x_2\eta, \\ f_2(x_1, x_2, \xi, \eta) &= 0.9x_1(2x_1^2 + 4x_1x_2 + 3x_2^2 - 0.5) \\ &+ 0.6x_1\xi + 0.6x_2\eta, \end{cases} \quad (12)$$

and construct the Markov process X by the following recurrence relations. First, ∂ is an absorbing state; second, if $X_k \neq \partial$, then $X_k = (X_k^1, X_k^2) \in K$ and we set

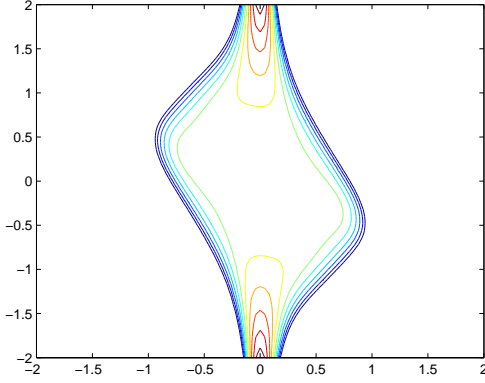
$$\begin{cases} X_{k+1}^1 &= f_1(X_k^1, X_k^2, \xi_k, \eta_k) \\ X_{k+1}^2 &= f_2(X_k^1, X_k^2, \xi_k, \eta_k) \end{cases} \quad (13)$$

if both right-hand sides are in $[-2, 2]$, whereas $X_{k+1} = \partial$ otherwise. Here $(\eta_k)_{k \geq 0}$ and $(\xi_k)_{k \geq 0}$ are sequences of iid standard normal random variables. In practice, the process X evolves according to the update law (13) within the set K , unless it is reset in the sink state ∂ whenever it leaves K .

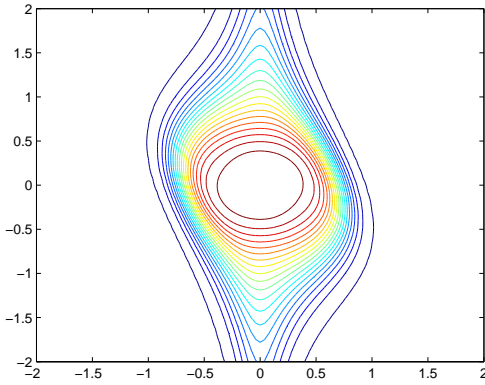
The goal is to study the stochastic attractivity of the origin, which is an absorbing set as per (12), and to find its ε -domains of attraction. We exploit Theorem 5, where we set $E = \partial$ and where clearly $c(\partial; \{0\}) = 0$. We are left with the solution of the problem for $x \in K$. Let us select $f(x) = \rho^2(x, 0)$: the pair $(f_{\leq 0.25}, f)$ is a stabilizing pair for the origin, moreover $r = 0.25$ as per Theorem 5. Since $c(\partial; \{0\}) = 0$, we obtain $|c(x; \{0\}) - w(x; D_{\varepsilon}, f_{\leq r})| \leq \varepsilon$ as per (11).

Note that f is not a superharmonic function: let us set $K = K' \cup (K')^c$, where $K' = \{x \in K : \mathcal{P}f(x) \leq f(x)\}$ is the largest subset of K where f has a superharmonic behavior. Since f is a square of the distance to the origin, one can imagine that the process X has convergent behavior if $x \in K'$ and a divergent one in the complement. The function $\mathcal{P}f - f$ measures how fast does the function f decrease along the dynamics of the process X .

The results on Figure 1 show the similarity of the level sets for the functions $\mathcal{P}f(x) - f(x)$ and $c(x; \{0\})$ on K . Moreover, the outcome shows that $c(x; \{0\})$ vanishes almost immediately outside of the set K' , which justifies our intuition about the divergent behavior of the process X on $(K')^c$.



(a) Level sets of the function $\mathcal{P}f - f$, going from the value -1 until 1 (the furthest from the origin), with a step of 0.2 .



