Expressiveness of probabilistic modal logics: a gradual approach☆

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Abstract

Logical characterizations of probabilistic bisimulation and simulation for Labelled Markov Processes were given by Desharnais et al. These results hold for systems defined on analytic state spaces and assume countably many labels in the case of bisimulation and finitely many labels in the case of simulation.

We revisit these results by giving simpler and more streamlined proofs. In particular, our proof for simulation has the same structure as the one for bisimulation, relying on a new result of a topological nature. We also propose a new notion of event simulation.

Our proofs assume countably many labels, and we show that the logical characterization of bisimulation may fail when there are uncountably many labels. However, with a stronger assumption on the transition functions (continuity instead of just measurability), we regain the logical characterization result for arbitrarily many labels. These results arose from a game-theoretic understanding of probabilistic simulation and bisimulation.

Keywords: labelled Markov process, probabilistic bisimulation, probabilistic simulation, probabilistic modal logic, analytic set, Polish space
2010 MSC: 28A05, 60J05, 60J10, 68Q10, 68Q85, 68Q87

1. Introduction

Probabilistic modal logic (PML) was introduced by Larsen and Skou in [LS91] as a counterpart of the classical Hennessy-Milner logic (HML) [HM85] for (reactive) probabilistic transition systems (PTSs). In such systems, labelled transitions are assigned probabilities so that for every label a, if a state has any a-labelled transitions then the probabilities of those transitions add up to 1. PML extends classical propositional logic (with constants for true and false, negation, binary conjunction and disjunction) with formulas of the form $\langle a\rangle_p\phi$, meaning that the probability of reaching a state

☆This is an extended and revised version of the conference paper [FKP17].

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[1]Supported by a grant from the Natural Sciences and Engineering Research Council of Canada.
[2]Supported by The Alan Turing Institute under the EPSRC grant EP/N510129/1 and the DeLTA project (ANR-16-CE40-0007).
[3]Supported by the Polish National Science Centre (NCN) grant 2012/07/E/ST6/03026.

that satisfies ϕ, after a transition labelled with a, is at least p. PML is *expressive*, i.e., its associated logical equivalence coincides with probabilistic bisimilarity.

As it turns out, certain restricted fragments of PML are already expressive. In [LS91], expressiveness was proved for a fragment without negation but extended with a special formula to detect the lack of transitions, on PTSs that satisfy the so-called *minimal deviation assumption*, i.e., ones where there is a global non-zero lower bound on probabilities of transitions. It is easy to see that this assumption implies that the PTS is finitely branching, i.e., that for every label a, every state can make only finitely many a-labelled transitions. The expressiveness proof in [LS91] is similar to the classical expressiveness proof for HML [HM85] in the case of (non-probabilistic) labelled transition systems. The only difference is the observation, called a "duality lemma" in [LS91], that under the minimal deviation assumption the relevant fragment of PML is essentially closed under negation.

In [DEP02], PML was studied in the more general setting of *labelled Markov processes*, where states form a measurable space rather than a discrete set. It was proved there, under the assumption that the state space is analytic, that *conjunctive PML*, where propositional connectives are restricted to truth and conjunction, is already expressive. This was surprising, as the analogous property fails for HML. The logical equivalence for conjunctive HML is simulation equivalence [vG01], strictly coarser than bisimilarity. Moreover, the minimal deviation assumption, and indeed the finite branching assumption, was dropped. That came at the price of a much more complicated proof that relied on measure-theoretic results, even for discrete systems. The authors of [DEP02] themselves noticed that "[t]he nature of the proof is quite different from proofs of other Hennessy-Milner type results".

These results were streamlined in [DDLP06], where an alternative definition of bisimulation, called event bisimulation, eliminated the need for analytic spaces. It appeared that the crucial measure-theoretic property relied on in the expressiveness proof was a version of Dynkin's π-λ theorem. An event bisimulation is not a relation between states but a σ-algebra, and to recover a more intuitive relational presentation properties of analytic spaces were used again.

In [JS10], conjunctive PML was cast in the framework of coalgebraic modal logic, based on the observation from [dVR99] that PTSs are coalgebras. Expressiveness of conjunctive PML was then derived from abstract theorems about coalgebraic logics, both for discrete PTSs and continuous space Markov processes. That was done in an unlabelled setting, but adding labels would not change the picture in an essential way. The expressiveness proofs were formulated in the abstract terminology of coalgebras and involved dual adjunctions between categories of sets (or measurable spaces) and meet-semilattices.

More recently, in [BM16], it was noticed that *disjunctive PML*, where propositional connectives are restricted to truth and disjunction, is also expressive. This is also surprising, as in the case of non-probabilistic systems the logical equivalence for disjunctive HML is trace equivalence [vG01], much coarser than bisimilarity. The expressiveness proof in [BM16] spanned several pages and was coalgebraic in flavour; it relied on an explicit construction of a final coalgebra for the behaviour functor of PTSs. It also restricted attention to finite branching, discrete systems. On the other hand, it did not rely on the π-λ theorem or any other measure-theoretic principle. A similar proof for conjunctive PML was also developed, based on coalgebras, but with no reliance on measure theory.

