# Structural Operational Semantics and Modal Logic, Revisited

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**Abstract**

A previously introduced combination of the bialgebraic approach to structural operational semantics with coalgebraic modal logic is re-examined and improved in some aspects. Firstly, a more abstract, conceptual proof of the main compositionality theorem is given, based on an understanding of modal logic as a study of coalgebras in slice categories of adjunctions. Secondly, a more concrete understanding of the assumptions of the theorem is provided, where proving compositionality amounts to finding a syntactic distributive law between two collections of predicate liftings.

*Keywords:* structural operational semantics, modal logic, coalgebra

## 1 Introduction

Compositionality of process equivalences is an important issue in the theory of Structural Operational Semantics (SOS; see e.g. [1,7]). Compositionality proofs for specific languages are often tedious, therefore plenty of meta-results have been proved that guarantee the compositionality of various equivalences by subjecting operational specifications to certain syntactic restrictions, called formats.

The process of inducing well-behaved transition systems from SOS specifications has been explained at the abstraction level of coalgebras, in the bialgebraic framework of [28]. There, a well-known SOS format called GSOS was understood as a type of distributive laws between behaviour and syntax endofunctors. The fact that LTS bisimilarity on GSOS-induced specifications is compositional, was explained at that level of generality.

One way to extend that approach to equivalences other than bisimilarities is to understand them as logical equivalences for some modal logics, and use a general coalgebraic approach to modal logic as developed, e.g., in [4,5,10,12,16,18,19,23,27]. In [15,17], such a combination of the bialgebraic approach with coalgebraic modal

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logic was presented. To prove that an equivalence defined by some logic on a transition system induced from an SOS specification is a congruence, one needs to equip logical formulas with a coalgebraic behaviour, and exhibit a "logical distributive law" of logical syntax over that behaviour that reflects the distributive law modeling the SOS specification.

Looking from some perspective, neither mathematical economy nor practical usability of [15,17] was entirely satisfactory. Firstly, the proof of the main compositionality result, albeit elementary, involved plenty of diagram chasing and inductive proofs, and in general was not very illuminating. Second, perhaps more painful deficiency, was that no intuitive general understanding of coalgebraic behaviour for logical formulas was provided. Although logical distributive laws for some specific kinds of logical behaviour were presented in an appealing, SOS-like manner, no concrete understanding of such laws for other types of behaviour was found. Also, no guidelines to finding behaviour functors for logical formulas were given, other than wild guessing. Checking that a candidate logical distributive law was correct involved heavy calculations of complex natural transformations, far removed from common understanding of formulas and processes.

This paper is an attempt to remove these two deficiencies to some degree. First, a more abstract, conceptual proof of the main compositionality theorem of [15] is provided (and the theorem is mildly generalized in the process). To this end, the interpretation of modal logic in coalgebras is understood as a functor from the category of coalgebras to a slice category of an adjunction. The compositionality theorem then follows from lifting that functor to structures that involve process syntax, via an adjoint lifting theorem.

Secondly, a concrete understanding of the compositionality theorem is provided for the important example where both processes and formulas live in the category of sets. Coalgebraic behaviour for formulas is explained in terms of predicate liftings [22,27], and logical distributive laws are syntactic distributive laws between two collections of liftings. Proving compositionality of a logical equivalence amounts to finding a suitable collection of liftings for process syntax, together with a set of equations that involve those liftings. Since the notion of predicate lifting is well studied and understood, this formulation should hopefully aid the understanding of our bialgebraic approach to logical compositionality.

The paper is structured as follows. In Sections 2 and 3, the bialgebraic approach to SOS and coalgebraic modal logic are briefly recalled. Section 4 studies endofunctors, algebras and coalgebras in slice categories of adjunctions, and culminates in a proof of the main compositionality theorem. Section 5 provides a concrete interpretation of the theorem in terms of predicate liftings. Finally, Sections 6 explains how both deficiencies mentioned above persist to some degree in the present formulation. Some proofs, not essential for the main line of reasoning, are relegated to Appendix.

Parts of the paper might be of interest also to those readers who do not care much about compositionality or SOS. For an explanation of coalgebraic modal logic in terms of (co)algebras in slice categories of adjunctions, without any involvement of process syntax, it is enough to read Sections 3, 4.1 and 4.2.

The reader is expected to be acquainted with basic category theory ([21] is a standard reference) and with the coalgebraic approach to theory of systems [26].

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## 2 SOS and distributive laws

In the context of Structural Operational Semantics, transition systems of various kinds are defined by structural induction using inference rules, and have closed terms over some signature as states. For example, given a fixed set $A$ of labels, the set of rules

$$\frac{}{a \xrightarrow{a} 0} \qquad \frac{x \xrightarrow{a} x' \quad y \xrightarrow{a} y'}{x \otimes y \xrightarrow{a} x' \otimes y'} \tag{1}$$

(where $a$ ranges over $A$) inductively defines a labeled transition system (LTS) on the set of closed terms over the grammar:

$$t ::= 0 \mid a \mid t \otimes t \qquad (a \in A). \tag{2}$$

In [28], this situation was expressed in the coalgebraic setting with the use of distributive laws and bialgebras for them. For example, rules (1) define a natural transformation $\lambda : \Sigma B \Longrightarrow B\Sigma$ (see e.g. [17] for a gentle explanation of this construction), where $\Sigma$ is the polynomial endofunctor on **Set** corresponding to the grammar (2), and $B = (\mathcal{P}_\omega -)^A$, where $\mathcal{P}_\omega$ is the finite powerset endofunctor; $B$-coalgebras are image-finite $A$-labeled transition systems (LTSs).

For any endofunctors $\Sigma$ and $B$ on a category $\mathcal{C}$, a transformation as above, called a distributive law of $\Sigma$ over $B$, induces an endofunctor $\Sigma_\lambda$ on the category $B$-**coalg** of $B$-coalgebras, and an endofunctor $B_\lambda$ on the category $\Sigma$-**alg** of $\Sigma$-algebras, acting on objects as follows:

$$\Sigma_\lambda( X \xrightarrow{h} BX ) = \Sigma X \xrightarrow{\Sigma h} \Sigma BX \xrightarrow{\lambda_X} B\Sigma X$$
$$B_\lambda( \Sigma X \xrightarrow{g} X ) = \Sigma BX \xrightarrow{\lambda_X} B\Sigma X \xrightarrow{Bg} BX$$

and as $\Sigma$ (resp. $B$) on morphisms. Clearly $\Sigma_\lambda$ lifts $\Sigma$ and $B_\lambda$ lifts $B$ along the respective forgetful functors $U_B : B$-**coalg** $\to \mathcal{C}$ and $U^\Sigma : \Sigma$-**alg** $\to \mathcal{C}$.

It is easy to see that a $\Sigma_\lambda$-algebra, or a $B_\lambda$-coalgebra, consists of a $\Sigma$-algebra $g$ and a $B$-coalgebra $h$ with the same carrier, so that the diagram:

$$\begin{array}{ccc} \Sigma X \xrightarrow{g} X \xrightarrow{h} BX \\ \Sigma h \downarrow \qquad\qquad \uparrow Bg \\ \Sigma BX \xrightarrow{\lambda_X} B\Sigma X \end{array} \tag{3}$$

commutes. Such structures are called $\lambda$-bialgebras (with carrier $X$), and a $\lambda$-bialgebra morphism is a map in $\mathcal{C}$ between the respective carriers that is simultaneously an algebra morphism and a coalgebra morphism; this defines a category $\lambda$-**bialg** of $\lambda$-bialgebras. There is an isomorphism of categories and a commuting

square of forgetful functors:

$$\lambda\textbf{-bialg} \cong \Sigma_\lambda\textbf{-alg} \cong B_\lambda\textbf{-coalg} \qquad\qquad \begin{array}{ccc} \lambda\textbf{-bialg} & \xrightarrow{U^{\Sigma_\lambda}} & B\textbf{-coalg} \\ \scriptstyle{U_{B_\lambda}}\downarrow & & \downarrow\scriptstyle{U_B} \\ \Sigma\textbf{-alg} & \xrightarrow[U^\Sigma]{} & \mathcal{C}. \end{array} \qquad (4)$$

If $g$ is an initial $\Sigma$-algebra, then there is a unique $B$-coalgebra $h$ such that (3) commutes, defined as the unique $\Sigma$-algebra morphism from $g$ to $B_\lambda(g)$. The result is an initial $\lambda$-bialgebra, and if $\lambda$ corresponds to an SOS specification as in (1), then $h$ corresponds to the transition system induced by the specification.