(b) Level sets of the function c , going from the value 0.05 until 1 (the closest to the origin), with a step of 0.05 .

Fig. 1. Level sets of function $\mathcal{P}f - f$ and of function c for the case study.

REFERENCES

- [1] A. Abate, J.-P. Katoen, J. Lygeros, and M. Prandini. Approximate model checking of stochastic hybrid systems. *European Journal of Control*, 16:624–641, December 2010.
- [2] C. Baier and J.-P. Katoen. *Principles of model checking*. The MIT Press, 2008.
- [3] R. Durrett. *Probability: Theory and Examples - Third Edition*. Duxbury Press, 2004.
- [4] Patrick Florchinger. Lyapunov-like techniques for stochastic stability. *SIAM Journal on Control and Optimization*, 33(4):1151–1169, 1995.
- [5] O. Hernández-Lerma and J. B. Lasserre. *Discrete-time Markov control processes*, volume 30 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 1996.
- [6] S.P. Meyn and R.L. Tweedie. *Markov chains and stochastic stability*. Springer Verlag, 1993.
- [7] L. Perko. *Differential equations and dynamical systems*, volume 7 of *Texts in Applied Mathematics*. Springer-Verlag, New York, third edition, 2001.
- [8] F. Ramponi, D. Chatterjee, S. Summers, and J. Lygeros. On the connections between PCTL and dynamic programming. In *Proceedings of the 13th ACM international conference on Hybrid Systems: Computation and Control*, pages 253–262, 2010.
- [9] D. Revuz. *Markov chains*, volume 11 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1984.
- [10] A. Shiryaev, B. Rozovskii, and G. Grimmett. *Optimal stopping rules*. Springer-Verlag Berlin Heidelberg, 2008.
- [11] I. Tkachev and A. Abate. On infinite-horizon probabilistic properties and stochastic bisimulation functions. In *Proceedings of the 50th*

IEEE Conference on Decision and Control and European Control Conference, pages 526–531, Orlando, FL, December 2011.

- [12] I. Tkachev and A. Abate. Regularization of Bellman Equations for Infinite-Horizon Probabilistic Properties. In *Proceedings of the 15th International Conference on Hybrid Systems: Computation and Control*, Beijing, China, April 2012.

V. APPENDIX: PROOFS

Proof: [Proof of Theorem 1] First of all, $\{X_n \in A\} = \bigcup_{n=0}^{\infty} \left\{ \prod_{k=n}^{\infty} 1_A(X_k) = 1 \right\}$ and $\{\tau_{A^c} = \infty\} = \left\{ \prod_{k=0}^{\infty} 1_A(X_k) = 1 \right\}$. Thus

$$\begin{aligned} h(x; A) &= P_x\{X_n \in A\} = P_x\left(\bigcup_{n=0}^{\infty} \left\{ \prod_{k=n}^{\infty} 1_A(X_k) = 1 \right\}\right) \\ &= \lim_n P_x\left\{ \prod_{k=n}^{\infty} 1_A(X_k) = 1 \right\} \\ &= \lim_n \mathcal{P}^n \left(P_x \left\{ \prod_{k=0}^{\infty} 1_A(X_k) = 1 \right\} \right) = \lim_n \mathcal{P}^n u(x; A) \end{aligned}$$

since $u(x; A) = P_x\{\tau_{A^c} = \infty\}$.

The first part of the statement is proved, so for the next part we mention that for all $x \in \mathcal{X}$, $A \in \mathcal{B}$

$$\begin{aligned} \mathcal{P}h(x; A) &= \int_{\mathcal{X}} \lim_n \mathcal{P}^n u(y; A) T(dy|x) \\ &= \lim_n \int_{\mathcal{X}} \mathcal{P}^n u(y; A) T(dy|x) \\ &= \lim_n \mathcal{P}^{n+1} u(x; A) = h(x; A) \end{aligned}$$

where we used dominated convergence theorem [3] to interchange the limit and the integral operators.

