Finite Abstractions of Networked Control Systems

Majid Zamani, Manuel Mazo Jr, and Alessandro Abate

Abstract— **In networked control systems (NCS), the communication between sensors, controllers, and actuators is supported by a shared communication channel that is subject to variable communication delays, limited bandwidth, packet losses, quantization errors, and other practical non-idealities. This work investigates the problem of constructively deriving symbolic models of NCS by simultaneously considering the mentioned network non-idealities. By employing the obtained symbolic models, one can completely automate the design of controllers enforcing rich logical specifications, e.g. formulae in linear temporal logic, over NCS.**

I. Introduction

Networked control systems (NCS) are becoming a ubiquitous element of many modern technologies because they provide a spatially distributed framework for control systems encompassing network elements, increasing on the one hand architecture flexibility and reducing on the other installation and maintenance costs. On the minus side, their analysis and synthesis require more involved studies because of new challenges introduced by non-idealities related to the network element, including quantization errors, limited bandwidth, packet dropouts, time-varying sampling/transmission intervals, time-varying communication delays, and communication constraints (scheduling protocols).

Recently, there have been numerous studies focusing mostly on the stability properties of NCS [1], [2], [3], [4], [5], [6], [7] and taking into account a subset of the aforementioned non-idealities. However, there are no significant results in the literature dealing with more sophisticated objectives, such as fully automated verification or synthesis of NCS, nor for more complex properties or specifications. Examples of relevant complex specifications include properties expressed as formulae in linear temporal logic (LTL) or as automata on infinite strings, which are by and large not amenable to be dealt with the existing techniques for NCS.

A promising approach to tackle these complex properties is the use of *symbolic models* [8]. Symbolic models are finite abstractions of the concrete models, where each abstract state corresponds to a collection of concrete states. Symbolic models allow us to leverage algorithmic machinery for controller synthesis of discrete systems [9] to automatically synthesize hybrid controllers enforcing complex specifications on the original concrete system [8]. The only results available in the literature on the construction of finite abstractions for NCS are the ones in [10], [11]. The results in [10], [11] consider the following network non-idealities simultaneously: quantization errors, limited bandwidth, packet dropouts, and time-varying communication delays. Nevertheless, the proposed results in [10], [11] present some limitations: they are restricted to grid-based abstractions, with evident scalability limitations; they only deal with static controllers, thus ruling out more general control architectures; the specifications need to be expressed in terms of some types of nondeterministic automata, which can be reductive; and they require to reformulate a given specification in an extended state-space fashion, in order to be semantically applicable to the proposed finite abstractions, which can be expensive. Furthermore, the implementations in [10], [11] assume that the sensors transmit messages only whenever a new controller update is received by the actuators, which in practice imposes the co-location of sensors and actuators.

In this paper, we provide a synthesis technique for constructing finite abstractions of NCS exclusively using existing finite abstractions of the plant. Therefore, one can use existing results in the literature to construct finite abstractions of the plant, such as the grid-based ones in [12], [13], or one partially based on grids [14] and without requiring state-space discretization, or specification-based approaches [15], and then construct the finite abstractions of the NCS from those. Given any type of finite abstraction for the plant, one can always leverage the results provided in this paper to provide a finite abstraction of the NCS. We consider the following network non-idealities simultaneously: quantization errors, limited bandwidth, packet dropouts, and time-varying communication delays. Moreover, in our proposal the transmission of updates from sensors and controllers are periodic, removing any location restriction of sensors, actuators, and/or controllers. This requires to explicitly take into account the possibility of out-of-order packet arrivals and message rejection, which means that older data will be neglected if more recent one is available. Furthermore, our proposed results do not restrict us to only work with static controllers, which in general are not sufficient to satisfy every LTL formula, see e.g. [16], [17]. Hence, one can also employ the results proposed here to study larger classes of temporal logic specifications without requiring any additional reformulation.

II. Control Systems & Stability/Completeness Notions

A. Notations

The symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}_0^+$ denote the set of natural, nonnegative integer, real, positive, and nonnegative

M. Zamani is with the Department of Design Engineering, Delft University of Technology, 2628 CE, Delft, The Netherlands. Email: m.zamani@tudelft.nl. URL: http://staff.tudelft.nl/en/m.zamani

M. Mazo Jr is with the Delft Center for Systems and Control, Delft University of Technology, 2628 CD, Delft, The Netherlands. Email: m.mazo@tudelft.nl. URL: http://www.dcsc.tudelft.nl/∼mmazo

A. Abate is with the Department of Computer Science, University of Oxford, OX1 3QD, Oxford, United Kingdom. Email: alessandro.abate@cs.ox.ac.uk. URL: http://www.cs.ox.ac.uk/people/alessandro.abate

real numbers, respectively. Given a set A, define $A^{n+1} = A \times A^n$ for any $n \in \mathbb{N}$. Given a vector $x \in \mathbb{R}^n$, we denote by x_i the i–th element of x, and by $\|x\|$ the infinity norm of x, namely, $\|x\| = \max\{|x_1|, |x_2|, ..., |x_n|\}$, where $|x_i|$ denotes the absolute value of x_i. Given an interval $[a, b] \subseteq \mathbb{R}$ with $a \leq b$, we denote by $[a; b]$ the set $[a, b] \cap \mathbb{N}$. We denote by $[\mathbb{R}^n]_\eta = \{a \in \mathbb{R}^n \mid a_i = k_i\eta,\ k_i \in \mathbb{Z},\ i = 1, \ldots, n\}$.

Given a measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty := (\text{ess})\sup\{\|f(t)\|, t \geq 0\}$. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed nonzero r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. We identify a relation $R \subseteq A \times B$ with the map $R : A \to 2^B$ defined by $b \in R(a)$ iff $(a, b) \in R$. Given a relation $R \subseteq A \times B$, R^{-1} denotes the inverse relation defined by $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$. When R is an equivalence relation[1] on a set A, we denote by $[a]$ the equivalence class of $a \in A$, by A/R the set of all equivalence classes, and by $\pi_R : A \to A/R$ the natural projection map taking a point $a \in A$ to its equivalence class $\pi(a) = [a] \in A/R$.

B. Control systems

The class of control systems that we consider in this paper is formalized in the following definition.

Definition 2.1: A *control system* is a tuple $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$, where:

- $\mathbb{R}^n$ is the state space;
- $\mathsf{U} \subseteq \mathbb{R}^m$ is the compact input set;
- $\mathcal{U}$ is a subset of the set of all measurable functions of time from intervals of the form $]a, b[\subseteq \mathbb{R}$ to U with $a < 0$ and $b > 0$;
- $f : \mathbb{R}^n \times \mathsf{U} \to \mathbb{R}^n$ is a continuous map which is locally Lipschitz continuous with respect to its first argument.

A curve $\xi :]a, b[\to \mathbb{R}^n$ is said to be a *trajectory* of Σ if there exists $\upsilon \in \mathcal{U}$ satisfying:

$$\dot{\xi}(t) = f(\xi(t), \upsilon(t)),$$

for almost all $t \in]a, b[$. Similarly, we refer to trajectories $\xi : [0, t] \to \mathbb{R}^n$ defined on closed domains $[0, t]$, $t \in \mathbb{R}^+$, with the understanding that there exists a trajectory $\xi' :]a, b[\to \mathbb{R}^n$ such that $\xi = \xi'|_{[0,t]}$ with $a < 0$ and $b > t$. We also write $\xi_{x\upsilon}(t)$ to denote the point reached at time t under the input υ from the initial condition $x = \xi_{x\upsilon}(0)$; the point $\xi_{x\upsilon}(t)$ is uniquely determined, since the assumptions on f ensure existence and uniqueness of trajectories [18].

A control system Σ is said to be forward complete if every trajectory is defined on an interval of the form $]a, \infty[$. Sufficient and necessary conditions for a control system to be forward complete can be found in [19].

[1]An equivalence relation $R \subseteq X \times X$ is a binary relation on a set X if it is reflexive, symmetric, and transitive.

C. Stability & completeness notions

Some of the existing results in the literature, briefly recalled in this paper, require certain stability and completeness properties on Σ. First, we recall a stability notion introduced in [20].

Definition 2.2: A control system Σ is incrementally input-to-state stable (δ-ISS) if it is forward complete and there exist a $\mathcal{KL}$ function β and a $\mathcal{K}_\infty$ function γ such that for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$, and any $\upsilon, \upsilon' \in \mathcal{U}$, the following condition is satisfied:

$$\|\xi_{x\upsilon}(t) - \xi_{x'\upsilon'}(t)\| \leq \beta(\|x - x'\|, t) + \gamma(\|\upsilon - \upsilon'\|_\infty). \tag{II.1}$$

We recall a completeness property, introduced in [13], which can be satisfied by more general classes of (even unstable) control systems.

Definition 2.3: A control system Σ is incrementally forward complete (δ-FC) if it is forward complete and there exist continuous functions $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $\gamma : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that for each fixed s, functions $\beta(r, s)$ and $\gamma(r, s)$ belong to class $\mathcal{K}_\infty$ with respect to r, and for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$, and any $\upsilon, \upsilon' \in \mathcal{U}$, the following condition is satisfied:

$$\|\xi_{x\upsilon}(t) - \xi_{x'\upsilon'}(t)\| \leq \beta(\|x - x'\|, t) + \gamma(\|\upsilon - \upsilon'\|_\infty, t). \tag{II.2}$$

The interested readers can find a characterization (resp. description) of δ-ISS (resp. δ-FC) in terms of the existence of so-called *incremental Lyapunov functions* in [20] (resp. [13]).

III. Systems & Approximate Equivalence Notions

We now recall the notion of systems, introduced in [8], that we use later to describe NCS as well as their finite abstractions.

Definition 3.1: A system is a tuple $S = (X, X_0, U, \longrightarrow, Y, H)$ consisting of: a (possibly infinite) set of states X; a (possibly infinite) set of initial states $X_0 \subseteq X$; a (possibly infinite) set of inputs U; a transition relation $\longrightarrow \subseteq X \times U \times X$; a set of outputs Y; and an output map $H : X \to Y$.