Dually, if a final $B$-coalgebra exists, it extends uniquely to a final $\lambda$-bialgebra. This immediately implies:

**Proposition 2.1** *For any $\lambda$-bialgebra $\Sigma X \xrightarrow{g} X \xrightarrow{h} BX$, the unique coalgebra morphism from $h$ to the final $B$-coalgebra is a $\Sigma$-algebra morphism from $g$.*

When $\mathcal{C} = \textbf{Set}$, two elements $x, y \in X$ for a given coalgebra $h : X \to BX$ are called *observationally equivalent* if they are identified by some coalgebra morphism. In particular, if final $B$-coalgebras exist, observational equivalence on $h$ is the kernel relation of the final coalgebra morphism from $h$. Thus Proposition 2.1, applied to initial bialgebras, means that observational equivalence on the transition system induced by a specification is a congruence, if the specification corresponds to a distributive law $\lambda$. [2]

The practical applicability of Proposition 2.1 as stated here is rather limited, since few interesting examples actually correspond to $\lambda$ as above. Already in [28] more general laws were studied, involving the free pointed endofunctor $\text{Id} + \Sigma$ and the free monad $\Sigma^*$, and the cofree copointed endofunctor $\text{Id} \times B$ and cofree comonad $B^\omega$ (assuming they exist). In particular, distributive laws of $\Sigma^*$ over $\text{Id} \times B$ correspond bijectively [20] to natural transformations $\lambda : \Sigma(\text{Id} \times B) \implies B\Sigma^*$, and for $B = (\mathcal{P}_\omega -)^A$, these correspond [2] to SOS specifications in the well-studied GSOS format [3]. Similarly, distributive laws of $\text{Id} + \Sigma$ over $B^\omega$ correspond to a format called safe-ntree in [28]. Proposition 2.1 is proved without much change for each of these more expressive laws; in fact, one does not need to prove each case separately, as each type of laws in question induces distributive laws of the monad $\Sigma^*$ over the comonad $B^\omega$ along the lines of [20], and Proposition 2.1 works for such laws as well, with essentially the same proof.

In this paper, only simple distributive laws $\lambda : \Sigma B \implies B\Sigma$ are considered. This is mainly to simplify the presentation and save space in the technical development in Section 4. The general case of distributive laws of monads over comonads is dealt with in an entirely analogous manner, but with additional checks to ensure the existence of certain (co)free (co)monads and the compatibility of (co)units and (co)multiplications, and is better left to an extended version of this paper.

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[2]  In [28], the congruence result is proved for coalgebraic bisimilarity [26] rather than observational equivalence; for that, the additional assumption of $B$ preserving weak pullbacks was needed.

## 3 Coalgebraic modal logic

An abstract approach to modal logics for coalgebras, based on adjunctions of contravariant functors, has attracted considerable attention (e.g., [4,5,10,12,16,18,27]). Assume an adjunction $S^{\mathrm{op}} \dashv T : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, with the intuition that objects of $\mathcal{D}$ are sets (or structures) of formulas, and objects of $\mathcal{C}$ are sets (or structures) of processes or states. For an endofunctor $B$ on $\mathcal{C}$, a coalgebraic modal logic for $B$-coalgebras is given by an endofunctor $L$ on $\mathcal{D}$ (the logical syntax), and a natural transformation $\rho : LT \Longrightarrow TB^{\mathrm{op}}$ (called a *connection*). Under the assumption that an initial $L$-algebra $a : L\Phi \to \Phi$ exists (intuitively, it is an algebra of logical formulas), the interpretation of logic $(L, \rho)$ on a given coalgebra $h : X \to BX$ is obtained by transposing along the adjunction $S^{\mathrm{op}} \dashv T$ the unique $L$-algebra morphism $s$ from $a$ to $Th \circ \rho_X$, as in the diagram:

$$
\begin{array}{ccc}
& & LTX \xleftarrow{\;Ls\;} L\Phi \\
& & \quad\downarrow{\scriptstyle \rho_X} \qquad\quad \downarrow{\scriptstyle a} \\
BX & & TBX \qquad {\scriptstyle \cong} \\
\;\uparrow{\scriptstyle h} & & \quad\downarrow{\scriptstyle Th} \\
X \xrightarrow{\;s^{\flat}\;} S\Phi & & TX \xleftarrow{\; \bar{s}\;} \Phi
\end{array}
\tag{5}
$$

where the left part is drawn in $\mathcal{C}$ and the right part in $\mathcal{D}$.

Take for example $\mathcal{C} = \mathcal{D} = \mathbf{Set}$ and $S = T = 2^{-}$ (where $2 = \{\mathtt{tt}, \mathtt{ff}\}$), and $B = (\mathcal{P}_{\omega}-)^A$. The trace fragment of Hennessy-Milner logic for $B$-coalgebras (i.e. LTSs) has syntax described by the grammar $\phi ::= \top \mid \langle a \rangle \phi$ that corresponds to the endofunctor $L\Phi = 1 + A \times \Phi$, and its standard semantics corresponds to $\rho_X : L2^X \to 2^{BX}$ defined by:

$$
\rho_X(\top)(b) = \mathtt{tt} \text{ always} \tag{6}
$$
$$
\rho_X(\langle a \rangle \phi)(b) = \mathtt{tt} \iff \exists y \in b(a).\ \phi(y) = \mathtt{tt}
$$

for any $X$. It is straightforward to check that (the kernel relation of) the interpretation of this $(L, \rho)$ on a $B$-coalgebra is the trace equivalence on it.

When searching for a logic for a given $B$, one may often restrict attention to endofunctors of a certain shape, without losing any generality. We now briefly recall an analysis from [16]. Connections $\rho : LT \Longrightarrow TB^{\mathrm{op}}$ are in bijective correspondence with natural transformations $\tilde{\rho} : L \Longrightarrow TB^{\mathrm{op}}S^{\mathrm{op}}$ by $\tilde{\rho} = \rho S^{\mathrm{op}} \circ L\eta$, where $\eta : \mathrm{Id}_{\mathcal{D}} \to TS^{\mathrm{op}}$ is the unit of $S^{\mathrm{op}} \dashv T$. If one insists on $L$ being finitary, then $\tilde{\rho}$ factors through the finitary restriction of $TB^{\mathrm{op}}S^{\mathrm{op}}$. The latter is defined by a coend formula; without losing generality one may replace the coend with a coproduct and, for $\mathcal{D} = \mathbf{Set}$, require a natural transformation

$$
\tilde{\rho} : L \Longrightarrow \coprod_{n \in \mathbb{N}} TBSn \times (-)^n.
$$

For $S = T = 2^{-}$ and $n \in \mathbb{N}$, elements of $TBSn$ are functions $\beta : B(2^n) \to 2$, which we call n-ary *B-modalities*. (By Yoneda Lemma, these bijectively correspond to

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natural transformations $\beta^{\mathcal{Y}} : (2^-)^n \Longrightarrow 2^{B-}$, i.e., polyadic predicate liftings of [27].)
As a result, a finitary $L$ with $\rho$ can be presented as a collection of $B$-modalities: a family $(L_n)_{n\in\mathbb{N}}$ of sets $L_n \subseteq 2^{B2^n}$ represents the polynomial endofunctor

$$L = \coprod_{n\in\mathbb{N}} L_n \times (-)^n, \tag{7}$$

with $\rho : L(2^-) \Longrightarrow 2^{B-}$ defined by copairing all predicate liftings $\beta^{\mathcal{Y}} : (2^-)^n \Longrightarrow 2^{B-}$ for each $n \in \mathbb{N}$ and $\beta \in L_n$.

For example, the trace logic for $B = (\mathcal{P}_\omega -)^A$ is represented by the following collection of modalities: $L_0 = \{\top\}$, $L_1 = \{\langle a \rangle \mid a \in A\}$, $L_n = \emptyset$ for $n > 1$, where $\top : B1 \to 2$ and $\langle a \rangle : B2 \to 2$ are defined by:

$$\top(b) = \mathtt{tt} \text{ always}, \qquad \langle a \rangle(b) = \mathtt{tt} \iff \mathtt{tt} \in b(a). \tag{8}$$

We mention in passing that this approach to logics suffers from practical expressivity problems similar to those mentioned in Section 2. For example, although a version of finitary Hennessy-Milner logic [8] for bisimilarity can be defined this way, it is rather unwieldy, with infinitely many modalities of arbitrary arities (see [16]). This is because logics based on $S = T = 2^-$ lack in-built support for propositional connectives, which must then be encoded as parts of complex modalities.

One way to avoid this problem is to change the adjunction $S^{\mathrm{op}} \dashv T$ in question (see [12] for examples). Another way is to consider, by analogy to distributive laws and SOS, more general types of connections. For example, one can allow ones like $\rho : LT \Longrightarrow T(\mathrm{Id} \times B)^{op}$, whereby propositional connectives such as $\wedge$ can easily be defined as simple modalities. One can even consider connections such as $\rho : LT \Longrightarrow T(B^\omega)^{\mathrm{op}}$ to describe e.g. Hennessy-Milner logic for weak bisimilarity.