The fact that the logical characterization result can be established with a purely positive logic was a surprise at the time of [DEP02]. It opened the door to the possibility that there could be a logical characterization of *simulation*. A clever example, due to Josée Desharnais [DGJP03], showed that this cannot be done with the same logic as the one used for bisimulation; one needs to add disjunction to the logic. A logical characterization of simulation was proved [DGJP03] for transition systems with *finitely many* labels. The main contribution of [DGJP03] was approximation theory which included a domain-theoretic treatment; the logical characterization result fell out of the domain theory results, and its proof was quite different from logical characterization results for bisimulation listed above. Desharnais [Des99] in her thesis gave a proof that avoided domain theory but it was restricted to the discrete case.

All this paints a complicated picture where essentially a single result (expressiveness of conjunctive PML) and its simple variants (disjunctive PML, logical characterization of simulation) are proved at various levels of generality (finitely branching, discrete, analytic state spaces) using different techniques (measure theory, coalgebra, domain theory) and proof strategies (following the classical HML argument as in [LS91], calculating system quotients as in [DEP02], or approximations as in [DGJP03]). Our first aim in this paper is to introduce some order into the story and explain how a classical HML expressiveness proof gradually generalizes to probabilistic systems: first to finitely branching PTS in Section 3, where the proof stays entirely elementary and does not use any measure-theoretic tools, then to arbitrary discrete PTSs in Section 4, where Dynkin's π-λ theorem is a useful tool, finally to labelled Markov processes on analytic spaces in Section 5, where additionally the Unique Structure Theorem becomes handy. The general structure of the proof remains the same for all these scenarios, and applies to conjunctive as well as to disjunctive PML. As a bonus, the same proof strategy applies to the logical characterization of simulation on labelled Markov processes (Section 6), avoiding the need for domain-theoretic methods and allowing for countably infinite sets of transition labels.

Technically speaking, the new results of this are: expressiveness of disjunctive PML beyond discrete PTSs, and the logical characterization of simulation on labelled Markov processes with infinitely many transition labels. But we believe that a far more important contribution is exposing a common structure in several distinct proofs and explaining the gradually emerging need for measure-theoretic technology as the generality of the setting increases.

As a by-product, techniques introduced in Section 6 naturally lead in Section 7 to a novel notion of *event simulation*, which is to probabilistic simulation as event bisimulation of [DDLP06] is to bisimulation. As it should, this notion is characterized by a suitable logic without assuming that the measurable space of states be analytic.

In most results mentioned so far it is assumed that the set of transition labels is countable. This assumption may seem a little strange, considering that it does not appear at all in the classical HML expressiveness theorem for nondeterministic systems. In Section 8 we study what happens when this assumption is dropped in the setting of labelled Markov processes. As it turns out, the assumption cannot be dropped in general, but it can be dropped for processes whose transition functions are continuous in the topological sense. An alternative approach to uncountable sets of labels was recently developed in [Gbu18], relying on a measurable space structure imposed on the set of labels.

Section 9 is essentially an extended side-remark, interesting in its own right. There we define a new notion of a two-player game that characterizes probabilistic bisimulation, just as classical bisimulation on nondeterministic systems is characterized by a well-known game. That final section does not have a direct technical connection to other parts of the paper, but it gives a different perspective on probabilistic bisimulation, and indeed it guided us in the search for simple expressiveness proofs for probabilistic logics.

Acknowledgments. We are very much indebted to Roman Pol, who showed us the proof of

Theorem 6.5 which had eluded us for a long time.

We would like to thank the Simons Institute for hosting the program Logical Structures in Computation during the Fall of 2016 where three of us were able to work together in a congenial atmosphere. We are grateful to Martin Otto and Thomas Colcombet for stimulating conversations in Berkeley.

2. Probabilistic systems and logics

We begin by formulating basic notions concerning probabilistic modal logics. For now we focus on the simple case of discrete systems as introduced by [LS91]. Later in Section 5 we will introduce the general setting of Labelled Markov Processes on analytic spaces.

Definition 2.1. A (reactive) *probabilistic transition system* (PTS) $\mathcal{S}$ with label set $\mathcal{A}$ is a structure $(X, \{\tau_a \mid a \in \mathcal{A}\})$, where X is a set and, for each $a \in \mathcal{A}$,

$$\tau_a : X \times X \to [0, 1]$$

is a function such that for each $x \in X$ the function $\tau_a(x, \cdot)$ is a sub-probability distribution on X, i.e., that

$$\sum_{y \in X} \tau_a(x, y) \leq 1.$$

In particular, we require the above sum to exist. Since an uncountable set of positive real numbers cannot have a well-defined sum, this implies that every PTS is "countably branching": for each $x \in X$ and $a \in \mathcal{A}$ the set of those $y \in X$ where $\tau_a(x, y) > 0$, is countable. If all these sets are in fact finite, we say that the PTS is *finitely branching*.

For now, we do not assume anything about the cardinality of X or $\mathcal{A}$; they may be infinite or even uncountable.

For $C \subseteq X$, we write $\tau_a(x, C)$ to mean $\sum_{y \in C} \tau_a(x, y)$.

Definition 2.2 (Larsen-Skou)**.** A *probabilistic bisimulation* on a PTS (X, τ) is an equivalence relation $R \subseteq X \times X$ such that for every xRy and every $a \in \mathcal{A}$,

$$\tau_a(x, C) = \tau_a(y, C)$$

for every equivalence class $C \in X/R$.