A transition $(x, u, x') \in \longrightarrow$ is also denoted by $x \xrightarrow{u} x'$. If $x \xrightarrow{u} x'$, state x' is called a u-successor of state x. We denote by $\mathbf{Post}_u(x)$ the set of all u-successors of a state x and by $U(x)$ the set of inputs $u \in U$ for which $\mathbf{Post}_u(x)$ is nonempty.

System S is said to be:

- *metric*, if the output set Y is equipped with a metric $\mathbf{d} : Y \times Y \to \mathbb{R}_0^+$;
- *finite* (or *symbolic*), if X and U are finite sets;
- *countable*, if X and U are countable sets;
- *deterministic*, if for any state $x \in X$ and any input $u \in U$, $|\mathbf{Post}_u(x)| \leq 1$;
- *nondeterministic*, if there exist a state $x \in X$ and an input $u \in U$ such that $|\mathbf{Post}_u(x)| > 1$;

Given a system $S = (X, X_0, U, \longrightarrow, Y, H)$, we denote by $|S|$ the size of S, defined as $|S| := |\longrightarrow|$, which is equal to the total number of transitions in S. Note that it is more reasonable to consider $|\longrightarrow|$ as the size of S rather

than $|X|$, as it is the transitions of S that are required to be stored rather than just the states of S.

We also recall the notions of (alternating) approximate (bi)simulation relation, introduced in [21], [22], which are useful to relate properties of NCS to those of their finite abstractions. First we recall the notions of approximate (bi)simulation relation, introduced in [21].

Definition 3.2: Let $S_a = (X_a, X_{a0}, U_a, \xrightarrow[a]{}, Y_a, H_a)$ and $S_b = (X_b, X_{b0}, U_b, \xrightarrow[b]{}, Y_b, H_b)$ be metric systems with the same output sets $Y_a = Y_b$ and metric $\mathbf{d}$. For $\varepsilon \in \mathbb{R}_0^+$, a relation $R \subseteq X_a \times X_b$ is said to be an ε-approximate simulation relation from S_a to S_b if the following three conditions are satisfied:

(i) for every $x_{a0} \in X_{a0}$, there exists $x_{b0} \in X_{b0}$ with $(x_{a0}, x_{b0}) \in R$;

(ii) for every $(x_a, x_b) \in R$, we have $\mathbf{d}(H_a(x_a), H_b(x_b)) \leq \varepsilon$;

(iii) for every $(x_a, x_b) \in R$, the existence of $x_a \xrightarrow[a]{u_a} x_a'$ in S_a implies the existence of $x_b \xrightarrow[b]{u_b} x_b'$ in S_b satisfying $(x_a', x_b') \in R$.

A relation $R \subseteq X_a \times X_b$ is said to be an ε-approximate bisimulation relation between S_a and S_b if R is an ε-approximate simulation relation from S_a to S_b and R^{-1} is an ε-approximate simulation relation from S_b to S_a.

System S_a is ε-approximately simulated by S_b, or S_b ε-approximately simulates S_a, denoted by $S_a \preceq^{\varepsilon}_{\mathcal{S}} S_b$, if there exists an ε-approximate simulation relation from S_a to S_b. System S_a is ε-approximate bisimilar to S_b, denoted by $S_a \cong^{\varepsilon}_{\mathcal{S}} S_b$, if there exists an ε-approximate bisimulation relation between S_a and S_b.

As explained in [22], for nondeterministic systems one needs to consider relationships that explicitly capture the adversarial nature of nondeterminism. Furthermore, these types of relations become crucial to enable the refinement of symbolic controllers [8].

Definition 3.3: Let $S_a = (X_a, X_{a0}, U_a, \xrightarrow[a]{}, Y_a, H_a)$ and $S_b = (X_b, X_{b0}, U_b, \xrightarrow[b]{}, Y_b, H_b)$ be metric systems with the same output sets $Y_a = Y_b$ and metric $\mathbf{d}$. For $\varepsilon \in \mathbb{R}_0^+$, a relation $R \subseteq X_a \times X_b$ is said to be an alternating ε-approximate simulation relation from S_a to S_b if conditions (i) and (ii) in Definition 3.2, as well as the following condition, are satisfied:

(iii) for every $(x_a, x_b) \in R$ and for every $u_a \in U_a(x_a)$ there exists some $u_b \in U_b(x_b)$ such that for every $x_b' \in \mathbf{Post}_{u_b}(x_b)$ there exists $x_a' \in \mathbf{Post}_{u_a}(x_a)$ satisfying $(x_a', x_b') \in R$.

A relation $R \subseteq X_a \times X_b$ is said to be an alternating ε-approximate bisimulation relation between S_a and S_b if R is an alternating ε-approximate simulation relation from S_a to S_b and R^{-1} is an alternating ε-approximate simulation relation from S_b to S_a.

System S_a is alternatingly ε-approximately simulated by S_b, or S_b alternatingly ε-approximately simulates S_a, denoted by $S_a \preceq^{\varepsilon}_{\mathcal{AS}} S_b$, if there exists an alternating ε-approximate simulation relation from S_a to S_b. System S_a is alternatingly ε-approximately bisimilar to S_b, denoted by $S_a \cong^{\varepsilon}_{\mathcal{AS}} S_b$, if there exists an alternating ε-approximate bisimulation relation between S_a and S_b.

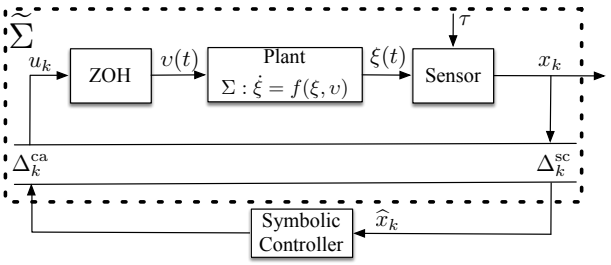

Fig. 1. Schematics of a networked control system $\widetilde{\Sigma}$.

Let us define a metric system $S_\tau(\Sigma) := (X_\tau, X_{\tau 0}, U_\tau, \xrightarrow[\tau]{}, Y_\tau, H_\tau)$, capturing all the information contained in the forward complete plant Σ at the sampling times:

- $X_\tau = \mathbb{R}^n$;
- $X_{\tau 0} = \mathbb{R}^n$;
- $U_\tau = \mathcal{U}$;
- $x_\tau \xrightarrow[\tau]{\upsilon_\tau} x_\tau'$ if there exists a trajectory $\xi_{x_\tau \upsilon_\tau} : [0, \tau] \to \mathbb{R}^n$ of Σ satisfying $\xi_{x_\tau \upsilon_\tau}(\tau) = x_\tau'$;
- $Y_\tau = \mathbb{R}^n / Q$ for some given equivalence relation $Q \subseteq X_\tau \times X_\tau$;
- $H_\tau = \pi_Q$.

Notice that the set of states and inputs of $S_\tau(\Sigma)$ are uncountable and that $S_\tau(\Sigma)$ is a deterministic system in the sense of Definition 3.1 since (cf. Subsection II-B) the trajectory of Σ is uniquely determined. We also assume that the output set Y_τ is equipped with a metric $\mathbf{d}_{Y_\tau} : Y_\tau \times Y_\tau \to \mathbb{R}_0^+$.

The interested readers can consult the results in [12] (resp. [13]) providing a finite abstraction $S_{\mathsf{q}}(\Sigma) := (X_{\mathsf{q}}, X_{\mathsf{q}0}, U_{\mathsf{q}}, \xrightarrow[\mathsf{q}]{}, Y_{\mathsf{q}}, H_{\mathsf{q}})$ for a δ-ISS (resp. δ-FC) control system Σ such that $S_{\mathsf{q}}(\Sigma) \cong^{\varepsilon}_{\mathcal{S}} S_\tau(\Sigma)$, or equivalently[2] $S_{\mathsf{q}}(\Sigma) \cong^{\varepsilon}_{\mathcal{AS}} S_\tau(\Sigma)$ (resp. $S_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_\tau(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} S_{\mathsf{q}}(\Sigma)$).

Remark 3.4: Consider the metric system $S_\tau(\Sigma)$ admitting an abstraction $S_{\mathsf{q}}(\Sigma)$. Since the plant Σ is forward complete, one can readily verify that given any state $x_\tau \in X_\tau$, there always exists a υ_τ-successor of x_τ for any $\upsilon_\tau \in U_\tau$. Hence, $U_\tau(x_\tau) = U_\tau$ for any $x_\tau \in X_\tau$. Therefore, without loss of generality, one can also assume that $U_{\mathsf{q}}(x_{\mathsf{q}}) = U_{\mathsf{q}}$ for any $x_{\mathsf{q}} \in X_{\mathsf{q}}$.

IV. Networked Control Systems

Consider a NCS $\widetilde{\Sigma}$ as depicted schematically in Figure 1 similar to the ones in [5, Figure 1] and [6, Figure 1]. The NCS $\widetilde{\Sigma}$ includes a plant Σ, a time-driven sampler, and an event-driven zero-order-hold (ZOH), which are described in more detail shortly. The forward complete plant $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ of a NCS is connected to a symbolic controller, explained in more detail in the next subsection, over a communication network that induces delays (Δ^{sc} and Δ^{ca}).

[2]Let us recall that the notions of alternating approximate (bi)simulation and of approximate (bi)simulation relation coincide when the systems involved are deterministic as per Definition 3.1.