Again, in this paper only simple connections $\rho : LT \Longrightarrow TB$ are considered; the issue of coalgebraic modal logics based on more complex connections is left for a separate study.

## 4 Compositionality for logical equivalences

Our main technical goal is to modify Proposition 2.1 to deal with logical equivalences rather than with observational equivalence. We shall prove (in Theorem 4.6) that under certain assumptions, for any $\lambda$-bialgebra $\Sigma X \xrightarrow{g} X \xrightarrow{h} BX$, the interpretation of logic $(L, \rho)$ on $h$ is a $\Sigma$-algebra morphism from $g$. To formulate the theorem and its proof, we introduce some basic notions and results regarding (co)algebras on slice categories of adjunctions. To structure the development to some degree, we shall begin with slice categories of functors, and later see what additional structure the adjunction $S^{\mathrm{op}} \dashv T$ introduces.

### 4.1 Slice categories

For a functor $T : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, the *slice category* $(\mathcal{D} \downarrow T)$ has:

- as objects, triples $\langle \Phi, X, s \rangle$ with $\Phi \in \mathcal{D}$, $X \in \mathcal{C}$ and $s : \Phi \to TX$ in $\mathcal{D}$,

- as maps from $\langle \Phi, X, s \rangle$ to $\langle \Psi, Y, r \rangle$, pairs $(g, f)$ with $g : \Phi \to \Psi$ in $\mathcal{D}$ and $f : Y \to X$ in $\mathcal{C}$ such that

$$
\begin{array}{ccc}
\Phi & \xrightarrow{\;s\;} & TX \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Tf} \\
\Psi & \xrightarrow{\;r\;} & TY
\end{array}
\tag{9}
$$

commutes.

There are obvious projection functors, denoted $\Pi_1 : (\mathcal{D} \downarrow T) \to \mathcal{D}$ and $\Pi_2 : (\mathcal{D} \downarrow T) \to \mathcal{C}^{\mathrm{op}}$.

If an initial object $0$ in $\mathcal{D}$ exists, it determines a full embedding $0^{\to} : \mathcal{C}^{\mathrm{op}} \to (\mathcal{D} \downarrow T)$ defined on objects by $0^{\to}(X) = \langle 0, X, 0 : 0 \to TX \rangle$, where the arrow $0$ is unique by initiality, and on arrows by $0^{\to}(g) = \langle \mathrm{id}_0, g \rangle$. It is easy to verify that:

**Proposition 4.1** $0^{\to}$ *is left adjoint to* $\Pi_2 : (\mathcal{D} \downarrow T) \to \mathcal{C}^{\mathrm{op}}$, *and the unit of the adjunction is the identity natural transformation.* $\qquad\square$

**Sliced endofunctors.** Assume endofunctors $B : \mathcal{C} \to \mathcal{C}$ and $L : \mathcal{D} \to \mathcal{D}$, and a natural transformation $\rho : LT \implies TB^{\mathrm{op}}$.

These ingredients define an endofunctor on $(\mathcal{D} \downarrow T)$, denoted $\widehat{\rho}$, as follows:

- on objects, $\widehat{\rho} \langle \Phi, X, r \rangle = \langle L\Phi, BX, \rho_X \circ Lr \rangle$
- on maps, $\widehat{\rho} \langle g, f \rangle = \langle Lg, Bf \rangle$.

It is easy to check that this is well-defined and functorial. Clearly $\widehat{\rho}$ lifts $L$ along $\Pi_1$, in the sense that $\Pi_1 \circ \widehat{\rho} = L \circ \Pi_1$. Similarly, $\widehat{\rho}$ lifts $B^{\mathrm{op}}$ along $\Pi_2$.

Endofunctors on $(\mathcal{D} \downarrow T)$ that arise in this way will be called *sliced (by $\rho$)*. Not every endofunctor on $(\mathcal{D} \downarrow T)$ is sliced in general, even if $T$ is well-behaved (for a counterexample, see the Appendix). However:

**Proposition 4.2** *Consider an endofunctor* $K : (\mathcal{D} \downarrow T) \to (\mathcal{D} \downarrow T)$ *such that for some* $L : \mathcal{D} \to \mathcal{D}$ *and* $B : \mathcal{C} \to \mathcal{C}$, *$K$ lifts $L$ along $\Pi_1$ and $B^{op}$ along $\Pi_2$. Then $K$ is sliced in a unique way.*

**Proof.** See the Appendix. $\qquad\square$

It immediately follows that sliced endofunctors are closed under composition. However, a more direct proof is possible: for $\rho : LT \implies TB^{\mathrm{op}}$ and $\rho' : L'T \implies T(B')^{\mathrm{op}}$, it is easy to check that the composite endofunctor $\widehat{\rho}\widehat{\rho'}$ is sliced by:

$$
\rho(B')^{\mathrm{op}} \circ L\rho' : LL'T \implies T(BB')^{\mathrm{op}}.
\tag{10}
$$

**Sliced natural transformations.** Assume connections $\rho : LT \implies TB^{\mathrm{op}}$ and $\rho' : L'T \implies TB'^{\mathrm{op}}$. Any two natural transformations $\alpha : L \implies L'$ and $\beta : B' \implies B$ such that

$$
\begin{array}{ccc}
LTS & \overset{\rho}{\Longrightarrow} & TB^{op} \\
{\scriptstyle \alpha T}\Downarrow & & \Downarrow{\scriptstyle T\beta^{op}} \\
L'T & \underset{\rho'}{\Longrightarrow} & TB'^{op}
\end{array}
\tag{11}
$$

commutes, give rise to a natural transformation $\alpha \circledast \beta : \widehat{\rho} \Longrightarrow \widehat{\rho'}$ defined by:

$$\alpha \circledast \beta_{\langle \Phi, X, s \rangle} = \langle \alpha_\Phi, \beta_X \rangle. \tag{12}$$

Not every transformation between sliced endofunctors is of this form (for a counterexample, see the Appendix). However, in Section 4.2 we shall show that this is the case if $T$ has a left adjoint.

**Algebras.** Given a connection $\rho : LT \Longrightarrow TB^{\mathrm{op}}$, a $\widehat{\rho}$-algebra is, equivalently, an $L$-algebra $g : L\Phi \to \Phi$ in $\mathcal{D}$, a $B$-coalgebra $h : X \to BX$ in $\mathcal{C}$, and a map $s : \Phi \to TX$, such that the diagram

$$
\begin{array}{ccc}
L\Phi \xrightarrow{\ Ls\ } LTX \xrightarrow{\ \rho_X\ } TBX \\
{\scriptstyle g}\Big\downarrow \qquad\qquad\qquad\quad \Big\downarrow{\scriptstyle Th} \\
\Phi \xrightarrow{\qquad\qquad s \qquad\qquad} TX
\end{array}
\tag{13}
$$

commutes in $\mathcal{D}$. Moreover, $\widehat{\rho}$-algebra morphisms are easily seen to be pairs of an $L$-algebra morphism and a $B$-coalgebra morphisms. In particular, there are evident projection functors $\Pi_1 : \widehat{\rho}\textbf{-alg} \to L\textbf{-alg}$ and $\Pi_2 : \widehat{\rho}\textbf{-alg} \to (B\textbf{-coalg})^{\mathrm{op}}$.

Let us pause for a moment to reflect on the meaning of $\widehat{\rho}$-algebras: they are $B$-coalgebras $h$ (systems) together with $L$-algebras $g$ (logical theories) interpreted in them (via $s$). For example, if $\mathcal{C} = \mathcal{D} = \textbf{Set}$ and $S = T = 2^-$, the function $s : \Phi \to TX$ in the carrier of a $\widehat{\rho}$-algebra is just a relation between $\Phi$ and $X$. For $B = (\mathcal{P}_\omega-)^A$ and $(L, \rho)$ as in (6), the equivalence relation on $X$ defined by this relation is always contained in trace equivalence on $h$, and coincides with it if $g$ is initial. Morphisms of $\widehat{\rho}$-algebras reflect these equivalence relations, implicitly present in their carriers. This suggests that when one wants to study coalgebras "up to" some logical equivalence, and when the task of finding an explicit coalgebraic presentation of these "up to" structures (such as in [11]) seems difficult or simply not worthwhile, one may try to resort to implicit modeling of logical equivalences by theories interpreted in coalgebras; this view is advocated e.g. in [23], and the present paper may be considered as an example application of it. One may argue that, just as structural operational semantics is a study of coalgebra in categories of algebras, coalgebraic modal logic is a study of (co)algebra in slice categories.