It is easy to see that the union $\approx$ of all bisimulations on a given PTS is itself a bisimulation; it is called the *probabilistic bisimilarity* relation.

Definition 2.3. Formulas of *probabilistic modal logic* (PML) are given by the grammar:

$$\phi ::= \top \mid \neg\phi \mid \phi \vee \phi \mid \phi \wedge \phi \mid \langle a \rangle_p \phi$$

where $p \in [0, 1] \cap \mathbb{Q}$. PML formulas can be interpreted in any PTS (X, τ), with the modality $\langle a \rangle_p$ interpreted as follows:

$$x \models \langle a \rangle_p \phi \iff \tau_a(x, \{y \in X \mid y \models \phi\}) > p \tag{1}$$

and propositional connectives interpreted as expected (in particular, $x \models \top$ for every $x \in X$). The set $\{y \in X \mid y \models \phi\}$ will be denoted $[\![\phi]\!]$.

In the context of a PTS (X, τ), we write $x \equiv y$ (and say that x and y are *logically equivalent*) if for every PML formula ϕ, $x \models \phi$ iff $y \models \phi$. We shall sometimes write $C \models \phi$ for some $C \in X/\equiv$, meaning that $x \models \phi$ for some (equivalently, for each) $x \in C$.

The following easy result means that PML is *sound* for probabilistic bisimilarity:

Theorem 2.4. *For any PTS (X, τ), and for any $x, y \in X$, if $x \approx y$ then $x \equiv y$.*

Proof. Straightforward induction on the structure of ϕ. □

The converse of Theorem 2.4 also holds, meaning that PML is *expressive* for probabilistic bisimulation. In the next section we shall state and prove this result for two fragments of PML. The *conjunctive fragment*, called $\mathrm{PML}_\wedge$, has formulas defined by:

$$\phi ::= \top \mid \phi \wedge \phi \mid \langle a \rangle_p \phi$$

Similarly, the *disjunctive fragment* $\mathrm{PML}_\vee$ is defined by:

$$\phi ::= \top \mid \phi \vee \phi \mid \langle a \rangle_p \phi$$

Both fragments are obviously sound by Theorem 2.4.

In the context of a PTS (X, τ) we write $x \preceq_\wedge y$ if for every $\mathrm{PML}_\wedge$ formula ϕ, $x \models \phi$ implies $y \models \phi$. (For the full PML this notation would make little sense, as that logic is closed under negation.) We write $x \equiv_\wedge y$ if $x \preceq_\wedge y$ and $y \preceq_\wedge x$. This is clearly an equivalence relation on X, and $\preceq_\wedge$ defines a partial order on the set of equivalence classes $X/\equiv_\wedge$.

Relations $\preceq_\vee$ and $\equiv_\vee$ are defined analogously for $\mathrm{PML}_\vee$.

3. The case of finitely branching systems

Theorem 3.1. *For any finitely branching PTS (X, τ), and for any $x, y \in X$, if $x \equiv_\wedge y$ then $x \approx y$.*

Proof. We show that $\equiv_\wedge$ is a probabilistic bisimulation on (X, τ). To this end, take some $x, y \in X$ and assume that there exists some $a \in \mathcal{A}$ such that $\tau_a(x, C) \neq \tau_a(y, C)$ for some $C \in X/\equiv_\wedge$. We need to prove that $x \not\equiv_\wedge y$.

For brevity, denote $\delta = \tau_a(x, \cdot)$ and $\gamma = \tau_a(y, \cdot)$. Among all $C \in X/\equiv_\wedge$ such that $\delta(C) \neq \gamma(C)$, pick one that is *maximal* with respect to $\preceq_\wedge$. This is possible since (X, τ) is finitely branching, therefore there are only finitely many $C \in X/\equiv_\wedge$ such that $\delta(C) \neq \gamma(C)$. Indeed, for all but finitely many C there is $\delta(C) = \gamma(C) = 0$.

For each class $C' \in X/\equiv_\wedge$ such that $\delta(C') > 0$ or $\gamma(C') > 0$ and such that $C \not\preceq_\wedge C'$, pick a $PML_\wedge$ formula $\phi_{C'}$ such that $C \models \phi_{C'}$ and $C' \not\models \phi_{C'}$. Then put ϕ to be the conjunction of all the $\phi_{C'}$. Note that the conjunction is finite thanks to finite branching, hence ϕ is a well-formed $\mathrm{PML}_\wedge$ formula.

An element $z \in [\![\phi]\!]$ cannot belong to any class C' as above, because $z \models \phi_{C'}$ and $C' \not\models \phi_{C'}$. So if $z \in [\![\phi]\!]$ then $z \in D$ for some D such that either $\delta(D) = \gamma(D) = 0$, or $C \preceq_\wedge D$. On the other hand, if $z \in D$ for some class D such that $C \preceq_\wedge D$, then $z \in [\![\phi]\!]$. This is simply because $C \models \phi_{C'}$ for each C', therefore $C \models \phi$ and $D \models \phi$.