Let f be a harmonic function such that $f \geq u$. The operator \mathcal{P} is clearly monotone, i.e. if $g' \leq g''$ then $\mathcal{P}g' \leq \mathcal{P}g''$, so $f = \mathcal{P}^n f \geq \mathcal{P}^n u$ for all $n \in \mathbb{N}_0$. The limit $n \rightarrow \infty$ yields: $f \geq h$, so h is the least harmonic majorant of u . Suppose that

$$\varepsilon = \inf_{x \in \mathcal{X}} |h(x; A) - u(x; A)| > 0$$

then $h(x; A) \geq u(x; A) + \varepsilon$ and hence

$$f(x) := h(x; A) - \varepsilon \geq u(x; A)$$

being in addition a harmonic function. But it contradicts with the fact that $f \geq h$, so (6) holds.

With regards to the function c , we have

$$\begin{aligned} \mathcal{P}c(x; A) &= \mathcal{P} \lim_{m \rightarrow \infty} h(x; \mathcal{O}_{1/m}(A)) \\ &= \lim_{m \rightarrow \infty} \mathcal{P}h(x; \mathcal{O}_{1/m}(A)) = c(x; A) \end{aligned} \quad (14)$$

since functions h are harmonic. \square

Proof: [Proof of Lemma 1] Note that $T(A|x) = \mathcal{P}1_A(x) = 1$ for all $x \in A$, so it is sufficient to prove that $T(A|x)$ is continuous on \bar{A} . Define

$$g(x) = \min\{\rho(x, \bar{A}), 1\}$$

so $g \in \mathcal{C}$. Put $f_n(x) = (1 - g(x))^n$ then $f_n(x) \in [0, 1]$ for all $x \in \mathcal{X}$, $n \geq 0$ and $f_n \in \mathcal{C}$. Moreover, $f_n(x) = 1$ for $x \in \bar{A}$ and $f_n \downarrow 1_{\bar{A}}$ pointwise.

Now, since $f_n(x) \geq 1_A(x)$ then $\mathcal{P}f_n(x) \geq \mathcal{P}1_A(x) = 1$ for all $x \in A$. Since X is weakly continuous, a function $\mathcal{P}f_n \in \mathcal{C}$ and hence $\mathcal{P}f_n(x) \geq 1$ for all $x \in \bar{A}$. By monotone convergence theorem we obtain $\mathcal{P}f_n(x) \downarrow \mathcal{P}1_{\bar{A}}(x) = T(\bar{A}|x)$, so $T(\bar{A}|x) = 1$ for all $x \in \bar{A}$, which proves the statement of the lemma. \square

Proof: [Proof of Lemma 2] If m_f is empty, it is absorbing by the definition. Hence, we assume that there is at least one $x \in m_f$. We have

$$0 \leq f(x) - \mathcal{P}f(x) = \int_{\mathcal{X}} (f_* - f(y))T(dy|x) \leq 0$$

where the left inequality holds since f is superharmonic and the right one holds because $f_* \leq f(y)$ for all $y \in \mathcal{X}$. As a result, $T(\{y \in \mathcal{X} : f(y) = f_*\}|x) = 1$. Or in other words, for any $x \in m_f$ we have $T(m_f|x) = 1$. \square

Proof: [Proof of Theorem 2] If u is harmonic then m_u is absorbing by Lemma 2. On the other hand, let m_u be absorbing. For $x \in A$ we have $u(x;A) = \mathcal{P}u(x;A)$. Now let $x \in A^c$, then $x \in m_u$ and so

$$\mathcal{P}u(x;A) = \int_{\mathcal{X}} u(y;A)T(dy|x) = \int_{m_u} u(y;A)T(dy|x) = 0$$

so $0 = u(x;A) = \mathcal{P}u(x;A)$ and hence u is harmonic. So u is the least harmonic majorant of itself and hence $h = u$. \square

Proof: [Proof of Lemma 4] The function $u_0(x;A) = 1_A(x)$ is u.s.c. since it is an indicator function of a closed set. For weakly continuous X it holds that $\mathcal{P}f$ is u.s.c. whenever f is u.s.c. [5]. As a result u_n is u.s.c. for all $n \geq 0$.