Back to the formal development: an alternative reading of (13) is that $s$ is a $L$-algebra morphism. In other words, $\widehat{\rho}$-algebras are morphisms between $L$-algebras of a certain shape. To formalize this, observe that $\rho$ induces a functor $\overline{T} : (B\textbf{-coalg})^{\mathrm{op}} \to L\textbf{-alg}$ defined by $\overline{T}( X \xrightarrow{\ h\ } BX ) = LTX \xrightarrow{\ \rho_X\ } TBX \xrightarrow{\ Th\ } TX$ on $B$-coalgebras, and as $T$ on $B$-coalgebra morphisms.

**Proposition 4.3** $\widehat{\rho}\textbf{-alg} \cong (L\textbf{-alg} \downarrow \overline{T})$.

**Proof.** We have already essentially noticed the correspondence on objects; morphisms are equally easy. $\qquad\square$

**Corollary 4.4** *If an initial $L$-algebra exists, then the projection functor $\Pi_2 : \widehat{\rho}\textbf{-alg} \to (B\textbf{-coalg})^{\mathrm{op}}$ has a left adjoint, and the unit of the adjunction is the identity natural transformation.*

**Proof.** Use Proposition 4.3 and apply Proposition 4.1. Given an initial $L$-algebra $a$, the left adjoint will be denoted $a^{\rightarrow}$. □

Note that the composition of $a^{\rightarrow}$ with the forgetful functor from $\widehat{\rho}$-**alg**:

$$(B\text{-}\mathbf{coalg})^{\mathrm{op}} \underset{\widehat{\Pi}_2}{\overset{a^{\rightarrow}}{\underset{\perp}{\rightleftarrows}}} \widehat{\rho}\text{-}\mathbf{alg} \xrightarrow{U^{\widehat{\rho}}} (\mathcal{D} \downarrow T)$$

corresponds almost entirely to the interpretation of coalgebraic modal logic in $B$-coalgebras, as constructed in (5). Indeed, the only step missing in this functorial presentation is the transposition of the semantic map $s$ from the initial $L$-algebra. For this final step, obviously, it is crucial that the functor $T$ has a left adjoint; we shall now proceed to develop our theory further with this additional assumption.

### 4.2  Slice categories of adjunctions

In this section, we shall assume that $T : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ has a left adjoint $S^{\mathrm{op}} : \mathcal{D} \to \mathcal{C}^{\mathrm{op}}$. The unit and counit of the adjunction $S^{\mathrm{op}} \dashv T$ will be denoted $\eta : \mathrm{Id} \Longrightarrow TS^{\mathrm{op}}$ and $\epsilon : S^{\mathrm{op}}T \Longrightarrow \mathrm{Id}$ respectively. Obviously then $T^{\mathrm{op}} : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ is left adjoint to $S : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}$, and $\eta^{\mathrm{op}} : T^{\mathrm{op}}S \Longrightarrow \mathrm{Id}$ and $\epsilon^{\mathrm{op}} : \mathrm{Id} \Longrightarrow ST^{\mathrm{op}}$ are the counit and the unit of the adjunction $T^{\mathrm{op}} \dashv S$. This adjoint situation means that there is a bijection

$$\mathcal{C}(X, S\Phi) \cong \mathcal{D}(\Phi, TX)$$

natural in $X \in \mathcal{C}$ and $\Phi \in \mathcal{D}$; we shall abuse notation and denote both sides of this bijection by $-^{\flat}$. A defining property of adjunctions is the isomorphism of slice categories: $(\mathcal{D} \downarrow T) \cong (\mathcal{C} \downarrow S)^{\mathrm{op}}$ (the isomorphism maps an object $\langle \Phi, X, s \rangle$ to $\langle X, \Phi, s^{\flat} \rangle$).

**Coalgebraic modal logic as a functor.** One immediate consequence of the adjunction assumption is that one can represent the entire modal logic interpretation construction (5) as a functor from the category of $B$-coalgebras:

$$(B\text{-}\mathbf{coalg})^{\mathrm{op}} \underset{\widehat{\Pi}_2}{\overset{g^{\rightarrow}}{\underset{\perp}{\rightleftarrows}}} \widehat{\rho}\text{-}\mathbf{alg} \xrightarrow{U^{\widehat{\rho}}} (\mathcal{D} \downarrow T) \cong (\mathcal{C} \downarrow S)^{\mathrm{op}} \tag{14}$$

However, the most useful consequences of that assumption appear when one decides to study *co*algebras for sliced endofunctors on $(\mathcal{D} \downarrow T)$.

**Coalgebras.** In the situation considered in Section 4.1, categories of coalgebras for sliced endofunctors have, in general, considerably less structure than those of algebras.

Consider endofunctors $\Sigma$ on $\mathcal{C}$ and $\Gamma$ on $\mathcal{D}$, and a connection $\zeta : \Gamma T \Longrightarrow T\Sigma^{\mathrm{op}}$ as in Section 4.1. A $\widehat{\zeta}$-coalgebra is a $\Gamma$-coalgebra $h : \Phi \to \Gamma\Phi$ in $\mathcal{D}$, a $\Sigma$-algebra $g : \Sigma X \to X$ in $\mathcal{C}$, and a map $s : \Phi \to TX$, such that the diagram

$$\begin{array}{ccc}
\Gamma\Phi & \xrightarrow{\Gamma s} \Gamma TX \xrightarrow{\zeta_X} & T\Sigma X \\
h \uparrow & & \uparrow Tg \\
\Phi & \xrightarrow{\quad s \quad} & TX
\end{array} \tag{15}$$

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commutes in $\mathcal{D}$, and a $\widehat{\zeta}$-coalgebra morphisms is a pair of a $\Gamma$-coalgebra morphism and a $\Sigma$-algebra morphism. This gives projection functors $\Pi_1 : \widehat{\zeta}\text{-}\mathbf{coalg} \to \Gamma\text{-}\mathbf{coalg}$ and $\Pi_2 : \widehat{\zeta}\text{-}\mathbf{coalg} \to (\Sigma\text{-}\mathbf{alg})^{\mathrm{op}}$.

In general, contrary to the situation of sliced algebras, the diagram (15) cannot be read as a coalgebra morphism (in [23], it was called a "twisted coalgebra morphism"). As a result, no property analogous to Proposition 4.3 holds for $\widehat{\zeta}$-coalgebras in general. However, additional structure appears when we assume a left adjoint $S^{\mathrm{op}} \dashv T$. Indeed, then connections $\zeta : \Gamma T \Longrightarrow T\Sigma^{\mathrm{op}}$ are in bijective correspondence with their *adjoint mates* [14] $\zeta^\star : \Sigma S \Longrightarrow S\Gamma^{\mathrm{op}}$, defined by transposing $\zeta S^{\mathrm{op}} \circ \Gamma\eta : \Gamma \Longrightarrow T\Sigma^{\mathrm{op}}S^{\mathrm{op}}$. It is straightforward to check that $\widehat{\zeta^\star}$ coincides with $(\widehat{\zeta})^{\mathrm{op}}$ along the isomorphism $(\mathcal{C} \downarrow S) \cong (\mathcal{D} \downarrow T)^{\mathrm{op}}$. In particular, this implies an isomorphism

$$\widehat{\zeta}\text{-}\mathbf{coalg} \cong (\widehat{\zeta^\star}\text{-}\mathbf{alg})^{\mathrm{op}}. \tag{16}$$

**Natural transformations are sliced.** Finally, the fact that $T$ has a left adjoint implies that all natural transformations between sliced endofunctors are sliced. Indeed, consider any $\rho : \Gamma T \Longrightarrow T\Sigma^{\mathrm{op}}$ and $\rho' : \Gamma'T \Longrightarrow T\Sigma'^{\mathrm{op}}$.

**Proposition 4.5** *If $S^{\mathrm{op}} \dashv T$ then natural transformations $\kappa : \widehat{\rho} \Longrightarrow \widehat{\rho'}$ are in one-to-one correspondence with pairs $\langle \alpha : \Gamma \Longrightarrow \Gamma', \beta : \Sigma' \Longrightarrow \Sigma \rangle$ such that $T\beta^{\mathrm{op}} \circ \rho = \rho' \circ \alpha T$ as in (11).*

**Proof.** (12) shows how to define $\kappa$ from $\alpha$ and $\beta$. For the other direction, see the Appendix. $\square$

### 4.3 Sliced distributive laws and compositionality

We now proceed to the study of bialgebras for distributive laws between sliced endofunctors.