Altogether this means that

$$\delta\left(\llbracket\phi\rrbracket\right) = \delta\left(\bigcup\{D \in X/{\equiv_\wedge} \mid C \preceq_\wedge D\}\right) = \delta(C) + \delta\left(\bigcup\{D \in X/{\equiv_\wedge} \mid C \preceq_\wedge D,\ C \neq D\}\right)$$

and similarly for γ. Note that for each D such that $C \preceq_\wedge D$ but $C \neq D$, there is $\delta(D) = \gamma(D)$, by maximality of C. On the other hand, we assumed that $\delta(C) \neq \gamma(C)$. Adding this up, we obtain $\delta\left(\llbracket\phi\rrbracket\right) \neq \gamma\left(\llbracket\phi\rrbracket\right)$.

Without loss of generality, assume $\delta\left(\llbracket\phi\rrbracket\right) > \gamma\left(\llbracket\phi\rrbracket\right)$ and pick $p \in \mathbb{Q}$ such that $\delta\left(\llbracket\phi\rrbracket\right) > p \geq \gamma\left(\llbracket\phi\rrbracket\right)$. We readily obtain $x \models \langle a\rangle_p\phi$ and $y \not\models \langle a\rangle_p\phi$, hence $x \not\equiv_\wedge y$ as requested. □

The following expressiveness proof for $\mathrm{PML}_\vee$ has essentially the same structure as the one above. The similarity is intentionally made apparent but left unexplained, to make both proofs self-contained. Proofs in Section 4 will exhibit more explicit symmetry.

Theorem 3.2. *For any finite branching PTS (X, τ), and for any $x, y \in X$, if $x \equiv_\vee y$ then $x \approx y$.*

Proof. We show that $\equiv_\vee$ is a probabilistic bisimulation on (X, τ). To this end, take some $x, y \in X$ and assume that there exists some $a \in \mathcal{A}$ such that $\tau_a(x, C) \neq \tau_a(y, C)$ for some $C \in X/{\equiv_\vee}$. We need to prove that $x \not\equiv_\vee y$.

For brevity, denote $\delta = \tau_a(x, \cdot)$ and $\gamma = \tau_a(y, \cdot)$. If $\delta(X) > \gamma(X)$, then pick a rational number p such that $\delta(X) > p \geq \gamma(X)$; it is easy so see that $x \models \langle a\rangle_p\top$ and $y \not\models \langle a\rangle_p\top$, therefore $x \not\equiv_\vee y$. The same formula distinguishes x and y if $\delta(X) < \gamma(X)$.

If $\delta(X) = \gamma(X)$, among all $C \in X/{\equiv_\vee}$ such that $\delta(C) \neq \gamma(C)$, pick one that is *minimal* with respect to $\preceq_\vee$. This is possible since (X, τ) is finitely branching, therefore there are only finitely many $C \in X/{\equiv_\vee}$ such that $\delta(C) \neq \gamma(C)$. Indeed, for all but finitely many C there is $\delta(C) = \gamma(C) = 0$.

For each class $C' \in X/{\equiv_\vee}$ such that $\delta(C') > 0$ or $\gamma(C') > 0$ and such that $C' \not\preceq_\vee C$, pick a $PML_\vee$ formula $\phi_{C'}$ such that $C' \models \phi_{C'}$ and $C \not\models \phi_{C'}$. Then put ϕ to be the disjunction of all the $\phi_{C'}$. Note that the disjunction is finite thanks to finite branching, hence ϕ is a well-formed $\mathrm{PML}_\vee$ formula.

An element $z \in \llbracket\phi\rrbracket$ cannot belong to C, because $z \models \phi_{C'}$ for some C', and $C \not\models \phi_{C'}$. For the same reason, it cannot belong to any class D such that $D \preceq_\vee C$. So if $z \in \llbracket\phi\rrbracket$ then $z \in C'$ for some C' such that $C' \not\preceq_\vee C$.

On the other hand, if $z \in C'$ for some class C' such that $C' \not\preceq_\vee C$, then either $\delta(C') = \gamma(C') = 0$, or $z \models \phi_{C'}$ hence $z \in \llbracket\phi\rrbracket$.

Altogether this means that

$$\delta\left(\llbracket\phi\rrbracket\right) = \delta\left(\bigcup\{C' \in X/{\equiv_\vee} \mid C' \not\preceq_\vee C\}\right) = \delta(X) - \delta(C) - \delta\left(\bigcup\{D \in X/{\equiv_\vee} \mid D \preceq_\vee C,\ D \neq C\}\right)$$

and similarly for γ. Note that for each D such that $D \preceq_\vee C$ but $D \neq C$, there is $\delta(D) = \gamma(D)$, by minimality of C. On the other hand, we assumed that $\delta(C) \neq \gamma(C)$ and $\delta(X) = \gamma(X)$. Adding this up, we obtain $\delta\left(\llbracket\phi\rrbracket\right) \neq \gamma\left(\llbracket\phi\rrbracket\right)$.

Without loss of generality, assume $\delta\left(\llbracket\phi\rrbracket\right) > \gamma\left(\llbracket\phi\rrbracket\right)$ and pick $p \in \mathbb{Q}$ such that $\delta\left(\llbracket\phi\rrbracket\right) > p \geq \gamma\left(\llbracket\phi\rrbracket\right)$. We readily obtain $x \models \langle a\rangle_p\phi$ and $y \not\models \langle a\rangle_p\phi$, hence $x \not\equiv_\vee y$ as requested. □

4. The case of discrete systems

So far we have only considered finitely branching PTSs. We shall drop this assumption now and consider arbitrary discrete PTSs, still according to Defn. 2.1. Logics $\mathrm{PML}_\wedge$ and $\mathrm{PML}_\vee$ remain as before, with finitary conjunction and disjunction only. However, we assume that the set of labels A is countable. This implies that there are only countably many formulas of $\mathrm{PML}_\wedge$ and $\mathrm{PML}_\vee$.