Since the convergence $u_n \rightarrow u$ is pointwise non-increasing, $u(x;A) = \inf_{n \geq 0} u_n(x;A)$ and so u is u.s.c. Finally, by (9) we have that l.a.s.(A) is a closed set and since it is a subset of a compact A , it is itself a compact. \square

Proof: [Proof of Theorem 3] 1) if A is trivial then $u(x;A) \equiv 0$ is harmonic and hence $h = u = 0$. On the other hand, since $h(x;A) \geq u(x;A) \geq 0$ for all $x \in \mathcal{X}$, the fact that $h = 0$ implies $u = 0$.

2) if X is weakly continuous and A is compact, A is trivial if and only if it is simple, by Lemma 3. Hence in that case 2) easily follows from 1).

3) clearly, $\{X_n \in A\} \subseteq \{X_n \rightarrow A\}$, thus $h(x;A) \leq c(x;A)$ for all $x \in \mathcal{X}$. If $c(x;A) \equiv 0$ then $h(x;A) \equiv 0$ and by 2) the set A is simple.

In the other direction, let us assume that A is a compact and simple set. Denote $h_m(x) = h(x; \mathcal{O}_{1/m}(A))$ so

$$c(x;A) = \lim_m h_m(x)$$

for all $x \in \mathcal{X}$, and the idea is to show that $h_m = 0$ for m big enough, provided A is simple.

First, we show that there is $M > 0$ such that $\overline{\mathcal{O}_{1/M}(A)}$ is a compact set. Since \mathcal{X} is locally compact, each $x \in A$ has a compact neighborhood, so put $\varepsilon : A \rightarrow \mathbb{R}$ be such that $\mathcal{O}_{\varepsilon(x)}(x)$ is contained in some compact set. We have that $\{\mathcal{O}_{\varepsilon(x)}(x) : x \in A\}$ is an open cover of A , thus there is an open subcover $\{\mathcal{O}_{\varepsilon_k}(x_k)\}_{k \leq n}$ with $\varepsilon_k := \varepsilon(x_k)$.

Now, $\overline{\mathcal{O}_{\varepsilon_k}(x_k)}$ is compact for each $k \leq n$, as closed subsets of compact sets. Hence the set

$$C = \bigcup_{k \leq n} \overline{\mathcal{O}_{\varepsilon_k}(x_k)}$$

is compact, and $\overline{\mathcal{O}_{\varepsilon'}(A)} \subseteq C$ where $\varepsilon' = \min_{k \leq n} \varepsilon_k$. We only need now to pick up $M > \frac{1}{\varepsilon'}$, then $\overline{\mathcal{O}_{1/M}(A)}$ is a compact as a closed subset of a compact C .

For $m \geq M$ we denote $B_m = \overline{\mathcal{O}_{1/m}(A)}$ and $B'_m = \text{l.a.s.}(B_m)$. Then we have $\mathcal{O}_{1/(m+1)}(A) \subseteq B_{m+1} \subseteq \mathcal{O}_{1/m}(A)$ and so $h_m(x) \leq h(x;B_m)$ for all $x \in \mathcal{X}$.

Second, let us show that there is $M' \geq M$ such that B_m is simple for all $m \geq M'$. Suppose contrary: namely that $B'_m \neq \emptyset$ for all $m \geq M$. By Lemma (4), B'_m are compact sets so $B' := \bigcap_{m=M}^{\infty} B'_m \subseteq A$ is not empty. For any $x \in B'$ it holds that $x \in B'_m$ for all $m \geq M$, thus $T(B'_m|x) = 1$. We have

$$T(B'|x) = T\left(\bigcap_{m=M}^{\infty} B'_m \middle| x\right) = \lim_m T(B'_m|x) = 1$$

which means that the set B' is a non-empty absorbing subset of A and contradicts with the simplicity of A .