**Sliced distributive laws.** For an adjunction $S^{\mathrm{op}} \dashv T : \mathcal{C}^{\mathrm{op}} \to D$, consider endofunctors $B, \Sigma$ on $\mathcal{C}$ and $L, \Gamma$ on $\mathcal{D}$, together with connections $\rho : LT \to TB^{\mathrm{op}}$ and $\zeta : \Gamma T \Longrightarrow T\Sigma^{\mathrm{op}}$ that define sliced endofunctors $\widehat{\rho}, \widehat{\zeta}$ on $(\mathcal{D} \downarrow T)$.

Now assume a distributive law of $\widehat{\rho}$ over $\widehat{\zeta}$, i.e., a natural transformation $\kappa : \widehat{\rho}\widehat{\zeta} \Longrightarrow \widehat{\zeta}\widehat{\rho}$. By (10), both $\widehat{\rho}\widehat{\zeta}$ and $\widehat{\zeta}\widehat{\rho}$ are sliced, and further by Proposition 4.5, $\kappa$ is of the form $\kappa = \chi \circledast \lambda$ (see (12)) for some distributive laws $\chi : L\Gamma \Longrightarrow \Gamma L$ and $\lambda : \Sigma B \Longrightarrow B\Sigma$ such that the hexagon of natural transformations commutes (cf. (11)):

$$
\begin{array}{ccccc}
L\Gamma T & \xLongrightarrow{L\zeta} & LT\Sigma^{\mathrm{op}} & \xLongrightarrow{\rho\Sigma^{\mathrm{op}}} & T(B\Sigma)^{\mathrm{op}} \\
{\scriptstyle \chi T}\Big\Vert & & & & \Big\Vert{\scriptstyle T\lambda^{\mathrm{op}}} \\
\Gamma LT & \xLongrightarrow[\Gamma\rho]{} & \Gamma TB^{\mathrm{op}} & \xLongrightarrow[\zeta B^{\mathrm{op}}]{} & T(\Sigma B)^{\mathrm{op}}.
\end{array}
\tag{17}
$$

**Sliced bialgebras.** As in Section 2, the law $\kappa$ defines endofunctors $\widehat{\rho}_\kappa$ on $\widehat{\zeta}\text{-}\mathbf{coalg}$ and $\widehat{\zeta}_\kappa$ on $\widehat{\rho}\text{-}\mathbf{alg}$, with an isomorphism of categories and a commuting square of

forgetful functors (cf. (4)):

$$\kappa\text{-}\mathbf{bialg} \cong \widehat{\rho}_\kappa\text{-}\mathbf{alg} \cong \widehat{\zeta}_\kappa\text{-}\mathbf{coalg} \qquad \begin{array}{ccc} \kappa\text{-}\mathbf{bialg} & \xrightarrow{U\widehat{\rho}_\kappa} & \widehat{\zeta}\text{-}\mathbf{coalg} \\ {\scriptstyle U_{\widehat{\zeta}_\kappa}}\downarrow & & \downarrow{\scriptstyle U_{\widehat{\zeta}}} \\ \widehat{\rho}\text{-}\mathbf{alg} & \xrightarrow{U\widehat{\rho}} & (\mathcal{D}\downarrow T). \end{array} \qquad (18)$$

To convey some intuition, it might be useful to provide a more concrete descriptions of $\kappa$-bialgebras. Each of these consists of a $\chi$-bialgebra $L\Phi \xrightarrow{k} \Phi \xrightarrow{l} \Gamma\Phi$, a $\lambda$-bialgebra $\Sigma X \xrightarrow{g} X \xrightarrow{h} BX$ and an arrow $s : \Phi \to TX$ in $\mathcal{D}$, such that the diagram:

$$\begin{array}{ccccc} L\Phi & \xrightarrow{\quad k \quad} & \Phi & \xrightarrow{\quad l \quad} & \Gamma\Phi \\ {\scriptstyle Ls}\downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \Gamma s} \\ LTX & \xrightarrow[\rho_X]{} TBX \xrightarrow[Th]{} & TX & \xrightarrow[Tg]{} T\Sigma X \xleftarrow[\zeta_X]{} & \Gamma TX \end{array} \qquad (19)$$

commutes. Morphisms of $\kappa$-bialgebras are pairs of a $\chi$- and a $\lambda$-bialgebra morphisms; in particular, there is an evident projection functor, which we will denote $\overline{\Pi_2} : \kappa\text{-}\mathbf{bialg} \to (\lambda\text{-}\mathbf{bialg})^{\mathrm{op}}$.

**Lifting coalgebraic modal logic.** Our immediate goal now is to exhibit a left adjoint to $\overline{\Pi_2}$. Note that the bottom row of (19) is not a $\chi$-bialgebra, so $\kappa$-**bialg** is not easily a slice category and the simple tactic of using Proposition 4.1 cannot be used. Instead, adjoint lifting can be used in the following way.

Since $\kappa$ acts as $\lambda$ on $\mathcal{C}$-components (see (12)), it is straightforward to check that $\widehat{\zeta}_\kappa$ acts as $\Sigma_\lambda$ on the $B$-coalgebra components of $\widehat{\rho}$-algebras and their morphisms; formally, $\Pi_2 \circ \widehat{\zeta}_\kappa = (\Sigma_\lambda)^{\mathrm{op}} \circ \Pi_2 : \widehat{\rho}\text{-}\mathbf{alg} \to (B\text{-}\mathbf{coalg})^{\mathrm{op}}$. This defines a lifting of $\Pi_2$ to a functor from $\widehat{\zeta}_\kappa\text{-}\mathbf{coalg}$ to $(\Sigma_\lambda\text{-}\mathbf{alg})^{\mathrm{op}}$ as in (A.1); it is straightforward to check that this lifted functor coincides with $\overline{\Pi_2}$, which justifies its name. We can now apply Proposition A.1 to

$$\begin{array}{ccc} (\lambda\text{-}\mathbf{bialg})^{\mathrm{op}} & \xleftarrow{\overline{\Pi_2}} & \kappa\text{-}\mathbf{bialg} \\ {\scriptstyle (U^{\Sigma_\lambda})^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle U_{\widehat{\zeta}_\kappa}} \\ (B\text{-}\mathbf{coalg})^{\mathrm{op}} & \xleftarrow{\Pi_2} & \widehat{\rho}\text{-}\mathbf{alg} \end{array}$$

and obtain a left adjoint $\overline{a^\rightarrow} \dashv \overline{\Pi_2}$. Combined with (18) and (16), this completes a lifting of the coalgebraic modal logic semantics (14) as in the diagram:

$$\begin{array}{ccccccc} (\lambda\text{-}\mathbf{bialg})^{\mathrm{op}} & \underset{\overline{\Pi_2}}{\overset{\overline{a^\rightarrow}}{\rightleftarrows}} {\scriptstyle\perp} & \kappa\text{-}\mathbf{bialg} & \xrightarrow{U\widehat{\rho}_\kappa} & \widehat{\zeta}\text{-}\mathbf{coalg} & \cong & (\widehat{\zeta^\star}\text{-}\mathbf{alg})^{\mathrm{op}} \\ {\scriptstyle (U^{\Sigma_\lambda})^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle U_{\widehat{\zeta}_\kappa}} & & \downarrow{\scriptstyle U_{\widehat{\zeta}}} & & \downarrow{\scriptstyle (U^{\widehat{\zeta^\star}})^{\mathrm{op}}} \\ (B\text{-}\mathbf{coalg})^{\mathrm{op}} & \underset{\overline{\Pi_2}}{\overset{a^\rightarrow}{\rightleftarrows}} {\scriptstyle\perp} & \widehat{\rho}\text{-}\mathbf{alg} & \xrightarrow{U\widehat{\rho}} & (\mathcal{D}\downarrow T) & \cong & (\mathcal{C}\downarrow S)^{\mathrm{op}} \end{array} \qquad (20)$$

Note that, by Corollary 4.4 and by the remark after Proposition A.1, the monad $\overline{\Pi_2} \circ \overline{a^\rightarrow}$ is (naturally isomorphic to) identity. This, together with the evident com-

muting square of forgetful and projection functors:

$$
\begin{array}{ccc}
(\lambda\text{-}\mathbf{bialg})^{\mathrm{op}} & \xrightarrow{(U_{B_\lambda})^{\mathrm{op}}} & (\Sigma\text{-}\mathbf{alg})^{\mathrm{op}} \\
\uparrow\overline{\Pi_2} & & \uparrow\Pi_2 \quad \overset{\Pi_1^{\mathrm{op}}}{\nwarrow} \\
\kappa\text{-}\mathbf{bialg} & \xrightarrow[U^{\widehat{\rho}_\kappa}]{} \widehat{\zeta}\text{-}\mathbf{coalg} & \cong \quad (\widehat{\zeta}^{\star}\text{-}\mathbf{alg})^{\mathrm{op}}
\end{array}
$$

means that the top row of (20) commutes with (the opposites of) forgetful functors from $\lambda$-**bialg** and $\widehat{\zeta}^{\star}$-**alg** to $\Sigma$-**alg**. Thus we arrive at the conclusion that for any $\lambda$-bialgebra $\Sigma X \xrightarrow{g} X \xrightarrow{h} BX$, the interpretation of logic $\rho$ on $h$ is a carrier of a $\widehat{\zeta}^{\star}$-algebra and a $\Sigma$-algebra morphism from $g$. Note that neither $\zeta$ nor $\chi$ is mentioned in this conclusion, so the most useful way to state this is:

**Theorem 4.6** *For any $S^{\mathrm{op}} \dashv T$, $\Sigma$, $B$, $\lambda$, $L$ and $\rho$ as above, if a $\Gamma$, $\zeta$ and $\kappa = \chi \circledast \lambda$ as above exist, then for any $\lambda$-bialgebra $\Sigma X \xrightarrow{g} X \xrightarrow{h} BX$, the interpretation of logic $\rho$ on $h$ is a $\Sigma$-algebra morphism from $g$.* $\square$

When applied to initial $\lambda$-bialgebras, Theorem 2 of [15] is obtained.