Definition 2.2 makes sense for this infinitely branching setting, so it seems natural to keep it as the definition of bisimulation. An alternative definition is:

Definition 4.1 (Desharnais-Edalat-Panangaden)**.** Probabilistic bisimulation on a discrete PTS (X, τ) is an equivalence relation $R \subseteq X \times X$ such that for every xRy and every $a \in \mathcal{A}$,

$$\tau_a(x, M) = \tau_a(y, M)$$

for every R-closed subset M of X.

Here we say that $M \subseteq X$ is *R-closed* if $x \in M$ and xRy implies $y \in M$, for all $x, y \in X$.

For now, this choice of definitions does not make much difference. The following fact is almost trivial in this setting but it is worth stating, as it will fail in Section 5.

Fact 4.2. *Definitions 2.2 and 4.1 are equivalent on discrete PTSs.*

Proof. Since every $C \in X/R$ is R-closed, clearly every bisimulation according to Definition 4.1 satisfies Definition 2.2. For the other direction, every R-closed set M is a disjoint union of classes in X/R. Since the functions $\tau_a(x, \cdot)$ and $\tau_a(y, \cdot)$ take nonzero values in only countably many points, in calculating $\tau_a(x, M)$ and $\tau_a(y, M)$ it is enough to consider a countable family of those classes, and $\tau_a(x, \cdot)$ and $\tau_a(y, \cdot)$ are additive with respect to countable disjoint unions. □

The definitions are equivalent, but it is useful to keep both of them in mind, as they both come handy in proving properties of bisimulations. First, a largest bisimulation (called bisimilarity and denoted $\approx$ as before) exists on any discrete PTS. Indeed, for any family $(R_i)_{i \in I}$ of bisimulations, the transitive closure R of the union of all the R_i is a bisimulation. This is easy to see using Definition 4.1, since then a set is R-closed if and only if it is R_i-closed for every $i \in I$.

Further, soundness Theorem 2.4 holds. To prove it, it is again convenient to use Definition 4.1 and proceed by induction on the structure of PML formulas, proving that for every formula ϕ the set $[\![\phi]\!]$ is $\approx$-closed.

Definition 2.2 is more convenient to prove expressiveness of $\mathrm{PML}_\wedge$ and $\mathrm{PML}_\vee$, generalizing Theorems 3.1 and 3.2. As it turns out, proofs of the generalized results become even more similar than before. The generality and similarity comes at the price of using the well known Dynkin's π-λ theorem, which we now recall.

A *π-system* is a family of subsets of a set X closed under finite intersections. A *λ-system* is a family that contains X and is closed under complement and countable disjoint unions. A *σ-algebra* is a family closed under complement and arbitrary countable unions (and, therefore, arbitrary countable intersections). For a family $\mathcal{E}$, let $\sigma(\mathcal{E})$ denote the least σ-algebra that contains $\mathcal{E}$.

Theorem 4.3 (Dynkin's π-λ theorem, [Dyn60, Bil95])**.** *For any π-system Π and a λ-system Λ on the same set X, if $\Pi \subseteq \Lambda$ then $\sigma(\Pi) \subseteq \Lambda$.* □

We shall also need an easy corollary of this. Define a *$\amalg$-system* to be a family of subsets closed under finite *unions*. Then:

Corollary 4.4. *For any $\rotatebox[origin=c]{180}{π}$-system $\amalg$ and a λ-system Λ on the same set X, if $\amalg \subseteq \Lambda$ then $\sigma(\amalg) \subseteq \Lambda$.* □

Proof. Assume $\amalg \subseteq \Lambda$, and let Π be the set of complements of the elements of $\amalg$. Clearly Π is a π-system and, since Λ is closed under complement, $\Pi \subseteq \Lambda$. By Theorem 4.3, $\sigma(\Pi) \subseteq \Lambda$. Finally, since σ-algebras are closed under complement, $\sigma(\amalg) = \sigma(\Pi)$. □

We are now ready to generalize Theorems 3.1 and 3.2. The following proof of Theorem 4.5 is very similar to the one in [DW14, Thm. 3], but it is worth stating here, as it makes a natural stepping stone from the entirely elementary Theorem 3.1 to the measure-theoretic Theorem 5.5 later.

Theorem 4.5. *For any PTS (X, τ), and for any $x, y \in X$, if $x \equiv_\wedge y$ then $x \approx y$.*

Proof. We show that $\equiv_\wedge$ is a probabilistic bisimulation on (X, τ), according to Definition 2.2. To this end, take some $x, y \in X$ and assume that there exists some $a \in \mathcal{A}$ such that $\tau_a(x, C) \neq \tau_a(y, C)$ for some $C \in X/\equiv_\wedge$. We need to prove that $x \not\equiv_\wedge y$.