The state measurements of the plant are sampled by a time-driven sampler at times $s_k := k\tau$, $k \in \mathbb{N}_0$, and we denote $x_k := \xi(s_k)$. The discrete-time control values computed by the symbolic controller at times s_k are denoted by u_k. Time-varying network-induced delays, i.e. the sensor-to-controller delay (Δ_k^{sc}) and the controller-to-actuator delay (Δ_k^{ca}), are included in the model. Moreover, packet dropouts in both channels of the network can be incorporated in the delays Δ_k^{sc} and Δ_k^{ca} as long as the maximum number of subsequent dropouts over the network is bounded [23]. Finally, the varying computation time needed to evaluate the symbolic controller is incorporated into Δ_k^{ca}. We assume that the time-varying delays are bounded and are integer multiples of the sampling time τ, i.e. $\Delta_k^{\text{sc}} := N_k^{\text{sc}}\tau$, where $N_k^{\text{sc}} \in [N_{\min}^{\text{sc}}; N_{\max}^{\text{sc}}]$, and $\Delta_k^{\text{ca}} := N_k^{\text{ca}}\tau$, where $N_k^{\text{ca}} \in [N_{\min}^{\text{ca}}; N_{\max}^{\text{ca}}]$, for some $N_{\min}^{\text{sc}}, N_{\max}^{\text{sc}}, N_{\min}^{\text{ca}}, N_{\max}^{\text{ca}} \in \mathbb{N}_0$. Under these assumptions, there is no difference in assuming that both the controller and the actuator act in an event-driven fashion (i.e. they respond instantaneously to newly arrived data) or in a time-driven fashion (i.e. they respond to newly arrived data at the sampling instants s_k). Furthermore, we model the occurrence of message rejection, i.e. the effect of older data being neglected because more recent data is available before the older data arrival, similar to the work in [5], [6]. The zero-order-hold (ZOH) function (see Figure 1) is placed before the plant Σ to transform the discrete-time control inputs u_k, $k \in \mathbb{N}_0$, to a continuous-time control input $\upsilon(t) = u_{k^*(t)}$, where $k^*(t) := \max\{k \in \mathbb{N}_0 \mid s_k + \Delta_k^{\text{ca}} \leq t\}$. As argued in [5], [6], within the sampling interval $[s_k, s_{k+1}[$ $\upsilon(t)$ can be explicitly described by

$$\upsilon(t) = u_{k+j_*-N_{\max}^{\text{ca}}}, \quad \text{for } t \in [s_k, s_{k+1}[\,, \tag{IV.1}$$

where $j_* \in [0; N_{\max}^{\text{ca}} - N_{\min}^{\text{ca}}]$ is defined as:

$$j_* = \widehat{f}\left(\widehat{N}_{N_{\min}^{\text{ca}}}, \ldots, \widehat{N}_{N_{\max}^{\text{ca}}}\right), \tag{IV.2}$$

where $\widehat{N}_k$, for $k \in [N_{\min}^{\text{ca}}; N_{\max}^{\text{ca}}]$, is the delay suffered by the control packet sent k samples before, namely $\widehat{N}_{N_{\max}^{\text{ca}}-i} = N_{k-N_{\max}^{\text{ca}}+i}^{\text{ca}}$ for any $i \in [0; N_{\max}^{\text{ca}} - N_{\min}^{\text{ca}}]$ and

$$\begin{aligned}&\widehat{f}\left(\widehat{N}_{N_{\min}^{\text{ca}}}, \ldots, \widehat{N}_{N_{\max}^{\text{ca}}}\right) \\ &\quad = \max\left\{\arg\min_j \widehat{g}\left(j, \widehat{N}_{N_{\min}^{\text{ca}}}, \ldots, \widehat{N}_{N_{\max}^{\text{ca}}}\right)\right\},\end{aligned} \tag{IV.3}$$

where

$$\begin{aligned}&\widehat{g}\Big(j, \widehat{N}_{N_{\min}^{\text{ca}}}, \ldots, \widehat{N}_{N_{\max}^{\text{ca}}}\Big) = \\ &\quad \min\Bigg\{\max\left\{0, \widehat{N}_{N_{\max}^{\text{ca}}-j} + j - N_{\max}^{\text{ca}}\right\}, \\ &\qquad \max\left\{0, \widehat{N}_{N_{\max}^{\text{ca}}-1-j} + j - N_{\max}^{\text{ca}} + 1\right\}, \\ &\qquad \ldots, \max\left\{0, \widehat{N}_{N_{\min}^{\text{ca}}} - N_{\min}^{\text{ca}}\right\}, 1\Bigg\},\end{aligned}$$

with $j \in [0; N_{\max}^{\text{ca}} - N_{\min}^{\text{ca}}]$. Note that the expression for the continuous-time control input in (IV.1) and (IV.2) takes into account the possible out-of-order packet arrivals and message rejection. For example, in Figure 3, the time-delays in the controller-to-actuator branch of the network are allowed to take values in $\{\tau, 2\tau, 3\tau\}$, resulting in a message rejection at time s_{k+2}. We refer the interested readers to references [5], [6] for more details on the proposed choices for j_* (IV.2), $\widehat{f}$, and $\widehat{g}$.

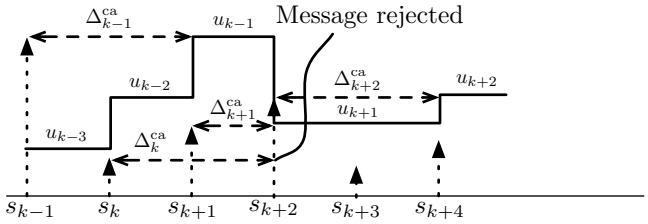

Fig. 2. Time-delays in the controller-to-actuator branch of the network with $\Delta_k^{\text{ca}} \in \{\tau, 2\tau, 3\tau\}$.

A. Symbolic controller

A symbolic controller is a mechanism that determines which inputs $u_k \in \mathsf{U}$ should be fed into the system Σ based on the observed states $x_k \in \mathbb{R}^n$. We refer the interested readers to [8] for a formal definition of symbolic controllers. Although for some LTL specifications such as safety or reachability it is sufficient to consider only static controllers (i.e. without memory) [24], we do not limit our work by this assumption. Hence the approach presented in what follows is applicable to dynamic controllers (i.e. a controller with memory) as well, which are required to address general LTL specifications [25]. Due to the presence of a ZOH, from now on we assume that the set $\mathcal{U}$ contains only curves that are constant over intervals of length $\tau \in \mathbb{R}^+$ and take values in U, i.e.:

$$\begin{aligned}&\mathcal{U} = \\ &\{\upsilon : \mathbb{R}_0^+ \to \mathsf{U} | \upsilon(t) = \upsilon((s-1)\tau), t \in [(s-1)\tau, s\tau[, s \in \mathbb{N}\}.\end{aligned} \tag{IV.4}$$

Correspondingly, one should update U_τ to $\mathcal{U}$ (IV.4) in the definition of $S_\tau(\Sigma)$ (cf. Section III).

Similar to what was assumed in the connection between controller and plant, we also consider the possible occurrence of message rejection for the measurement data sent from the sensor to the symbolic controller. The symbolic controller uses $\widehat{x}_k$ as an input at the sampling times $s_k := k\tau$, where

$$\widehat{x}_k = x_{k+\ell_*-N_{\max}^{\text{sc}}}, \tag{IV.5}$$

where $\ell_* \in [0; N_{\max}^{\text{sc}} - N_{\min}^{\text{sc}}]$ is defined as:

$$\ell_* = \widehat{f}\left(\widetilde{N}_{N_{\min}^{\text{sc}}}, \ldots, \widetilde{N}_{N_{\max}^{\text{sc}}}\right), \tag{IV.6}$$

where $\widetilde{N}_k$, for $k \in [N_{\min}^{\text{sc}}; N_{\max}^{\text{sc}}]$, is the delay suffered by the measurement packet sent k samples ago, namely $\widetilde{N}_{N_{\max}^{\text{sc}}-i} = N_{k-N_{\max}^{\text{sc}}+i}^{\text{sc}}$ for any $i \in [0; N_{\max}^{\text{sc}} - N_{\min}^{\text{sc}}]$, and $\widehat{f}$ is the function appearing in (IV.3). Note that the expression for the input of the controller in (IV.5) and (IV.6) takes into account possible out-of-order packet arrivals and message rejection. Again, we refer the interested readers to [5], [6] for more details on the proposed choices for ℓ_* in (IV.6).

B. Describing NCS as metric systems

Given $S_\tau(\Sigma)$ and the NCS $\widetilde{\Sigma}$, now consider the metric system $S(\widetilde{\Sigma}) := (X, X_0, U, \longrightarrow, Y, H)$, capturing all the information contained in the NCS $\widetilde{\Sigma}$, where:

- $X = \{X_\tau \cup q\}^{N^{\text{sc}}_{\max}} \times U_\tau^{N^{\text{ca}}_{\max}} \times [N^{\text{sc}}_{\min}; N^{\text{sc}}_{\max}]^{N^{\text{sc}}_{\max}} \times [N^{\text{ca}}_{\min}; N^{\text{ca}}_{\max}]^{N^{\text{ca}}_{\max}}$, where q is a dummy symbol;
- $X_0 = \Big\{\big(x_0, q, \dots, q, \upsilon_0, \dots, \upsilon_0, N^{\text{sc}}_{\max}, \dots, N^{\text{sc}}_{\max}, N^{\text{ca}}_{\max}, \dots, N^{\text{ca}}_{\max}\big) \mid x_0 \in X_{\tau 0}, \upsilon_0 \in U_\tau\Big\}$;
- $U = U_\tau$;
- $\Big(x_1, \dots, x_{N^{\text{sc}}_{\max}}, \upsilon_1, \dots, \upsilon_{N^{\text{ca}}_{\max}}, \widetilde{N}_1, \dots, \widetilde{N}_{N^{\text{sc}}_{\max}}, \widehat{N}_1, \dots, \widehat{N}_{N^{\text{ca}}_{\max}}\Big) \xrightarrow{\upsilon} \Big(x', x_1, \dots, x_{N^{\text{sc}}_{\max}-1}, \upsilon, \upsilon_1, \dots, \upsilon_{N^{\text{ca}}_{\max}-1}, \widetilde{N}, \widetilde{N}_1, \dots, \widetilde{N}_{N^{\text{sc}}_{\max}-1}, \widehat{N}, \widehat{N}_1, \dots, \widehat{N}_{N^{\text{ca}}_{\max}-1}\Big)$ for all $\widetilde{N} \in [N^{\text{sc}}_{\min}; N^{\text{sc}}_{\max}]$ and all $\widehat{N} \in [N^{\text{ca}}_{\min}; N^{\text{ca}}_{\max}]$ if there exists a transition $x_1 \xrightarrow[\tau]{\upsilon_{N^{\text{ca}}_{\max}-j_*}} x'$ in $S_\tau(\Sigma)$ where $j_* = \widehat{f}\Big(\widehat{N}_{N^{\text{ca}}_{\min}}, \dots, \widehat{N}_{N^{\text{ca}}_{\max}}\Big)$, defined in (IV.2);
- $Y = Y_\tau \times Y_\tau$;
- $H\Big(x_1, \dots, x_{N^{\text{sc}}_{\max}}, \upsilon_1, \dots, \upsilon_{N^{\text{ca}}_{\max}}, \widetilde{N}_1, \dots, \widetilde{N}_{N^{\text{sc}}_{\max}}, \widehat{N}_1, \dots, \widehat{N}_{N^{\text{ca}}_{\max}}\Big) = \big(H_\tau(x_1), H_\tau\big(x_{N^{\text{sc}}_{\max}-\ell_*}\big)\big)$, where $\ell_* = \widehat{f}\Big(\widetilde{N}_{N^{\text{sc}}_{\min}}, \dots, \widetilde{N}_{N^{\text{sc}}_{\max}}\Big)$ is defined in (IV.6). With a slight abuse of notation, we assume that $H_\tau(q) := q$.