# 5 Logical distributive laws over Set

Theorem 4.6 can be used to prove that a logical equivalence (defined by syntax $L$ and semantics $\rho$) on a transition system ($B$-coalgebra) induced by a structural operational specification (defined by $\lambda$) is a congruence (with respect to syntax $\Sigma$). To use the theorem, one needs to find three additional ingredients: an endofunctor $\Gamma$ and natural transformations $\zeta$ and $\chi$ such that (17) commutes. So far we have provided no intuitive meaning of these ingredients. This is the purpose of this section, where we restrict attention to the dual adjunction $\mathcal{C} = \mathcal{D} = \mathbf{Set}$, $S = T = 2^{-}$.

## 5.1 *Distributive laws and predicate liftings*

The search for $\Gamma$ and $\zeta$ for a given $\Sigma$ is entirely analogous to the search for modal logics $L$ and $\rho$ for a given $B$, as described in Section 3. One may therefore restrict attention to functors of the form

$$
\Gamma = \coprod_{n \in \mathbb{N}} \Gamma_n \times (-)^n, \tag{21}
$$

where $\Gamma_n \subseteq 2^{\Sigma 2^n}$. Recall that we may safely assume that $L$ is of a similar form (7). This means that, once $\Gamma$ with $\zeta$ were chosen, the last missing ingredient $\chi$ for Theorem 4.6 is a distributive law $\chi$ between polynomial functors. Such laws can be presented as systems of equations, as follows.

Suppose $\Gamma$ and $L$ are as in (21) and (7), presented by families of $\Sigma$- and $B$-modalities $(\Gamma_n)_{n \in \mathbb{N}}$ and $(L_n)_{n \in \mathbb{N}}$ respectively. Then a distributive law $\chi : L\Gamma \Longrightarrow \Gamma L$

is equivalent to a family of equations of the form:

$$\beta(\sigma_1(x_{11}, \ldots, x_{1m_1}), \ldots, \sigma_n(x_{n1}, \ldots, x_{nm_n}))$$
$$=$$
$$\sigma(\beta_1(y_{11}, \ldots, y_{1l_1}), \ldots, \beta_k(y_{k1}, \ldots, y_{kl_k})),$$

$$(22)$$

where:

- $\beta \in L_n$, $\sigma_i \in \Gamma_{m_i}$, $\sigma \in \Gamma_k$ and $\beta_i \in L_{l_i}$,
- all variables $x_{ij}$ are distinct,
- every variable $y_{ij}$ occurs on the left side.

The latter two conditions determine a function $v : l \to m$, where $m = \sum_{i=1}^n m_i$ and $l = \sum_{i=1}^k l_i$ are arities of both sides of the equation.

To define a distributive law $\chi$, the family must contain exactly one equation for each combination $\beta, \sigma_1, \ldots, \sigma_n$ of a $B$-modality (of arity, say, $n$) and a sequence of $\Sigma$-modalities (of length $n$).

We shall now formulate the condition (17), necessary for the application of Theorem 4.6, in terms of modalities and equations. To this end, first note that each $\beta, \sigma_1, \ldots, \sigma_n$ as on the left hand side of (22), defines a $B\Sigma$-modality of arity $m = \sum_{i=1}^n m_i$, (where $\sigma_i \in \Gamma_{m_i}$), which, following [27], will be denoted $\beta \odot (\sigma_1, \ldots, \sigma_n) : B\Sigma(2^m) \to 2$. Moreover, the composite polynomial endofunctor $L\Gamma$ is represented by the collection of all such composite modalities, and the connection $\rho\Sigma^{\mathrm{op}} \circ L\zeta$ as in (17) (see also (10)) is obtained by copairing all the corresponding predicate liftings $(\beta \odot (\sigma_1, \ldots, \sigma_n))^{\mathcal{Y}}$. Similarly one can define a lifting $\sigma \odot (\beta_1, \ldots, \beta_k) : \Sigma B(2^l) \to 2$, from the right hand side of (22).

Since $L\Gamma$ is a polynomial functor, the condition (17) can be checked by cases, for each (left hand side of) equation (22). Each case amounts to checking that the following square of natural transformations commutes:

$$
\begin{array}{ccc}
(2^-)^m & \xrightarrow{(\beta \odot (\sigma_1, \ldots, \sigma_n))^{\mathcal{Y}}} & 2^{B\Sigma-} \\
{\scriptstyle (2^-)^v} \Big\Vert & & \Big\Vert {\scriptstyle 2^\lambda} \\
(2^-)^l & \xrightarrow[(\sigma \odot (\beta_1, \ldots, \beta_k))^{\mathcal{Y}}]{} & 2^{\Sigma B-}.
\end{array}
$$

By Yoneda lemma, this amounts to checking the equality of two $\Sigma B$-modalities of arity $m$:

$$(\beta \odot (\sigma_1, \ldots, \sigma_n)) \circ \lambda_{2^m} = (\sigma \odot (\beta_1, \ldots, \beta_k)) \circ \Sigma B(2^v) \qquad (23)$$

This condition can be intuitively explained as follows. For a fixed set $X$, if predicates on $X$ are substituted for variables $x_{i1}, \ldots, x_{im_i}$, then the expression $\sigma(x_{i1}, \ldots, x_{im_i})$ on the left hand side of (22) defines a predicate on $\Sigma X$; similarly, the entire left hand side defines a predicate on $B\Sigma X$ from a collection of $m$ predicates on $X$. Further, the right hand side (together with the function $v : l \to m$ implicit in the equation) defines a predicate on $\Sigma B X$. Now the condition (23) means that the former predicate coincides with the latter when precomposed with $\lambda$.

### 5.2  A toy example

Consider $\Sigma$, $B$ and $\lambda$ as in (2) and (1) in Section 2. Consider also, as $L$ and $\rho$, the trace equivalence defined in (6), represented by the collection of modalities given in (8). To apply Theorem 4.6 to infer the compositionality of trace equivalence for the language defined by (1), one needs to find a collection $(\Gamma_n)_{n \in \mathbb{N}}$ of $\Sigma$-modalities, and an collection of equations (22), such that the condition (23) holds for each equation.

As a first attempt, one might try the empty collection ($\Gamma_n = \emptyset$ for $n \in \mathbb{N}$), i.e., no $\Sigma$-modalities. There is only one left hand side of (22) to take care of: the 0-ary $B$-modality $\top$. Unfortunately, however, there are no possible right hand sides of (22) at all, therefore no equation for $\top$ can be written.

To amend this, one can include an "always true" $\Sigma$-modality $\mathbb{T} : \Sigma 1 \to 2$ to $\Gamma_0$, formally defined by $\mathbb{T}(t) = \mathtt{tt}$ always. Then one can write an equation for $\top$:

$$\top = \mathbb{T} \tag{24}$$

and the condition (23) holds. Unfortunately now there are more left hand sides to take care of: no appropriate equation can be written for $\langle a \rangle(\mathbb{T})$.

The latter expression denotes a 0-ary $B\Sigma$-modality that, intuitively, checks whether some $a$-successor of a process exists. To express a corresponding (along $\lambda$) $\Sigma B$-modality, one may add, for each $a \in A$, a new unary $\Sigma$-modality $a \vee [\otimes] : \Sigma 2 \to 2$ to $\Gamma_2$, formally defined by: $a \vee [\otimes](t) = \mathtt{tt} \iff t \in \{a \ , \ \mathtt{tt} \otimes \mathtt{tt}\}$, and write an equation:

$$\langle a \rangle \mathbb{T} = a \vee [\otimes](\langle a \rangle \top). \tag{25}$$

Intuitively, a process has an $a$-successor if and only if it is the process $a$ or it is of the form $p \otimes q$ such that both $p$ and $q$ have $a$-successors. Formally, the condition (23) holds for this equation.