For brevity, denote $\delta = \tau_a(x, \cdot)$ and $\gamma = \tau_a(y, \cdot)$. If $\delta(X) > \gamma(X)$, then pick a rational number p such that $\delta(X) > p \geq \gamma(X)$; it is easy so see that $x \models \langle a \rangle_p \top$ and $y \not\models \langle a \rangle_p \top$, therefore $x \not\equiv_\wedge y$. The same formula distinguishes x and y if $\delta(X) < \gamma(X)$.

If $\delta(X) = \gamma(X)$ then pick any $C \in X/\equiv_\wedge$ such that $\delta(C) \neq \gamma(C)$. Let Φ be the set of all formulas that hold for states in C. Clearly:

$$C = \left(\bigcap_{\phi \in \Phi} [\![\phi]\!] \right) \cap \left(\bigcap_{\phi \notin \Phi} (X - [\![\phi]\!]) \right). \tag{2}$$

Define

$$\Pi = \{[\![\phi]\!] \mid \phi \in \mathrm{PML}_\wedge\} \qquad \text{and} \qquad \Lambda = \{Y \subseteq X \mid \delta(Y) = \gamma(Y)\}.$$

It is easy to see that Π is a π-system and Λ is a λ-system (in particular, Λ is closed under complement since $\delta(X) = \gamma(X)$). Moreover, since there are only countably many formulas, the intersections in (2) are countable and so $C \in \sigma(\Pi)$. Since by assumption $C \notin \Lambda$, we have $\sigma(\Pi) \not\subseteq \Lambda$, hence (by Theorem 4.3) $\Pi \not\subseteq \Lambda$. In other words, there exists a $\mathrm{PML}_\wedge$ formula ϕ such that $\delta([\![\phi]\!]) \neq \gamma([\![\phi]\!])$.

Without loss of generality, assume $\delta([\![\phi]\!]) > \gamma([\![\phi]\!])$ and pick $p \in \mathbb{Q}$ such that $\delta([\![\phi]\!]) > p \geq \gamma([\![\phi]\!])$. We readily obtain $x \models \langle a \rangle_p \phi$ and $y \not\models \langle a \rangle_p \phi$, hence $x \not\equiv_\wedge y$ as requested. □

Theorem 4.6. *For any PTS (X, τ), and for any $x, y \in X$, if $x \equiv_\vee y$ then $x \approx y$.*

Proof. Proceed exactly as for Theorem 4.5, but instead of Π define

$$\amalg = \{[\![\phi]\!] \mid \phi \in \mathrm{PML}_\vee\},$$

notice that $\amalg$ is a $\rotatebox[origin=c]{180}{π}$-system and use Corollary 4.4 instead of Theorem 4.3. □

5. The case of Labelled Markov Processes

To go beyond the setting of discrete-space, countably branching systems, we need to review some definitions and concepts from measure theory and topology. We assume that the reader is familiar with concepts like: σ-algebra, measurable functions, (sub)probability measures, topology and continuity. For an introduction to these topics in the context of Labelled Markov Processes, see e.g. [Pan09].

Given a topological space X, the σ-algebra induced by its open sets (or, equivalently, its closed sets) is called the *Borel algebra*; we will always work with Borel algebras of topological spaces. We call them Borel spaces.

A topological space is said to be *separable* if it has a countable dense subset. For metric spaces this is equivalent to having a countable base of open sets. A *Polish space* is the topological space underlying a complete separable metric space. Note that a space like the open interval $(0, 1)$ which is not complete in its usual metric is nevertheless Polish, since it can be given a complete metric that produces the same topology. If X, Y are Polish spaces and $f : X \to Y$ is a continuous function then the image $f(X) \subset Y$ is an *analytic space*. The class of analytic spaces is not altered if we allow f to be measurable instead of continuous or if we take the image of a Borel set instead of all of X.

The following definition generalizes Definition 2.1.

Definition 5.1. A **labelled Markov process** (LMP) $\mathcal{S}$ with label set $\mathcal{A}$ is a structure

$$(X, \Sigma, \{\tau_a \mid a \in \mathcal{A}\}),$$

where (X, Σ) is a Borel space and, for each $a \in \mathcal{A}$,

$$\tau_a : X \times \Sigma \longrightarrow [0, 1]$$

is a function such that:

- for each fixed $x \in X$, the set function $\tau_a(x, \cdot)$ is a sub-probability measure, and
- for each fixed $C \in \Sigma$ the function $\tau_a(\cdot, C)$ is measurable.

One interprets $\tau_a(x, C)$ as the probability of the process starting in state x making an a-labelled transition into one of the states in C.

The logic PML and its fragments $\mathrm{PML}_\wedge$ and $\mathrm{PML}_\vee$ are defined as in Section 2. Formally, for the equation (1) to make sense, for any formula ϕ the set $[\![\phi]\!] \subseteq X$ must be measurable, i.e., $[\![\phi]\!] \in \Sigma$. This is, however, proved by easy induction on the structure of formulas.

Following [DEP02], bisimulation on labelled Markov processes is defined by analogy to Definition 4.1:

Definition 5.2 (Desharnais-Edalat-Panangaden)**.** Probabilistic bisimulation on a labelled Markov process (X, Σ, τ) is an equivalence relation R such that whenever xRy then

$$\tau_a(x, M) = \tau_a(y, M)$$

for each $a \in \mathcal{A}$ and each R-closed set $M \in \Sigma$.