Note that the choice of the set of state X in $S(\widetilde{\Sigma})$ allows one to keep track of the measurement and control packets and the corresponding delays suffered by them, in order to consider out-of-order packet arrivals and message rejections.

Let us remark that the set of states and inputs of $S(\widetilde{\Sigma})$ are uncountable and that $S(\widetilde{\Sigma})$ is a nondeterministic system in the sense of Definition 3.1, since depending on the values of $\widetilde{N}$ and $\widehat{N}$, more than one υ-successor of any state of $S(\widetilde{\Sigma})$ may exist.

Remark 4.1: Note that the output value of any state of $S(\widetilde{\Sigma})$ is a pair: the first entry is the output of the plant available at the sensors at times $s_k := k\tau$, and the second one is the output of the plant available at the controller at the same times s_k taking into consideration the occurrence of message rejection (cf. see Figure 1 for the pair of outputs). With the output map defined as we suggest, the synthesis of controllers should be performed using the first entries of the output pairs to define the satisfaction of properties. This is so because usually specifications are expressed in terms of the outputs exhibited by the plant, i.e. what is available at the sensors before the network. However, the controller refinement (and any interconnection analysis) should make use of the second entry of the output pairs as those are the outputs received by the controllers. In the present paper we do not dive further into these issues, which are left as object of future research.

We assume that the output set Y is equipped with the metric $\mathbf{d}_Y$ that is induced by the metric $\mathbf{d}_{Y_\tau}$, as the following: given any $\mathsf{x} := \Big(x_1, \dots, x_{N^{\text{sc}}_{\max}}, \upsilon_1, \dots, \upsilon_{N^{\text{ca}}_{\max}}, \widetilde{N}_1, \dots, \widetilde{N}_{N^{\text{sc}}_{\max}}, \widehat{N}_1, \dots, \widehat{N}_{N^{\text{ca}}_{\max}}\Big)$ and $\mathsf{x}' := \Big(x'_1, \dots, x'_{N^{\text{sc}}_{\max}}, \upsilon'_1, \dots, \upsilon'_{N^{\text{ca}}_{\max}}, \widetilde{N}'_1, \dots, \widetilde{N}'_{N^{\text{sc}}_{\max}}, \widehat{N}'_1, \dots, \widehat{N}'_{N^{\text{ca}}_{\max}}\Big)$ in X, we set

$$\mathbf{d}_Y\left(H\left(\mathsf{x}\right), H\left(\mathsf{x}'\right)\right) = \mathbf{d}_Y\left((x_1, x_k), (x'_1, x'_k)\right) := \max\left\{\mathbf{d}_{Y_\tau}\left(H_\tau\left(x_1\right), H_\tau\left(x'_1\right)\right), \mathbf{d}_{Y_\tau}\left(H_\tau\left(x_k\right), H_\tau\left(x'_k\right)\right)\right\}, \tag{IV.7}$$

for some given $k \in [N^{\text{sc}}_{\min}; N^{\text{sc}}_{\max}]$, where we extend the metric $\mathbf{d}_{Y_\tau}$ such that $\mathbf{d}_{Y_\tau}(H_\tau(x), H_\tau(q)) = +\infty$ for any $x \in \mathbb{R}^n$ and $\mathbf{d}_{Y_\tau}(H_\tau(q), H_\tau(q)) = 0$. Hence, two states of $S(\widetilde{\Sigma})$ are ε-close if not only their first entries are ε-close but also if the second entries are ε-close too.

V. Symbolic Models for NCS

This section contains the main contributions of the paper. We show the existence and construction of symbolic models for NCS by using existing symbolic models for the plant Σ, namely, $S_{\mathsf{q}}(\Sigma) := (X_{\mathsf{q}}, X_{\mathsf{q}0}, U_{\mathsf{q}}, \underset{\mathsf{q}}{\longrightarrow}, Y_{\mathsf{q}}, H_{\mathsf{q}})$.

Define the following system:

$$S_*(\widetilde{\Sigma}) := \Big(X_*, X_{*0}, U_*, \underset{*}{\longrightarrow}, Y_*, H_*\Big),$$

where

- $X_* = \{X_{\mathsf{q}} \cup q\}^{N^{\text{sc}}_{\max}} \times U_{\mathsf{q}}^{N^{\text{ca}}_{\max}} \times [N^{\text{sc}}_{\min}; N^{\text{sc}}_{\max}]^{N^{\text{sc}}_{\max}} \times [N^{\text{ca}}_{\min}; N^{\text{ca}}_{\max}]^{N^{\text{ca}}_{\max}}$;
- $X_{*0} = \Big\{\big(x_{*0}, q, \dots, q, u_{*0}, \dots, u_{*0}, N^{\text{sc}}_{\max}, \dots, N^{\text{sc}}_{\max}, N^{\text{ca}}_{\max}, \dots, N^{\text{ca}}_{\max}\big) \mid x_{*0} \in X_{\mathsf{q}0}, u_{*0} \in U_{\mathsf{q}}\Big\}$;
- $U_* = U_{\mathsf{q}}$;
- $\Big(x_{*1}, \dots, x_{*N^{\text{sc}}_{\max}}, u_{*1}, \dots, u_{*N^{\text{ca}}_{\max}}, \widetilde{N}_{*1}, \dots, \widetilde{N}_{*N^{\text{sc}}_{\max}}, \widehat{N}_{*1}, \dots, \widehat{N}_{*N^{\text{ca}}_{\max}}\Big) \xrightarrow[*]{u_*} \Big(x'_*, x_{*1}, \dots, x_{*(N^{\text{sc}}_{\max}-1)}, u_*, u_{*1}, \dots, u_{*(N^{\text{ca}}_{\max}-1)}, \widetilde{N}_*, \widetilde{N}_{*1}, \dots, \widetilde{N}_{*(N^{\text{sc}}_{\max}-1)}, \widehat{N}_*, \widehat{N}_{*1}, \dots, \widehat{N}_{*(N^{\text{ca}}_{\max}-1)}\Big)$ for all $\widetilde{N}_* \in [N^{\text{sc}}_{\min}; N^{\text{sc}}_{\max}]$ and all $\widehat{N}_* \in [N^{\text{ca}}_{\min}; N^{\text{ca}}_{\max}]$ if there exists transition $x_{*1} \xrightarrow[\mathsf{q}]{u_{*(N^{\text{ca}}_{\max}-j_*)}} x'_*$ in $S_{\mathsf{q}}(\Sigma)$ where $j_* = \widehat{f}\Big(\widehat{N}_{*N^{\text{ca}}_{\min}}, \dots, \widehat{N}_{*N^{\text{ca}}_{\max}}\Big)$, defined in (IV.2);
- $Y_* = Y_{\mathsf{q}} \times Y_{\mathsf{q}}$;
- $H_*\big(x_{*1}, \dots, x_{*N^{\text{sc}}_{\max}}, u_{*1}, \dots, u_{*N^{\text{ca}}_{\max}}, \widetilde{N}_{*1}, \dots, \widetilde{N}_{*N^{\text{sc}}_{\max}}, \widehat{N}_{*1}, \dots, \widehat{N}_{*N^{\text{ca}}_{\max}}\big) = \big(H_{\mathsf{q}}\left(x_{*1}\right), H_{\mathsf{q}}\left(x_{*(N^{\text{sc}}_{\max}-\ell_*)}\right)\big)$ where $\ell_* = \widehat{f}\Big(\widetilde{N}_{*N^{\text{sc}}_{\min}}, \dots, \widetilde{N}_{*N^{\text{sc}}_{\max}}\Big)$, defined in (IV.6). With a slight abuse of notation, we set $H_{\mathsf{q}}(q) := q$.

Note that $S_*(\widetilde{\Sigma})$ has the same structure as $S(\widetilde{\Sigma})$, but it is computed using a symbolic model of $S_\tau(\Sigma)$, namely, $S_{\mathsf{q}}(\Sigma)$. It can be readily seen that the system $S_*(\widetilde{\Sigma})$ is countable or symbolic if the system $S_{\mathsf{q}}(\Sigma)$ is countable or symbolic, respectively. Although $S_{\mathsf{q}}(\Sigma)$ may be a deterministic system, $S_*(\widetilde{\Sigma})$ is always a nondeterministic one, since depending on the possible delays in both channels of the network (i.e. the values of $\widetilde{N}_*$ and $\widehat{N}_*$), more than one u_*-successor of any state of $S_*(\widetilde{\Sigma})$ may exist.

We can now state the main results of the paper.

Theorem 5.1: Consider a NCS $\widetilde{\Sigma}$ and suppose that there exists an abstraction $S_{\mathsf{q}}(\Sigma)$ such that $S_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_\tau(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} S_{\mathsf{q}}(\Sigma)$. Then we have $S_*(\widetilde{\Sigma}) \preceq^{\varepsilon}_{\mathcal{AS}} S(\widetilde{\Sigma}) \preceq^{\varepsilon}_{\mathcal{S}} S_*(\widetilde{\Sigma})$.
The proof of Theorem 5.1 is provided in the Appendix.