However, there is a slight problem here: formally, the right hand side of this equation is not of the form allowed in (22), as $\langle a \rangle \top$ is not a modality used in $L$. A principled solution to this problem would be to allow composite $B$-modalities on the right sides of equations; i.e., consider distributive laws $\chi : L\Gamma \implies \Gamma L^*$, just as complex types of distributive laws are considered in the theory of SOS (see Section 2). Another solution is to simply add the missing (0-ary) $B$-modalities $\langle a \rangle \top$ to $L_0$ and proceed to find further equations. Changing a logic to prove its compositionality is an awkward step, but in this case it does not cause any serious harm, as the logical equivalence of the resulting logic is still trace equivalence. Formally, one then needs to provide suitable equations with 0-ary modalities $\langle a \rangle \top$ on the left hand side, but this is now straightforward:

$$\langle a \rangle \top = a \vee [\otimes](\langle a \rangle \top). \tag{26}$$

To complete the picture, one still needs to come up with equations for left hand sides such as $\langle a \rangle(b \vee [\otimes]x)$. This is solved by adding yet another, unary modality $[\otimes]$ to $\Gamma_1$, defined by $[\otimes](t) = \mathtt{tt} \iff t = \mathtt{tt} \otimes \mathtt{tt}$, with equations

$$\langle a \rangle(b \vee [\otimes]x)) = [\otimes]\langle a \rangle x, \qquad \langle a \rangle[\otimes]x = [\otimes]\langle a \rangle x.$$

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These, together with (24–26), form a complete family of equations for our chosen $\Sigma$-modalities:

$$\Gamma_0 = \{\mathbb{T}\} \qquad \Gamma_1 = \{[\otimes]\} \cup \{a \vee [\otimes] \mid a \in A\} \qquad \Gamma_n = \emptyset \text{ for } n > 1$$

and the condition (23) holds for each equation, hence we can use Theorem 4.6 to conclude that trace equivalence is compositional for (1). The same result was used as an example in [15,17]; however, our crude understanding of $\Gamma$ and $\chi$ there resulted in unnecessarily rich logical behaviours and complicated distributive laws.

### 5.3  Compositionality for expressive logics

An important question about the robustness of our approach to compositionality is whether Theorem 4.6 is a generalization of Proposition 2.1, i.e., whether it covers observational equivalence without any loss of generality. Under mild conditions (such as finitarity of $B$) studied in [16], observational equivalence on $B$-coalgebras is a logical equivalence for some logic $(L, \rho)$ (such logic is called expressive). If this is the case, then the conclusion of Proposition 2.1 is a special case of the conclusion of Theorem 4.6. However, is there an expressive logic that satisfies the assumptions of Theorem 4.6?

We shall now give a partial positive answer to this question: we restrict attention to $\mathcal{C} = \mathcal{D} = \mathbf{Set}$ and $S = T = 2^-$, polynomial process syntax functors $\Sigma$, and finitary $B$ that preserve finite sets. In the general case the question is left open.

For our special case, observational equivalence for $B$-coalgebras is defined by the expressive logic $(L, \rho)$ presented by the collection of *all* finitary $B$-modalities, i.e., by $L_n = 2^{B2^n}$ for $n \in \mathbb{N}$. We shall now show that this logic satisfies the assumptions of Theorem 4.6 when one takes $\Gamma$ and $\zeta$ presented by all $\Sigma$-modalities, i.e., $\Gamma_n = 2^{\Sigma 2^n}$ for $n \in \mathbb{N}$.

To this end we need, for every $B$-modality $\beta \in L_n$ and $\Sigma$-modalities $\sigma_1, \ldots, \sigma_n$, to present the $\Sigma B$-modality $(\beta \odot (\sigma_1, \ldots, \sigma_n)) \circ \lambda_{2^m}$ (where $m$ is the sum of arities of the $\sigma_i$) in the form

$$(\sigma \odot (\beta_1, \ldots, \beta_k)) \circ \Sigma B2^v \tag{27}$$

for some $\sigma \in \Gamma_k$ and $\beta_1, \ldots, \beta_k$ $B$-modalities with sum of arities $l$, and $v : l \to m$ a function between arities. Under our assumptions this can be done without any analysis of $\beta$, $\sigma_i$ or $\lambda$, by the following result:

**Proposition 5.1** *If $\Sigma$ is polynomial and $B$ preserves finite sets then every $\Sigma B$-modality of arity $m$ can be decomposed as in* (27).

**Proof.** Let $\Sigma = \coprod_{i \in I} (-)^{n_i}$. Consider any $\gamma : \Sigma B(2^m) \to 2$. Put $k = |B(2^m)|$ (note that $B(2^m)$ is finite). For any $b \in B(2^m)$, define $\beta_b : B(2^m) \to 2$ by $\beta_b(b') = \mathtt{tt} \iff b = b'$. Then define $\sigma : \Sigma(2^k) \to 2$ by:

$$\sigma(\iota_i(p_1, \ldots, p_{n_i})) = \mathtt{tt}$$
$$\iff$$
$$\forall b_1, \ldots, b_{n_i} \in B2^m. \ \big((\forall j = 1..n_i. \ p_j(b_j) = \mathtt{tt}) \implies \gamma(\iota_i(b_1, \ldots, b_{n_i})) = \mathtt{tt}\big),$$

where $\iota_i : (-)^{n_i} \implies \Sigma$ ranges over the coproduct injections into $\Sigma$. This gives a composite modality $\sigma \circledcirc (\beta_b)_{b \in B2^m}$ of arity $m \times k$, and it turns out that

$$\gamma = (\sigma \circledcirc (\beta_b)_{b \in B2^m}) \circ \Sigma B2^\pi$$

where $\pi : m \times k \to m$ is the evident projection. $\qquad\square$

# 6  Future work

Unfortunately, both deficiencies that this paper aims at removing, still persist to some extent in the present formulation. On the abstract level, there clearly is a 2-categorical treatment of coalgebraic modal logic waiting to be discovered and combined with the one developed in [24] for bialgebras. Connections $\rho$ and $\zeta$ are simply morphisms of endofunctors, just as $\lambda$ and $\chi$ are endomorphisms on them; also sliced distributive laws $\chi \circledast \lambda$ are distributive law morphisms in the sense of [24]. There is clearly more structure in the story than currently explained.

On the concrete level, some more specific guidelines for finding suitable collections of $\Sigma$-modalities are much needed. Last but not least, more examples of logical distributive laws, and their relation to other work on SOS compositionality such as [6], need to be shown.

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# A  Adjoint lifting

The following theorem is standard; a proof of it (more precisely, its dual) can be found in [9], see also [13].

Consider endofunctors $B$ on $\mathcal{C}$ and $B'$ on $\mathcal{C}'$, together with a functor $R : \mathcal{C} \to \mathcal{C}'$. A natural transformation $\alpha : RB \implies B'R$ induces a functor $\overline{R} : B\textbf{-coalg} \to B'\textbf{-coalg}$ defined by:

$$\overline{R}( X \xrightarrow{\ h\ } BX ) = RX \xrightarrow{\ Rh\ } RBX \xrightarrow{\ \alpha_X\ } B'RX . \tag{A.1}$$

Then, for the commuting diagram:

$$
\begin{array}{ccc}
B'\textbf{-coalg} & \xleftarrow{\ \overline{R}\ } & B\textbf{-coalg} \\
\downarrow{\scriptstyle U_{B'}} & & \downarrow{\scriptstyle U_B} \\
\mathcal{C}' & \xleftarrow{\ R\ } & \mathcal{C},
\end{array}
$$

the following holds:

**Proposition A.1** *If $\alpha$ is a natural isomorphism than a left adjoint $L \dashv R$ induces a left adjoint $\overline{L} \dashv \overline{R}$.*

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Moreover, the adjunction $\overline{L} \dashv \overline{R}$ lifts $L \dashv R$ along the respective functors. In particular, since $U_{B'}$ reflects isomorphisms, this implies that if the unit of $L \dashv R$ is a natural isomorphism then so is the unit of $\overline{L} \dashv \overline{R}$.