Finally, since for some $M' \geq M$ sets $B_m, m \geq M'$ are compact and simple, $h(x;B_m) = 0$ by 2). On the other hand, $0 = h(x;B_m) \geq h_m(x)$ and so $c(x;A) = \lim_m h_m(x) = 0$. \square

Proof: [Proof of Corollary 1] 1) For all $m \geq 1/\delta$ we have $h(x; \mathcal{O}_{1/m}(A)) = 0$ and hence $c(x;A) = 0$ for all $x \in \mathcal{X}$. As a result, (3) does not hold for A .

2) Follows directly from the statement 3), Theorem 3. \square

Proof: [Proof of Lemma 5] Let us introduce an excessive operator on \mathbb{B} as $\mathcal{Q}f(x) = \max\{f(x), \mathcal{P}f(x)\}$. From [10, Lemma 6, Lemma 8, p. 43] it follows that if $f \in \mathbb{B}$ and $g(x) = \lim_{n \rightarrow \infty} \mathcal{Q}^n g(x)$ then

$$P_x \left\{ \lim_n g(X_n) = \limsup_n f(X_n) \right\} = 1. \quad (16)$$

We take $f = 1_{A^c}$ so by [11, Theorem 1] we obtain:

$$P_x \left\{ \lim_n v(X_n; A^c) = \limsup_n 1_{A^c}(X_n) \right\} = 1.$$

Now we use $u(x;A) = 1 - v(x;A^c)$ to obtain the first equality. Furthermore, $h(x;A) = \lim_n \mathcal{P}^n u(x;A) = \lim_n \mathcal{Q}^n u(x;A)$ since u is subharmonic, $\mathcal{P}^n u$ are subharmonic for all $n \geq 0$. Hence, $\mathcal{Q} \mathcal{P}^n u = \mathcal{P}^{n+1} u$ for all $n \geq 0$ and we obtain the second equality from (16).

For any x , P_x -a.s. we have that $\lim_n u(X_n;A) = 1$ and $\lim_n h(X_n;A) = 1$ if and only if $\liminf_n 1_A(X_n) = 1$ and the latter statement is $\{X_n \in A\}$. Recall of the definition of h finishes the proof. \square

Proof: [Proof of Lemma 6] Suppose, that $\lim_n u(x_n;A) = 1$ and $x_n \not\rightarrow A$, i.e. $\limsup_n \rho(x_n, \text{l.a.s.}(A)) > 0$. Since $u(x;A) = 0$ for all $x \in A^c$, we have $x_n \in A$. Exploiting the compactness of A we obtain that there is a convergent subsequence $x'_n \rightarrow x' \in A$ such that $\lim_n \rho(x'_n, \text{l.a.s.}(A)) > 0$ but $\lim_n u(x'_n;A) = 1$.

In Lemma 4 we proved that the function $u(x;A)$ is u.s.c. so $u(x';A) = 1$ and hence $x' \in \text{l.a.s.}(A)$ but $\rho(x', \text{l.a.s.}(A)) > 0$ which leads us to a contradiction. \square

Proof: [Proof of Theorem 4] Let $A \in \mathcal{B}$ be a stable absorbing set and U_A be as in Definition 5. Since U_A is compact and $A = \text{l.a.s.}(U_A)$ by Lemma 4 we obtain that A is compact too. Moreover, since U_A is a compact neighborhood of A , each $x \in A$ has a compact neighborhood and similarly to the proof of Theorem 3 we pick up M such that $\overline{\mathcal{O}_{1/M}(A)}$ is a compact set. Thus for all $m \geq M$ the set $\overline{\mathcal{O}_{1/m}(A)}$ is compact.