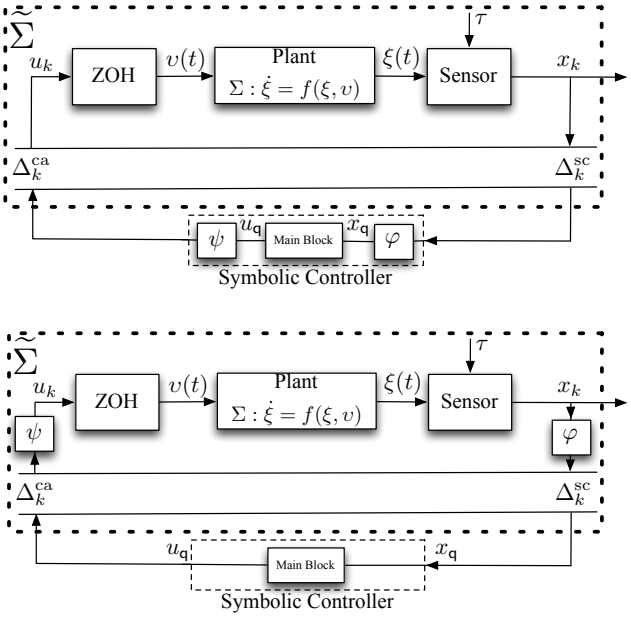

Fig. 3. Shifting functions φ and ψ to the other sides of the communication network.

Corollary 5.2: Consider a NCS $\widetilde{\Sigma}$ and suppose that there exists an abstraction $S_{\mathsf{q}}(\Sigma)$ such that $S_{\mathsf{q}}(\Sigma) \cong^{\varepsilon}_{\mathcal{AS}} S_{\tau}(\Sigma)$. Then we have $S_*(\widetilde{\Sigma}) \cong^{\varepsilon}_{\mathcal{AS}} S(\widetilde{\Sigma})$.

Proof: Using Theorem 5.1 one gets that $S_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_{\tau}(\Sigma)$ implies $S_*(\widetilde{\Sigma}) \preceq^{\varepsilon}_{\mathcal{AS}} S(\widetilde{\Sigma})$ equipped with the alternating ε-approximate simulation relation $\widetilde{R}$ as defined in the proof of Theorem 5.1. In a similar way, one can show that $S_{\tau}(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_{\mathsf{q}}(\Sigma)$ implies $S(\widetilde{\Sigma}) \preceq^{\varepsilon}_{\mathcal{AS}} S_*(\widetilde{\Sigma})$ equipped with the alternating ε-approximate simulation relation $\widetilde{R}^{-1}$ which completes the proof. ■

Remark 5.3: With reference to the formal definition of symbolic controllers in [8], one can readily verify the existence of two static functions $\varphi : X_{\tau} \to X_{\mathsf{q}}$ and $\psi : U_{\mathsf{q}} \to \mathsf{U}$, inside the symbolic controllers, associating to any $x_{\tau} \in X_{\tau}$ one symbol $x_{\mathsf{q}} \in X_{\mathsf{q}}$ and to any symbol $u_{\mathsf{q}} \in U_{\mathsf{q}}$ one control value $u_{\tau} \in \mathsf{U}$, respectively, as shown in Figure 3. Since the functions φ and ψ are static, without violating the main results one can shift those functions toward sensor and actuator in the NCS as shown in Figure 3. If $S_{\mathsf{q}}(\Sigma)$ is symbolic, then U_{q} and X_{q} are finite sets. Hence, one can automatically take care of limited bandwidth constraints without introducing additional quantization errors. As also noted in [10], [11], for the grid-based symbolic abstractions $S_{\mathsf{q}}(\Sigma)$ proposed in [12], [13], one has: $\psi = 1_{U_{\mathsf{q}}}$ and $\varphi : x \to [x]_{\eta}$, where $[x]_{\eta} \in [\mathbb{R}^n]_{\eta}$ such that $\|x - [x]_{\eta}\| \leq \eta/2$ for a given state space quantization parameter $\eta \in \mathbb{R}^+$.

We refer the interested readers to Subsection 5.1 in [26] providing similar results as the ones in Theorem 5.1 and Corollary 5.2 when the symbolic controller is static.

Remark 5.4: One of the distinguishing contributions of our work with respect to [10], [11] is that our results do not hinge on any assumption on the controller. Hence, the provided abstractions here are amenable to any available synthesis techniques and tools, e.g. PESSOA [27] and SPIN [28] (whether resulting in static or dynamic controllers). Contrary, the authors in [10], [11] were forced to provide specific constructions for both the controller synthesis and the reformulation of the specification, due to the employed assumptions and the convolved output values of the proposed abstractions.

VI. Space Complexity Analysis

We compare the results provided here with the ones provided in [10], [11] in terms of the size of the proposed finite abstractions. To obtain a fair comparison, we also impose a grid-based finite abstraction for the plant Σ using the same sampling time and quantization parameters as the ones in [10], [11]. By assuming that we are only interested in the dynamics of Σ on a compact set $\mathsf{D} \subset \mathbb{R}^n$, the size of the set of states of the finite abstractions, provided in [10], [11], is:

$$|X_{\star}| = \sum_{i \in \{\{1\} \cup [N_{\min}; N_{\max}]\}} \left| [\mathsf{D}]_{\eta} \right|^i,$$

where $N_{\min} = N^{\mathrm{sc}}_{\min} + N^{\mathrm{ca}}_{\min}$ and $N_{\max} = N^{\mathrm{sc}}_{\max} + N^{\mathrm{ca}}_{\max}$. Meanwhile, the size of the set of states for the abstractions provided by Theorem 5.1 and Corollary 5.2, is at most:

$$|X_*| = \left(\left| [\mathsf{D}]_{\eta} \right| + 1 \right)^{N^{\mathrm{sc}}_{\max}} \cdot \left| [\mathsf{U}]_{\mu} \right|^{N^{\mathrm{ca}}_{\max}} \cdot (N^{\mathrm{sc}}_{\max} - N^{\mathrm{sc}}_{\min} + 1)^{N^{\mathrm{sc}}_{\max}} \cdot (N^{\mathrm{ca}}_{\max} - N^{\mathrm{ca}}_{\min} + 1)^{N^{\mathrm{ca}}_{\max}},$$

where $[\mathsf{D}]_{\eta} = \mathsf{D} \cap [\mathbb{R}^n]_{\eta}$ and $[\mathsf{U}]_{\mu} = \mathsf{U} \cap [\mathbb{R}^m]_{\mu}$ for some quantization parameters $\eta, \mu \in \mathbb{R}^+$.

One can easily verify that the size of the symbolic models proposed in [10], [11] is at most:

$$\begin{aligned} \left| S_{\star}(\widetilde{\Sigma}) \right| &= |X_{\star}| \cdot |[\mathsf{U}]_{\mu}| \cdot (N_{\max} - N_{\min} + 1) \cdot K \qquad \text{(VI.1)} \\ &= \left(\sum_{i \in \{\{1\} \cup [N_{\min}; N_{\max}]\}} \left| [\mathsf{D}]_{\eta} \right|^i \right) \cdot |[\mathsf{U}]_{\mu}| \cdot (N_{\max} - N_{\min} + 1) \cdot K, \end{aligned}$$

where K is the maximum number of u-successors of any state of the symbolic model $S_{\mathsf{q}}(\Sigma)$ for $u \in [\mathsf{U}]_{\mu}$. Note that with the results proposed in [12] one has $K = 1$ because $S_{\mathsf{q}}(\Sigma)$ is a deterministic system, while with the ones proposed in [13] one has $K \geq 1$ because $S_{\mathsf{q}}(\Sigma)$ is a nondeterministic system and the value of K depends on the functions β and γ in (II.2), see [13] for more details. The size of the symbolic models provided in this paper is at most:

$$\begin{aligned} \left| S_*(\widetilde{\Sigma}) \right| &= |X_*| \cdot |[\mathsf{U}]_{\mu}| \cdot (N^{\mathrm{sc}}_{\max} - N^{\mathrm{sc}}_{\min} + 1) \cdot (N^{\mathrm{ca}}_{\max} - N^{\mathrm{ca}}_{\min} + 1) \cdot K \qquad \text{(VI.2)} \\ &= \left(\left| [\mathsf{D}]_{\eta} \right| + 1 \right)^{N^{\mathrm{sc}}_{\max}} \cdot \left| [\mathsf{U}]_{\mu} \right|^{N^{\mathrm{ca}}_{\max}+1} \cdot (N^{\mathrm{sc}}_{\max} - N^{\mathrm{sc}}_{\min} + 1)^{N^{\mathrm{sc}}_{\max}+1} \cdot (N^{\mathrm{ca}}_{\max} - N^{\mathrm{ca}}_{\min} + 1)^{N^{\mathrm{ca}}_{\max}+1} \cdot K, \end{aligned}$$

with the same K as in (VI.1). For the sake of fair comparison, one should compare the size in (VI.1) with the one in (6.3) in [26] because in both symbolic models $S_{\star}(\widetilde{\Sigma})$ and $\overline{S}_*(\widetilde{\Sigma})$ in [26] it is assumed that the symbolic controllers are static. It can be readily verified that if $\left| [\mathsf{D}]_{\eta} \right|$ is much bigger than

$\left|[\mathsf{U}]_\mu\right|$ $\left(\left|[\mathsf{D}]_\eta\right| >> \left|[\mathsf{U}]_\mu\right|\right)$ which is often the case, $\left|\overline{S}_*(\widetilde{\Sigma})\right|$ can be much smaller than $\left|S_\star(\widetilde{\Sigma})\right|$ specially for large values of $N_{\max}$. The symbolic model $S_*(\widetilde{\Sigma})$ can also have a smaller size for large values of $N_{\max}$ and for $\left|[\mathsf{D}]_\eta\right| >> \left|[\mathsf{U}]_\mu\right|$, as shown in the following numerical example.