# B  Proofs

## B.1  Section 4.1: Not every endofunctor on $(\mathcal{D} \downarrow T)$ is sliced.

One important counterexample is the biextensional collapse construction on Chu spaces [25], seen as an endofunctor on $\mathbf{Chu}(\mathbf{Set}, 2) = (\mathbf{Set} \downarrow 2^-)$. For a simpler counterexample, consider $\mathcal{C} = \mathbf{Set}^{\mathrm{op}}$, $\mathcal{D} = \mathbf{Set}$ and $T = \mathrm{Id}$. Then $(\mathcal{D} \downarrow T) = \mathbf{Ar}(\mathbf{Set})$, the arrow category of $\mathbf{Set}$. Now consider an endofunctor $Q : \mathbf{Ar}(\mathbf{Set}) \to \mathbf{Ar}(\mathbf{Set})$ defined by:

- on objects, $Q(s : X \to Y) = m : Z \to Y$, where $X \xrightarrow{e} Z \xrightarrow{m} Y$ is the epi-mono factorization of $s$,

- on arrows, a pair $\langle f : X \to X', g : Y \to Y' \rangle$ such that

$$
\begin{array}{ccc}
X & \xrightarrow{s} & Y \\
\downarrow{f} & & \downarrow{g} \\
X' & \xrightarrow{s'} & Y'
\end{array}
$$

commutes, is mapped to $\langle z, g \rangle$ as in the diagram:

$$
\begin{array}{ccccc}
X & \xrightarrow{e} & Z & \xrightarrow{m} & Y \\
\downarrow{f} & & \downarrow{z} & & \downarrow{g} \\
X' & \xrightarrow{e'} & Z' & \xrightarrow{m'} & Y',
\end{array}
$$

where $z$ exists uniquely by the epi-mono factorization system of $\mathbf{Set}$.

Functoriality of $Q$ is ensured by the factorization system as well.

However, $Q$ does not lift any functor on $\mathbf{Set}$ along $\Pi_2$, since it might give different results for different functions (objects in $\mathbf{Ar}(\mathbf{Set})$) even if they have the same domain.

## B.2  Section 4.1: Proof of Prop. 4.2.

For any object $X \in \mathcal{C}$, consider the object $\langle TX, X, \mathrm{id}_X \rangle \in (\mathcal{D} \downarrow T)$ and define

$$
\rho_X = \pi_3(K \langle TX, X, \mathrm{id}_X \rangle).
$$

Then $\rho : LT \Longrightarrow TB^{op}$ is natural. Indeed, take any $f : X \to Y$ in $\mathcal{C}$. The square

$$
\begin{array}{ccc}
TY & =\!\!=\!\!= & TY \\
\downarrow{Tf} & & \downarrow{Tf} \\
TX & =\!\!=\!\!= & TX
\end{array}
$$

trivially commutes, hence

$$\langle Tf, f\rangle : \langle TY, Y, \mathrm{id}_Y\rangle \to \langle TX, X, \mathrm{id}_X\rangle$$

is a valid morphism in $(\mathcal{D}\downarrow T)$. But then also

$$K\langle Tf, f\rangle : K\langle TY, Y, \mathrm{id}_Y\rangle \to K\langle TX, X, \mathrm{id}_X\rangle$$

must be a valid morphism. Since $K$ lifts $L$ and $B^{op}$, there is $K\langle Tf, f\rangle = \langle LTf, Bf\rangle$ and by (9) the naturality square

$$
\begin{array}{ccc}
LTY & \xrightarrow{\;\rho_Y\;} & TBY \\
{\scriptstyle LTf}\downarrow & & \downarrow{\scriptstyle TBf} \\
LTX & \xrightarrow[\;\rho_X\;]{} & TBX
\end{array}
$$

commutes.

Moreover, $K = \widehat{\rho}$. To see this, it is enough to show, for any object $\langle \Phi, X, s\rangle$, that

$$\pi_3 K\langle \Phi, X, s\rangle = \pi_3 K\langle TX, X, \mathrm{id}_X\rangle \circ Ls.$$

To this end, notice that the square

$$
\begin{array}{ccc}
\Phi & \xrightarrow{\;s\;} & TX \\
{\scriptstyle s}\downarrow & & \| {\scriptstyle T\mathrm{id}_X} \\
TX & \xrightarrow[\mathrm{id}_{TX}]{} & TX
\end{array}
$$

obviously commutes, hence

$$\langle s, \mathrm{id}_X\rangle : \langle \Phi, X, s\rangle \to \langle TX, X, \mathrm{id}_X\rangle$$

is a valid morphism in $(\mathcal{D}\downarrow T)$. But then also

$$K\langle s, \mathrm{id}_X\rangle : K\langle \Phi, X, s\rangle \to K\langle TX, X, \mathrm{id}_X\rangle$$

must be a valid morphism. Since $K$ lifts $L$ and $B^{op}$, there is $K\langle s, \mathrm{id}_X\rangle = \langle Ls, \mathrm{id}_{BX}\rangle$ and by definition of morphisms in $(\mathcal{D}\downarrow T)$, the square

$$
\begin{array}{ccc}
L\Phi & \xrightarrow{\;\pi_3 K\langle \Phi, X, s\rangle\;} & TBX \\
{\scriptstyle Ls}\downarrow & & \| \\
LTX & \xrightarrow[\pi_3 K\langle TX, X, \mathrm{id}_X\rangle]{} & TBX
\end{array}
$$

commutes; but this is exactly the required equation. □

*B.3   Section 4.1: Not every natural transformation between sliced endofunctors is sliced.*

For a counterexample, take $\mathcal{C} = \mathbf{1}$, $\mathcal{D} = \mathbf{Set}$ and $T = C_2$ (the constant functor at a two-element set). Then $(\mathcal{D}\downarrow T) = \mathbf{Set}/2$ is the category of sets over a two-element

set 2. Now consider endofunctors $K = \text{Id}$ (the identity functor) and $K' = C_{\text{id}_2}$ (the constant functor at $\text{id}_2$) on $(\mathcal{D} \downarrow T)$. It is easy to see that both functors are sliced.

Define $\kappa : K \Longrightarrow K'$ by $\kappa_{s:X \to 2} = s$. It is easy to see that this is well-defined as a morphism in $(\mathcal{D} \downarrow T)$; to show naturality, assume any $s : X \to 2$ and $r : Y \to 2$ and some $f : X \to Y$ such that $r \circ f = s$. The naturality square of $\kappa$ at $f$ is:

$$
\begin{array}{ccc}
s & \xrightarrow{\ f\ } & r \\
{\scriptstyle \kappa_s = s}\Big\downarrow & & \Big\downarrow{\scriptstyle \kappa_r = r} \\
\text{id}_2 & \xrightarrow[\ \text{id}_2\ ]{} & \text{id}_2
\end{array}
$$

and this commutes immediately by the assumption on $f$.

However, the above $\kappa$ is not sliced. To see this, take any two distinct functions $s, r : X \to 2$ for some set $X$. Then obviously $\kappa_s \neq \kappa_r$, therefore a purported $\alpha : \text{Id} \Longrightarrow 2$ in **Set** cannot be defined on $X$.

### B.4  Section 4.2: Proof of Proposition 4.5.

(12) shows how to define $\kappa$ from $\alpha$ and $\beta$. For the other direction, for any $\kappa : \widehat{\rho} \Longrightarrow \widehat{\rho'}$, define

$$\alpha_\Phi = \Pi_1 \kappa_{\langle \Phi, S\Phi, \eta_\Phi \rangle} \qquad \beta_X = \Pi_2 \kappa_{\langle TX, X, \text{id}_{TX} \rangle}$$

for any $X \in \mathcal{C}$ and $\Phi \in \mathcal{D}$, where $\eta : \text{Id} \to TS^{\text{op}}$ is the unit of $S^{\text{op}} \dashv T$.

To check the naturality of $\alpha$, for any $f : \Phi \to \Psi$ in $\mathcal{D}$ consider the first component of the naturality square of $\kappa$ at $\langle f, S^{\text{op}} f \rangle : \langle \Phi, S\Phi, \eta_\Phi \rangle \to \langle \Psi, S\Psi, \eta_\Psi \rangle$, which is a well-defined morphism in $(\mathcal{D} \downarrow T)$ by naturality of $\eta$. For the naturality of $\beta$, for any $g : X \to Y$ in $\mathcal{C}$ consider the second component of the naturality square of $\kappa$ at $\langle Tg, g \rangle : \langle TY, Y, \text{id}_{TY} \rangle \to \langle TX, X, \text{id}_{TX} \rangle$, which is trivially a well-defined morphism in $(\mathcal{D} \downarrow T)$. The equation $T\beta^{op} \circ \rho = \rho' \circ \alpha T$ follows, for any given $X \in \mathcal{C}$, from the fact that the component of $\kappa$ at $\langle TX, X, \text{id}_{TX} \rangle$ is a well-defined morphism. Finally, it is straightforward to check that the construction of $\alpha$ and $\beta$ from $\kappa$ is mutually inverse with (12).