Let us consider any $m \geq M$. By Lemma 6 we obtain that $u(x_n; \overline{\mathcal{O}_{1/m}(A)}) \rightarrow 1$ implies $x_n \rightarrow A$. Let us show that the reverse statement is also true. Since \mathcal{X} is a metric space, it is equivalent to show that for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $\rho(x,A) < \delta(\varepsilon)$ implies

$$u(x; \overline{\mathcal{O}_{1/m}(A)}) \geq 1 - \varepsilon. \quad (17)$$

We fix $\varepsilon > 0$ and denote $f(x) := 1 - u(x; U_A)$. Since f is a superharmonic function with a range in $[0, 1]$, we obtain that the process $(f(X_n))_{n \geq 0}$ is a non-negative \mathbb{P}_x -supermartingale for all $x \in \mathcal{X}$ [10]. Hence, the Doob's inequality [3] holds:

$$\mathbb{P}_x \left\{ \sup_{n \in \mathbb{N}_0} f(X_n) > r \right\} \leq \frac{1}{r} f(x) \quad (18)$$

for all $x \in \mathcal{X}$ and $r > 0$. In the level sets notation, the inequality (18) takes the form $u(x; f_{\leq r}) \geq 1 - \frac{1}{r} f(x)$.

Let us show that the stability of A implies an existence of $r > 0$ such that $f_{\leq r} \subseteq \overline{\mathcal{O}_{1/m}(A)}$. Indeed, if it would not be true, then we were able to pick up a sequence $x_k \notin \overline{\mathcal{O}_{1/m}(A)}$ such that $f(x_k) \leq 1/k$. Clearly,

$$\lim_k u(x_k; U_A) = 1 - \lim_k f(x_k) = 1$$

but $d(x_k, A) \geq \frac{1}{m}$ which contradicts with Lemma 6.

It follows from the existence of r that

$$u(x; \overline{\mathcal{O}_{1/m}(A)}) \geq u(x; f_{\leq r}) \geq 1 - \frac{1}{r} f(x).$$

Leveraging the stability of A again, we obtain that there is $\delta > 0$ such that $f(x) \leq r\varepsilon$ for all $x \in \mathcal{O}_\delta(A)$ and hence for all such x the inequality (17) holds.

As a result, for any $m \geq M$ we obtain that $u(x_n; \overline{\mathcal{O}_{1/m}(A)}) \rightarrow 1$ if and only if $x_n \rightarrow A$. Since

$$h(x; \overline{\mathcal{O}_{1/m}(A)}) = \mathbb{P}_x \left\{ u(X_n; \overline{\mathcal{O}_{1/m}(A)}) \rightarrow 1 \right\} = \mathbb{P}_x \{ X_n \rightarrow A \}$$

we obtain for all $m', m'' \geq M$ and $x \in \mathcal{X}$:

$$h(x; \overline{\mathcal{O}_{1/m'}(A)}) = h(x; \overline{\mathcal{O}_{1/m''}(A)}).$$

Furthermore, since

$$h(x; \overline{\mathcal{O}_{1/m}(A)}) \geq h(x; \mathcal{O}_{1/m}(A)) \geq h(x; \overline{\mathcal{O}_{1/(m+1)}(A)}).$$

for all $m \geq M$ and $x \in \mathcal{X}$, we obtain

$$c(x; A) = \lim_m h(x; \overline{\mathcal{O}_{1/m}(A)}) = h(x; \overline{\mathcal{O}_{1/m'}(A)})$$

for all $m' \geq M$ and $x \in \mathcal{X}$. Moreover, $u(x_n; U_A) \rightarrow 1$ if and only if $x_n \rightarrow A$, so $c(x; A) = h(x; U_A)$.

To finish the proof of the theorem we observe that $h(x; \overline{\mathcal{O}_{1/m'}(A)}) \geq u(x; \overline{\mathcal{O}_{1/m'}(A)})$ and for the latter we proved that it converges to 1 on any sequence which converges to A . As a result, A is stochastically attractive. \square

Proof: [Proof of Lemma 7] Suppose that such f and U_A exist. From the local version of Doob's inequality [11, Theorem 6] it follows:

$$u(x; f_{\leq r}) \geq 1 - \frac{1}{r} f(x)$$

for all $x \in \mathcal{X}$. As a result, $\lim_{x \rightarrow A} u(x; U_A) = 1$ since $u(x; U_A) \geq u(x; f_{\leq r})$.