Example 6.1: Consider a plant Σ such that $\mathsf{D} = [-1,1] \times [-1,1]$, $\mathsf{U} = [0,1]$, $\eta = 0.1$, and $\mu = 1$. Assume that the delays in different parts of the network are as the following: $N^{\text{sc}}_{\min} = 1$, $N^{\text{sc}}_{\max} = 2$, $N^{\text{ca}}_{\min} = 2$, and $N^{\text{ca}}_{\max} = 3$. Using equations (VI.1), (VI.2), and (6.3) in [26], one obtains:

$$\left|S_\star(\widetilde{\Sigma})\right| = 6.1594 \times 10^{13} K, \ \left|S_*(\widetilde{\Sigma})\right| = 3.2932 \times 10^{8} K,$$
$$\left|\overline{S}_*(\widetilde{\Sigma})\right| = 1.8662 \times 10^{7} K.$$

It can be readily verified that the sizes of our proposed abstractions $S_*(\widetilde{\Sigma})$ and $\overline{S}_*(\widetilde{\Sigma})$ in [26] are roughly 2×10^5 and 3×10^6 times smaller than the one of $S_\star(\widetilde{\Sigma})$, proposed in [10], [11], respectively.

Remark 6.2: Note that in Remark 5.2 in [10], the authors suggest a more concise representation for their proposed finite abstractions of NCS in order to reduce the space complexity. However, this representation is only applicable if the plant Σ is δ-ISS. Therefore, for general classes of plants Σ of NCS, our proposed approach is potentially more appropriate in terms of the size of the abstractions, especially for large values of $N_{\max}$.

VII. Discussion

In this paper we have provided a construction of symbolic models for NCS, subject to variable communication delays, quantization errors, packet losses, and limited bandwidth, using available symbolic models for the plant. Furthermore, our approach allows us to treat general specifications expressed as formulae in LTL or as automata on infinite strings without requiring additional reformulations. Finally, we have shown that the proposed methodology also results, in general, in smaller abstractions than similar approaches in the literature [10], [11].

Future work will concentrate on: 1) providing efficient implementations of the symbolic models, the construction of which has been shown in this work; 2) the construction of symbolic models for NCS by considering some probabilistic structure on the transmission intervals, communication delays, and packet dropouts; 3) construction of symbolic models for NCS by considering additional network non-idealities, in particular time-varying sampling/transmission intervals.

References

[1] N. W. Bauer, P. J. H. Maas, and W. P. M. H. Heemels, "Stability analysis of networked control systems: a sum of squares approach," *Automatica*, vol. 48, no. 8, pp. 1514–1524, 2012.

[2] H. Gao, T. Chen, and J. Lam, "A new delay system approach to network-based control," *Automatica*, vol. 44, no. 1, pp. 39–52, 2008.

[3] R. Alur, A. D'Innocenzo, K. H. Johansson, G. J. Pappas, and G. Weiss, "Compositional modeling and analysis of multi-hop control networks," *IEEE Transactions on Automatic Control*, vol. 56, no. 10, pp. 2345–2357, 2011.

[4] D. Antunes, J. P. Hespanha, and C. Silvestre, "Volterra integral approach to impulsive renewal systems: Application to networked control," *IEEE Transactions on Automatic Control*, vol. 57, no. 3, pp. 607–619, March 2012.

[5] M. B. G. Cloosterman, N. van de Wouw, W. P. M. H. Heemels, and H. Nijmeijer, "Stability of networked control systems with uncertain time-varying delays," *IEEE Transactions on Automatic Control*, vol. 54, no. 7, pp. 1575–1580, July 2009.

[6] N. van de Wouw, D. Nesic, and W. P. M. H. Heemels, "A discrete-time framework for stability analysis of nonlinear networked control systems," *Automatica*, vol. 48, no. 6, pp. 1144–1153, June 2012.

[7] D. Nesic and D. Liberzon, "A unified framework for design and analysis of networked and quantized control systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 4, pp. 732–747, 2009.

[8] P. Tabuada, *Verification and Control of Hybrid Systems, A symbolic approach.* Springer US, 2009.

[9] O. Maler, A. Pnueli, and J. Sifakis, "On the synthesis of discrete controllers for timed systems," *in Proceedings of 12th Annual Symposium on Theoretical Aspects of Computer Science*, vol. 900, pp. 229–242, 1995.

[10] A. Borri, G. Pola, and M. Di Benedetto, "A symbolic approach to the design of nonlinear networked control systems," *in Proceedings of 15th International Conference on Hybrid Systems: Computation and Control*, pp. 255–264, April 2012.

[11] ——, "Integrated symbolic design of unstable nonlinear networked control systems," *in Proceedings of 51th IEEE Conference on Decision and Control*, December 2012.

[12] A. Girard, G. Pola, and P. Tabuada, "Approximately bisimilar symbolic models for incrementally stable switched systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 1, pp. 116–126, 2009.

[13] M. Zamani, G. Pola, M. M. Jr., and P. Tabuada, "Symbolic models for nonlinear control systems without stability assumptions," *IEEE Transactions on Automatic Control*, vol. 57, no. 7, pp. 1804–1809, July 2012.

[14] M. Zamani, I. Tkachev, and A. Abate, "Bisimilar symbolic models for stochastic control systems without state-space discretization," in *Proceedings of the 17th International Conference on Hybrid Systems: Computation and Control*, April 2014, to appear.

[15] B. Yordanov, J. Tumova, I. Cerna, J. Barnat, and C. Belta, "Formal analysis of piecewise affine systems through fomula-guided refinement," *Automatica*, vol. 49, no. 1, pp. 261–266, January 2013.

[16] B. Jobstmann, A. Griesmayer, and R. Bloem, "Program repair as a game," in *Computer Aided Verification*. Springer, 2005, pp. 226–238.

[17] W. Thomas, "On the synthesis of strategies in infinite games," *in Proceedings of 12th Annual Symposium on Theoretical Aspects of Computer Science*, vol. 900, pp. 1–13, 1995.

[18] E. D. Sontag, *Mathematical control theory*, 2nd ed. New York: Springer-Verlag, 1998, vol. 6.

[19] D. Angeli and E. D. Sontag, "Forward completeness, unboundedness observability, and their Lyapunov characterizations," *Systems and Control Letters*, vol. 38, pp. 209–217, 1999.

[20] D. Angeli, "A Lyapunov approach to incremental stability properties," *IEEE Transactions on Automatic Control*, vol. 47, no. 3, pp. 410–421, 2002.

[21] A. Girard and G. J. Pappas, "Approximation metrics for discrete and continuous systems," *IEEE Transactions on Automatic Control*, vol. 25, no. 5, pp. 782–798, 2007.

[22] G. Pola and P. Tabuada, "Symbolic models for nonlinear control systems: alternating approximate bisimulations," *SIAM Journal on Control and Optimization*, vol. 48, no. 2, pp. 719–733, 2009.

[23] W. P. M. H. Heemels and N. van de Wouw, "Stability and stabilization of networked control systems," in *Networked Control Systems*, ser. Lecture Notes in Control and Information Sciences, A. Bemporad, W. P. M. H. Heemels, and M. Johansson, Eds. Springer London, 2010, vol. 406, pp. 203–253.

[24] A. Girard, "Synthesis using approximately bisimilar abstractions: state-feedback controllers for safety specifications," *in Proceedings of 13th International Conference on Hybrid Systems: Computation and Control*, pp. 111–120, April 2010.

[25] C. Baier and J. P. Katoen, *Principles of model checking*. The MIT Press, April 2008.

[26] M. Zamani, M. Mazo Jr., and A. Abate, "Symbolic models for networked control systems," *arXiv: 1401.6396*, January 2014.

[27] M. Mazo Jr., A. Davitian, and P. Tabuada, "PESSOA: A tool for embedded control software synthesis," in *Computer Aided Verification (CAV)*, ser. LNCS, T. Touili, B. Cook, and P. Jackson, Eds., vol. 6174. Springer-Verlag, July 2010, pp. 566–569.

[28] G. Holzmann, *The SPIN model checker: Primer and reference manual*, 1st ed. Addison-Wesley Professional, 2003.

VIII. Appendix

Proof: [Proof of Theorem 5.1] We start by proving $S_*(\widetilde{\Sigma}) \preceq^{\varepsilon}_{\mathcal{AS}} S(\widetilde{\Sigma})$. Since $S_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_\tau(\Sigma)$, there exists an alternating ε-approximate simulation relation R from $S_{\mathsf{q}}(\Sigma)$ to $S_\tau(\Sigma)$. Consider the relation $\widetilde{R} \subseteq X_* \times X$ defined by $(\mathsf{x}_*, \mathsf{x}) \in \widetilde{R}$, where $\mathsf{x}_* = \Big(x_{*1}, \ldots, x_{*N^{\mathrm{sc}}_{\max}}, u_{*1}, \ldots, u_{*N^{\mathrm{ca}}_{\max}}, \widetilde{N}_{*1}, \ldots, \widetilde{N}_{*N^{\mathrm{sc}}_{\max}}, \widehat{N}_{*1}, \ldots, \widehat{N}_{*N^{\mathrm{ca}}_{\max}}\Big)$ and $\mathsf{x} = \Big(x_1, \ldots, x_{N^{\mathrm{sc}}_{\max}}, \upsilon_1, \ldots, \upsilon_{N^{\mathrm{ca}}_{\max}}, \widetilde{N}_1, \ldots, \widetilde{N}_{N^{\mathrm{sc}}_{\max}}, \widehat{N}_1, \ldots, \widehat{N}_{N^{\mathrm{ca}}_{\max}}\Big)$, if and only if $\widetilde{N}_{*i} = \widetilde{N}_i$, $\forall i \in [1; N^{\mathrm{sc}}_{\max}]$, $\widehat{N}_{*j} = \widehat{N}_j$, $\forall j \in [1; N^{\mathrm{ca}}_{\max}]$, $(x_{*k}, x_k) \in R$, $\forall k \in [1; N^{\mathrm{sc}}_{\max}]$, and for each u_{*i} and the corresponding υ_i there exists $x'_* \in \mathbf{Post}_{u_{*i}}(x_*)$ such that $(x'_*, \xi_{x\upsilon_i}(\tau)) \in R$ for any $i \in [1; N^{\mathrm{ca}}_{\max}]$ and any $(x_*, x) \in R$. Note that if $U_\tau = U_{\mathsf{q}}$ and they are finite then the last condition of the relation $\widetilde{R}$ is nothing more than requiring $u_{*i} = \upsilon_i$ for any $i \in [1; N^{\mathrm{ca}}_{\max}]$.