It might be tempting to generalize Definition 2.2 instead, and to postulate the equality

$$\tau_a(x, C) = \tau_a(y, C)$$

not for arbitrary measurable R-closed sets, but only for measurable equivalence classes $C \in X/R$. The resulting definition, however, would be rather ill-behaved, as the following example shows.

Example 5.3. Denote $X = \{p, q\} \cup [0, 1]$. Equip X with the smallest σ-algebra that makes all Borel sets of $[0, 1]$ as well as the singletons $\{p\}$ and $\{q\}$ measurable. Denote by μ the Lebesgue[4] probability measure on $[0, 1]$.

Consider a singleton action set $\mathcal{A} = \{a\}$. Define a function $\tau_a : X \times \Sigma \to [0, 1]$ as follows:

$$\begin{array}{rcll} \tau_a(p, C) & = & \mu(C \cap [0,1]) & \\ \tau_a(q, C) & = & 0 & \\ \tau_a(x, C) & = & 0 & \text{for each } x \in [0,1]. \end{array}$$

It is easy to check that (X, Σ, τ) is a well-formed labelled Markov process.

Let Δ denote the equality relation on X. For each $x, y \in [0, 1]$, define an equivalence relation on X:

$$R_{(x,y)} = \Delta \cup \{(p, q), (q, p)\} \cup \{(x, y), (y, x)\}.$$

Since all equivalence classes of $R_{(x,y)}$ are finite, we have that

$$\tau_a(p, C) = \tau_a(q, C) = 0$$

for every equivalence class C. As a result, if bisimulations were defined by analogy to Definition 2.2 as explained above, every $R_{(x,y)}$ would be a bisimulation. However, the union of all these relations:

$$R = \bigcup_{x,y \in [0,1]} R_{(x,y)} = [0, 1]^2 \cup \{p, q\}^2$$

although itself an equivalence relation, would not be a bisimulation. Indeed, one of its equivalence classes is $[0, 1]$, and

$$\tau_a(p, [0, 1]) = 1 \neq 0 = \tau(q, [0, 1]). \tag{3}$$

This means that bisimulations on labelled Markov processes, defined by analogy to Definition 2.2, would not be closed under (transitive closures) of unions, and largest bisimulations would not necessarily exist. Moreover, the logic PML and its fragments would not be sound: here, the states p and q would be related by a bisimulation (indeed, by every relation $R_{(x,y)}$), but they are clearly distinguished by the formula $\langle a \rangle_{0.5} \top$.

Note that (3) implies that none of the relations $R_{(x,y)}$ are bisimulations according to Definition 5.2, since the set $[0, 1]$ is closed under every one of them.

For these reasons, from now on we stick to Definition 5.2. With this definition, largest bisimulations do exist and PML is sound without any further assumptions, with arguments as given in Section 4.

Expressiveness, however, is more problematic. If for $\mathrm{PML}_\wedge$ one tries to repeat the argument from the proof of Theorem 4.5 using $\equiv_\wedge$-closed measurable sets M instead of classes $C \in X/\!\equiv_\wedge$, one hits a problem trying to rewrite equation (2): how to present an arbitrary $\equiv_\wedge$-closed, measurable set using countable unions, intersections and complements of sets of the form $[\![\phi]\!]$?

In [DEP02], expressiveness of $\mathrm{PML}_\wedge$ was proved under the assumption that the underlying space (X, Σ) is *analytic*. As observed in [DGJP03], the key property of analytic spaces that allows an expressiveness proof to go through is the following:

[4] We mean the usual measure on $[0, 1]$ which assigns to intervals their length. This is usually extended to the Lebesgue σ-algebra, i.e. the one obtained by completing the Borel σ-algebra with respect to this measure. However, we are just using this measure on the Borel sets.

Theorem 5.4 (Unique Structure Theorem, [Arv76]). *In any analytic space (X, Σ), for every countable family $\mathcal{E} \subseteq \Sigma$ such that $X \in \mathcal{E}$, every measurable $\equiv_{\mathcal{E}}$-closed subset of X is an element of $\sigma(\mathcal{E})$.* □

Here and in the following, $\equiv_{\mathcal{E}}$ is the relation of equivalence up to $\mathcal{E}$, i.e., $x \equiv_{\mathcal{E}} y$ if and only if, for every $Y \in \mathcal{E}$, $x \in Y$ iff $y \in Y$.

One can see that Theorem 5.4 is precisely what is needed for the equation (2) to be generalized to any $\equiv_\wedge$-closed measurable set. The entire expressiveness proof, essentially copying the argument of Theorem 4.5, becomes quite short:

Theorem 5.5. *For any labelled Markov process (X, Σ, τ) where (X, Σ) is analytic and $\mathcal{A}$ is countable, and for any $x, y \in X$, if $x \equiv_\wedge y$ then $x \approx y$.*

Proof. We show that $\equiv_\wedge$ is a probabilistic bisimulation on (X, Σ, τ), according to Definition 5.2. To this end, take some $x, y \in X$ and assume that there exists some $a \in \mathcal{A}$ such that $\tau_a(x, M) \neq \tau_a(y, M)$ for some $\equiv_\wedge$-closed set $M \in \Sigma$. We need to prove that $x \not\equiv_\wedge y$.