Now, let A be a stable absorbing set. Then there is a compact neighborhood U_A of A such that $A = \text{l.a.s.}(U_A)$. Clearly $(U_A, 1 - u(x; U_A))$ is a stabilizing pair. \square

Proof: [Proof of Lemma 8] Suppose that $\inf_{x \in \mathcal{X}} h(x; A) >$

0. Clearly, $\mathbb{P}_x \left\{ \lim_{n \rightarrow \infty} h(X_n; A) = 0 \right\} = 0$ and hence by Lemma 5 we obtain $h(x; A) \equiv 1$. Applying the same argument to the case $\sup h(x; A) < 0$ we obtain $h(x; A) \equiv 0$.

Proof: [Proof of Theorem 5] From the proof of Theorem 4 we obtain that $c(x; A) = h(x; U_A)$. Moreover, since $\lim_{x \rightarrow A} f(x) = 0$ we obtain that there is $m \in \mathbb{N}$ such that $\overline{\mathcal{O}_{1/m}(A)} \subseteq f_{\leq r}$ and since

$$\overline{\mathcal{O}_{1/(m+1)}(A)} \subseteq f_{\leq r} \subseteq U_A$$

we obtain that $c(x; A) = h(x; f_{\leq r})$.

Let us denote $\tau^\varepsilon = \inf \{ n \geq 0 : X_n \in f_{\leq \varepsilon} \cup E \}$. Then:

$$h(x) = \mathbb{P}_x \{ X_n \in f_{\leq r}, \tau^\varepsilon = \infty \} + \mathbb{P}_x \{ X_n \in f_{\leq r}, \tau^\varepsilon < \infty \}.$$

For the first term we have:

$$\mathbb{P}_x \{ X_n \in f_{\leq r}, \tau^\varepsilon = \infty \} = \mathbb{P}_x \{ X_n \in f_{\leq r} \setminus f_{\leq \varepsilon}, \tau^\varepsilon = \infty \} = 0.$$

To prove it we show that $f_{\leq r} \setminus f_{\leq \varepsilon}$ is trivial. Since $\lim_{x \rightarrow A} f(x) = 0$ there is $m \in \mathbb{N}$ such that $\overline{\mathcal{O}_{1/m}(A)} \subseteq f_{\leq \varepsilon}$ so $U_A \setminus \overline{\mathcal{O}_{1/m}(A)}$ is compact and simple, hence trivial. Hence $f_{\leq r} \setminus f_{\leq \varepsilon} \subseteq U_A \setminus \overline{\mathcal{O}_{1/m}(A)}$ is trivial as well. For the second term:

$$\mathbb{P}_x \{ X_n \in f_{\leq r}, \tau^\varepsilon < \infty \} = \int_E h(y) \nu(x, dy) + \int_{f_{\leq \varepsilon}} h(y) \nu(x, dy)$$

where $\nu(x, B) = \mathbb{P}_x \{ X_{\tau^\varepsilon} \in B, \tau^\varepsilon < \infty \}$. We obtain:

$$\inf_{y \in f_{\leq \varepsilon}} h(y) w(x; D_{\varepsilon}, f_{\leq \varepsilon}) \leq h(x) \leq \sup_{y \in E} h(y) + w(x; D_{\varepsilon}, f_{\leq \varepsilon})$$

since $\inf_{y \in E} h(y) = 0$. Now, $h(x) \geq u(x; f_{\leq r}) \geq 1 - \frac{1}{r} f(x)$ and hence for all $x \in f_{\leq \varepsilon}$ we have $h(x) \geq 1 - \varepsilon$, which finishes the proof. \square