Consider $\mathsf{x}_{*0} := (x_{*0}, q, \ldots, q, u_{*0}, \ldots, u_{*0}, N^{\mathrm{sc}}_{\max}, \ldots, N^{\mathrm{sc}}_{\max}, N^{\mathrm{ca}}_{\max}, \ldots, N^{\mathrm{ca}}_{\max}) \in X_{*0}$. Due to the relation R, there exist $x_0 \in X_{\tau 0}$ such that $(x_{*0}, x_0) \in R$ and $\upsilon_0 \in U_\tau$ such that there exists $x'_* \in \mathbf{Post}_{u_{*0}}(x_*)$ satisfying $(x'_*, \xi_{x\upsilon_0}(\tau)) \in R$ for any $(x_*, x) \in R$. Hence, by choosing $\mathsf{x}_0 := (x_0, q, \ldots, q, \upsilon_0, \ldots, \upsilon_0, N^{\mathrm{sc}}_{\max}, \ldots, N^{\mathrm{sc}}_{\max}, N^{\mathrm{ca}}_{\max}, \ldots, N^{\mathrm{ca}}_{\max}) \in X_0$, one gets $(\mathsf{x}_{*0}, \mathsf{x}_0) \in \widetilde{R}$ and condition (i) in Definition 3.3 is satisfied.

Now consider any $(\mathsf{x}_*, \mathsf{x}) \in \widetilde{R}$, where $\mathsf{x}_* = \Big(x_{*1}, \ldots, x_{*N^{\mathrm{sc}}_{\max}}, u_{*1}, \ldots, u_{*N^{\mathrm{ca}}_{\max}}, \widetilde{N}_{*1}, \ldots, \widetilde{N}_{*N^{\mathrm{sc}}_{\max}}, \widehat{N}_{*1}, \ldots, \widehat{N}_{*N^{\mathrm{ca}}_{\max}}\Big)$ and $\mathsf{x} = \Big(x_1, \ldots, x_{N^{\mathrm{sc}}_{\max}}, \upsilon_1, \ldots, \upsilon_{N^{\mathrm{ca}}_{\max}}, \widetilde{N}_1, \ldots, \widetilde{N}_{N^{\mathrm{sc}}_{\max}}, \widehat{N}_1, \ldots, \widehat{N}_{N^{\mathrm{ca}}_{\max}}\Big)$. Since $\widetilde{N}_{*i} = \widetilde{N}_i$, $\forall i \in [1; N^{\mathrm{sc}}_{\max}]$, and $\widehat{N}_{*j} = \widehat{N}_j$, $\forall j \in [1; N^{\mathrm{ca}}_{\max}]$, and using definitions of $S_*(\widetilde{\Sigma})$ and $S(\widetilde{\Sigma})$, one obtains $H_*(\mathsf{x}_*) = (x_{*1}, x_{*k})$ and $H(\mathsf{x}) = (x_1, x_k)$, for some $k \in [N^{\mathrm{sc}}_{\min}; N^{\mathrm{sc}}_{\max}]$ (cf. Definitions $S_*(\widetilde{\Sigma})$ and $S(\widetilde{\Sigma})$). Since $(x_{*i}, x_i) \in R$, $\forall i \in [1; N^{\mathrm{sc}}_{\max}]$, one gets $\mathsf{d}_{Y_\tau}(H_{\mathsf{q}}(x_{*i}), H_\tau(x_i)) \leq \varepsilon$, $\forall i \in [1; N^{\mathrm{sc}}_{\max}]$. Therefore, $\mathbf{d}_Y(H_*(\mathsf{x}_*), H(\mathsf{x})) = \max\{\mathsf{d}_{Y_\tau}(H_{\mathsf{q}}(x_{*1}), H_\tau(x_1)), \mathsf{d}_{Y_\tau}(H_{\mathsf{q}}(x_{*k}), H_\tau(x_k))\} \leq \varepsilon$ and condition (ii) in Definition 3.3 is satisfied.

Let us now show that condition (iii) in Definition 3.3 holds. Consider any $(\mathsf{x}_*, \mathsf{x}) \in \widetilde{R}$, where $\mathsf{x}_* = \Big(x_{*1}, \ldots, x_{*N^{\mathrm{sc}}_{\max}}, u_{*1}, \ldots, u_{*N^{\mathrm{ca}}_{\max}}, \widetilde{N}_{*1}, \ldots, \widetilde{N}_{*N^{\mathrm{sc}}_{\max}}, \widehat{N}_{*1}, \ldots, \widehat{N}_{*N^{\mathrm{ca}}_{\max}}\Big)$, $\mathsf{x} = \Big(x_1, \ldots, x_{N^{\mathrm{sc}}_{\max}}, \upsilon_1, \ldots, \upsilon_{N^{\mathrm{ca}}_{\max}}, \widetilde{N}_1, \ldots, \widetilde{N}_{N^{\mathrm{sc}}_{\max}}, \widehat{N}_1, \ldots, \widehat{N}_{N^{\mathrm{ca}}_{\max}}\Big)$. Consider any $u_* \in U_*(\mathsf{x}_*) = U_{\mathsf{q}}$. Using the relation R, there exist $\upsilon \in U(\mathsf{x}) = U_\tau$ and $\dot{x}_* \in \mathbf{Post}_{u_*}(x_*)$ such that $(\dot{x}_*, \xi_{x\upsilon}(\tau)) \in R$ for any $(x_*, x) \in R$. Now consider any $\mathsf{x}' = \Big(x', x_1, \ldots, x_{N^{\mathrm{sc}}_{\max}-1}, \upsilon, \upsilon_1, \ldots, \upsilon_{N^{\mathrm{ca}}_{\max}-1}, \widetilde{N}, \widetilde{N}_1, \ldots, \widetilde{N}_{N^{\mathrm{sc}}_{\max}-1}, \widehat{N}, \widehat{N}_1, \ldots, \widehat{N}_{N^{\mathrm{ca}}_{\max}-1}\Big) \in \mathbf{Post}_\upsilon(\mathsf{x}) \subseteq X$ for some $\widetilde{N} \in [N^{\mathrm{sc}}_{\min}; N^{\mathrm{sc}}_{\max}]$ and $\widehat{N} \in [N^{\mathrm{ca}}_{\min}; N^{\mathrm{ca}}_{\max}]$ where $x' = \xi_{x_1\upsilon_k}(\tau)$ for some given $k \in [N^{\mathrm{ca}}_{\min}; N^{\mathrm{ca}}_{\max}]$ (cf. Definition $S(\widetilde{\Sigma})$). Because of the relation R, there exists $x'_* \in \mathbf{Post}_{u_{*k}}(x_{*1})$ in $S_{\mathsf{q}}(\Sigma)$ such that $(x'_*, x') \in R$. Hence, due to the definition $S_*(\widetilde{\Sigma})$, one can choose $\mathsf{x}'_* = \Big(x'_*, x_{*1}, \ldots, x_{*\left(N^{\mathrm{sc}}_{\max}-1\right)}, u_*, u_{*1}, \ldots, u_{*\left(N^{\mathrm{ca}}_{\max}-1\right)}, \widetilde{N}, \widetilde{N}_1, \ldots, \widetilde{N}_{N^{\mathrm{sc}}_{\max}-1}, \widehat{N}, \widehat{N}_1, \ldots, \widehat{N}_{N^{\mathrm{ca}}_{\max}-1}\Big) \in \mathbf{Post}_{u_*}(\mathsf{x}_*) \subseteq X_*$. Due to the relation R, one can readily verify that $\mathbf{d}_{Y_\tau}(H_{\mathsf{q}}(x'_*), H_\tau(x')) \leq \varepsilon$. Since $\mathbf{d}_{Y_\tau}(H_{\mathsf{q}}(x_{*j}), H_\tau(x_j)) \leq \varepsilon$, $\forall j \in [1; N^{\mathrm{sc}}_{\max} - 1]$, one gets $\mathbf{d}_Y(H_*(\mathsf{x}'_*), H(\mathsf{x}')) = \max\{\mathsf{d}_{Y_\tau}(H_{\mathsf{q}}(x'_*), H_\tau(x')), \mathsf{d}_{Y_\tau}(H_{\mathsf{q}}(x_{*k}), H_\tau(x_k))\} \leq \varepsilon$, for some given[3] $k \in [N^{\mathrm{sc}}_{\min} - 1; N^{\mathrm{sc}}_{\max} - 1]$ (cf. Definitions $S_*(\widetilde{\Sigma})$ and $S(\widetilde{\Sigma})$). Hence, $(\mathsf{x}'_*, \mathsf{x}') \in \widetilde{R}$ implying that condition (iii) in Definition 3.3 holds.

Now we prove $S(\widetilde{\Sigma}) \preceq^{\varepsilon}_{\mathcal{S}} S_*(\widetilde{\Sigma})$. Since $S_\tau(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} S_{\mathsf{q}}(\Sigma)$, there exists an ε-approximate simulation relation R from $S_\tau(\Sigma)$ to $S_{\mathsf{q}}(\Sigma)$. Consider the relation $\widetilde{R} \subseteq X \times X_*$ defined by $(\mathsf{x}, \mathsf{x}_*) \in \widetilde{R}$, where $\mathsf{x} = \Big(x_1, \ldots, x_{N^{\mathrm{sc}}_{\max}}, \upsilon_1, \ldots, \upsilon_{N^{\mathrm{ca}}_{\max}}, \widetilde{N}_1, \ldots, \widetilde{N}_{N^{\mathrm{sc}}_{\max}}, \widehat{N}_1, \ldots, \widehat{N}_{N^{\mathrm{ca}}_{\max}}\Big)$ and $\mathsf{x}_* = \Big(x_{*1}, \ldots, x_{*N^{\mathrm{sc}}_{\max}}, u_{*1}, \ldots, u_{*N^{\mathrm{ca}}_{\max}}, \widetilde{N}_{*1}, \ldots, \widetilde{N}_{*N^{\mathrm{sc}}_{\max}}, \widehat{N}_{*1}, \ldots, \widehat{N}_{*N^{\mathrm{ca}}_{\max}}\Big)$, if and only if $\widetilde{N}_i = \widetilde{N}_{*i}$, $\forall i \in [1; N^{\mathrm{sc}}_{\max}]$, $\widehat{N}_j = \widehat{N}_{*j}$, $\forall j \in [1; N^{\mathrm{ca}}_{\max}]$, $(x_k, x_{*k}) \in R$, $\forall k \in [1; N^{\mathrm{sc}}_{\max}]$, and for each υ_i and the corresponding u_{*i} there exists a $x'_* \in \mathbf{Post}_{u_{*i}}(x_*)$ such that $(\xi_{x\upsilon_i}(\tau), x'_*) \in R$ for any $i \in [1; N^{\mathrm{ca}}_{\max}]$ and any $(x, x_*) \in R$. Note that if $U_\tau = U_{\mathsf{q}}$ and they are finite then the last condition of the relation $\widetilde{R}$ is nothing more than requiring $u_{*i} = \upsilon_i$ for any $i \in [1; N^{\mathrm{ca}}_{\max}]$.

Consider $\mathsf{x}_0 := (x_0, q, \ldots, q, \upsilon_0, \ldots, \upsilon_0, N^{\mathrm{sc}}_{\max}, \ldots, N^{\mathrm{sc}}_{\max}, N^{\mathrm{ca}}_{\max}, \ldots, N^{\mathrm{ca}}_{\max}) \in X_0$. Due to the relation R, there exist $x_{*0} \in X_{*0}$ such that $(x_0, x_{*0}) \in R$ and $u_{*0} \in U_{\mathsf{q}}$ such that there exists $x'_* \in \mathbf{Post}_{u_{*0}}(x_*)$ satisfying $(\xi_{x\upsilon_0}(\tau), x'_*) \in R$ for any $(x, x_*) \in R$. Hence, by choosing $\mathsf{x}_{*0} := (x_{*0}, q, \ldots, q, u_{*0}, \ldots, u_{*0}, N^{\mathrm{sc}}_{\max}, \ldots, N^{\mathrm{sc}}_{\max}, N^{\mathrm{ca}}_{\max}, \ldots, N^{\mathrm{ca}}_{\max}) \in X_{*0}$, one gets $(\mathsf{x}_0, \mathsf{x}_{*0}) \in \widetilde{R}$ and condition (i) in Definition 3.2 is satisfied.

Now consider any $(\mathsf{x}, \mathsf{x}_*) \in \widetilde{R}$, where $\mathsf{x} = \Big(x_1, \ldots, x_{N^{\mathrm{sc}}_{\max}}, \upsilon_1, \ldots, \upsilon_{N^{\mathrm{ca}}_{\max}}, \widetilde{N}_1, \ldots, \widetilde{N}_{N^{\mathrm{sc}}_{\max}}, \widehat{N}_1, \ldots, \widehat{N}_{N^{\mathrm{ca}}_{\max}}\Big)$ and $\mathsf{x}_* = \Big(x_{*1}, \ldots, x_{*N^{\mathrm{sc}}_{\max}}, u_{*1}, \ldots, u_{*N^{\mathrm{ca}}_{\max}}, \widetilde{N}_{*1}, \ldots, \widetilde{N}_{*N^{\mathrm{sc}}_{\max}}, \widehat{N}_{*1}, \ldots, \widehat{N}_{*N^{\mathrm{ca}}_{\max}}\Big)$. Since $\widetilde{N}_i = \widetilde{N}_{*i}$, $\forall i \in [1; N^{\mathrm{sc}}_{\max}]$, and $\widehat{N}_j = \widehat{N}_{*j}$, $\forall j \in [1; N^{\mathrm{ca}}_{\max}]$, and using definitions of $S(\widetilde{\Sigma})$ and $S_*(\widetilde{\Sigma})$, one obtains $H(\mathsf{x}) = (x_1, x_k)$ and $H_*(\mathsf{x}) = (x_{*1}, x_{*k})$, for some $k \in [N^{\mathrm{sc}}_{\min}; N^{\mathrm{sc}}_{\max}]$ (cf. Definitions $S_*(\widetilde{\Sigma})$ and $S(\widetilde{\Sigma})$). Since $(x_i, x_{*i}) \in R$, $\forall i \in [1; N^{\mathrm{sc}}_{\max}]$, one gets $\mathsf{d}_{Y_\tau}(H_\tau(x_i), H_{\mathsf{q}}(x_{*i})) \leq \varepsilon$, $\forall i \in [1; N^{\mathrm{sc}}_{\max}]$. Therefore, $\mathbf{d}_Y(H(\mathsf{x}), H_*(\mathsf{x}_*)) = \max\{\mathsf{d}_{Y_\tau}(H_\tau(x_1), H_{\mathsf{q}}(x_{*1})), \mathsf{d}_{Y_\tau}(H_\tau(x_k), H_{\mathsf{q}}(x_{*k}))\} \leq \varepsilon$ and condition (ii) in Definition 3.2 is satisfied.

Let us now show that condition (iii) in Definition 3.2 holds. Consider any $(\mathsf{x}, \mathsf{x}_*) \in \widetilde{R}$, where $\mathsf{x} = \Big(x_1, \ldots, x_{N^{\mathrm{sc}}_{\max}}, \upsilon_1, \ldots, \upsilon_{N^{\mathrm{ca}}_{\max}}, \widetilde{N}_1, \ldots, \widetilde{N}_{N^{\mathrm{sc}}_{\max}}, \widehat{N}_1, \ldots, \widehat{N}_{N^{\mathrm{ca}}_{\max}}\Big)$, $\mathsf{x}_* = \Big(x_{*1}, \ldots, x_{*N^{\mathrm{sc}}_{\max}}, u_{*1}, \ldots, u_{*N^{\mathrm{ca}}_{\max}}, \widetilde{N}_{*1}, \ldots, \widetilde{N}_{*N^{\mathrm{sc}}_{\max}}, \widehat{N}_{*1}, \ldots, \widehat{N}_{*N^{\mathrm{ca}}_{\max}}\Big)$. Consider any $\upsilon \in U(\mathsf{x}) = U_\tau$. Using the relation R, there exist $u_* \in U_*(\mathsf{x}_*) = U_{\mathsf{q}}$ and $\dot{x}_* \in \mathbf{Post}_{u_*}(x_*)$ such that $(\xi_{x\upsilon}(\tau), \dot{x}_*) \in R$ for any $(x, x_*) \in R$. Now consider any $\mathsf{x}' = \Big(x', x_1, \ldots, x_{N^{\mathrm{sc}}_{\max}-1}, \upsilon, \upsilon_1, \ldots, \upsilon_{N^{\mathrm{ca}}_{\max}-1}, \widetilde{N}, \widetilde{N}_1, \ldots, \widetilde{N}_{N^{\mathrm{sc}}_{\max}-1}, \widehat{N}, \widehat{N}_1 \ldots, \widehat{N}_{N^{\mathrm{ca}}_{\max}-1}\Big) \in \mathbf{Post}_\upsilon(\mathsf{x}) \subseteq X$ for some $\widetilde{N} \in [N^{\mathrm{sc}}_{\min}; N^{\mathrm{sc}}_{\max}]$ and $\widehat{N} \in [N^{\mathrm{ca}}_{\min}; N^{\mathrm{ca}}_{\max}]$ where $x' = \xi_{x\upsilon_k}(\tau)$ for some given $k \in [N^{\mathrm{ca}}_{\min}; N^{\mathrm{ca}}_{\max}]$ (cf. Definition $S(\widetilde{\Sigma})$). Because of the relation R, there exists $x'_* \in \mathbf{Post}_{u_{*k}}(x_{*1})$ in $S_{\mathsf{q}}(\Sigma)$ such that $(x', x'_*) \in R$. Hence, due to the definition $S_*(\widetilde{\Sigma})$, one can choose $\mathsf{x}'_* = \Big(x'_*, x_{*1}, \ldots, x_{*\left(N^{\mathrm{sc}}_{\max}-1\right)}, u_*, u_{*1}, \ldots, u_{*\left(N^{\mathrm{ca}}_{\max}-1\right)}, \widetilde{N}, \widetilde{N}_1, \ldots, \widetilde{N}_{N^{\mathrm{sc}}_{\max}-1}, \widehat{N}, \widehat{N}_1, \ldots, \widehat{N}_{N^{\mathrm{ca}}_{\max}-1}\Big) \in \mathbf{Post}_{u_*}(\mathsf{x}_*) \subseteq X_*$. Due to the relation R, one can readily verify that $\mathbf{d}_{Y_\tau}(H_\tau(x'), H_{\mathsf{q}}(x'_*)) \leq \varepsilon$. Since $\mathbf{d}_{Y_\tau}(H_\tau(x_j), H_{\mathsf{q}}(x_{*j})) \leq \varepsilon$, $\forall j \in [1; N^{\mathrm{sc}}_{\max} - 1]$, one gets $\mathbf{d}_Y(H(\mathsf{x}'), H_*(\mathsf{x}'_*)) = \max\{\mathsf{d}_{Y_\tau}(H_\tau(x'), H_{\mathsf{q}}(x'_*)), \mathsf{d}_{Y_\tau}(H_\tau(x_k), H_{\mathsf{q}}(x_{*k}))\} \leq \varepsilon$, for some given[3] $k \in [N^{\mathrm{sc}}_{\min} - 1; N^{\mathrm{sc}}_{\max} - 1]$ (cf. Definitions $S_*(\widetilde{\Sigma})$ and $S(\widetilde{\Sigma})$). Hence, $(\mathsf{x}', \mathsf{x}'_*) \in \widetilde{R}$ implying that condition (iii) in Definition 3.2 holds, which completes the proof. ∎

[3]Note that if $N^{\mathrm{sc}}_{\min} = 0$, then $x_{*(-1)} = x'_*$ and $x_{-1} = x'$.