Denote $\delta = \tau_a(x, \cdot)$ and $\gamma = \tau_a(y, \cdot)$. If $\delta(X) > \gamma(X)$, then pick a rational number p such that $\delta(X) > p \geq \gamma(X)$; it is easy so see that $x \models \langle a \rangle_p \top$ and $y \not\models \langle a \rangle_p \top$, therefore $x \not\equiv_\wedge y$. The same formula distinguishes x and y if $\delta(X) < \gamma(X)$.

If $\delta(X) = \gamma(X)$ then pick any $\equiv_\wedge$-closed $M \in \Sigma$ such that $\delta(M) \neq \gamma(M)$. Define

$$\Pi = \{[\![\phi]\!] \mid \phi \in \mathrm{PML}_\wedge\} \qquad \text{and} \qquad \Lambda = \{Y \in \Sigma \mid \delta(Y) = \gamma(Y)\}.$$

It is easy to see that Π is a π-system and Λ is a λ-system (in particular, Λ is closed under complement since $\delta(X) = \gamma(X)$). Clearly, $\equiv_\Pi = \equiv_\wedge$. Since $\top \in \mathrm{PML}_\wedge$, we have $X \in \Pi$. Moreover, since there are only countably many formulas, Π is countable and, by Theorem 5.4, $M \in \sigma(\Pi)$. Since by assumption $M \notin \Lambda$, we have $\sigma(\Pi) \not\subseteq \Lambda$, hence (by Theorem 4.3) $\Pi \not\subseteq \Lambda$. In other words, there exists a $\mathrm{PML}_\wedge$ formula ϕ such that $\delta([\![\phi]\!]) \neq \gamma([\![\phi]\!])$.

Without loss of generality, assume $\delta([\![\phi]\!]) > \gamma([\![\phi]\!])$ and pick $p \in \mathbb{Q}$ such that $\delta([\![\phi]\!]) > p \geq \gamma([\![\phi]\!])$. We readily obtain $x \models \langle a \rangle_p \phi$ and $y \not\models \langle a \rangle_p \phi$, hence $x \not\equiv_\wedge y$ as requested. □

With the insights of Section 4, expressiveness of $\mathrm{PML}_\vee$ is now immediate:

Theorem 5.6. *For any labelled Markov process (X, Σ, τ) where (X, Σ) is analytic and $\mathcal{A}$ is countable, and for any $x, y \in X$, if $x \equiv_\vee y$ then $x \approx y$.*

Proof. Proceed exactly as for Theorem 5.5, but instead of Π define

$$\amalg = \{[\![\phi]\!] \mid \phi \in \mathrm{PML}_\vee\},$$

notice that $\amalg$ is a $\rotatebox{180}{$\pi$}$-system and use Corollary 4.4 instead of Theorem 4.3. □

6. Simulation on Labelled Markov Processes

For a preorder R on a set X, we say that $C \subseteq X$ is *R-closed* if $x \in C$ and xRy implies $y \in C$, for all $x, y \in X$.

Definition 6.1 (Simulation). A probabilistic simulation on a labelled Markov process (X, Σ, τ) is a preorder relation R such that whenever xRy then

$$\tau_a(x, M) \leq \tau_a(y, M)$$

for each $a \in \mathcal{A}$ and each R-closed set $M \in \Sigma$.

We say that x is simulated by y, denoted $x \lesssim y$, if there exists a simulation R such that xRy. As in the case of bisimulation, it is easy to see that (the transitive closure of) the union of an arbitrary family of simulations is again a simulation; as a consequence, $\lesssim$ is itself a simulation relation, indeed the largest simulation on a given labelled Markov process.

The logic $\mathrm{PML}_{\curlyvee\!\curlywedge}$ is the union of $\mathrm{PML}_{\wedge}$ and $\mathrm{PML}_{\vee}$:

$$\phi \;=\; \top \;\mid\; \phi \wedge \phi \;\mid\; \phi \vee \phi \;\mid\; \langle a \rangle_p \phi.$$

We write $x \leq_{\curlyvee\!\curlywedge} y$ to say that every formula in $\mathrm{PML}_{\curlyvee\!\curlywedge}$ satisfied by x is also satisfied by y.

Our proof of the logical characterization of simulation is completely analogous to the one for bisimulation. It is enough to replace the two main ingredients (Theorems 4.3 and 5.4) by new ones, Theorems 6.4 and Theorems 6.5 below. Theorem 6.4 (see Section 6.1) is similar and easier to prove than Theorem 4.3, and it should be treated as folklore. Theorem 6.5 (see Section 6.2), on the other hand, is a new, stronger version of Theorem 5.4.

6.1. Positive Monotone Class Theorem

A *lattice of sets* is a family of subsets of a set X closed under finite unions and intersections.[5] A *monotone class* is a family closed under unions of increasing chains and under intersections of decreasing chains. For a family $\mathcal{E}$, let $M(\mathcal{E})$ denote the least monotone class that contains $\mathcal{E}$. A σ-*lattice of sets* is a family of sets closed under countable unions and countable intersections. For a family $\mathcal{E}$, let $L(\mathcal{E})$ denote the least σ-lattice of sets that contains $\mathcal{E}$.

Lemma 6.2. For any lattice of sets $\mathcal{E}$, the family $M(\mathcal{E})$ is a lattice of sets.

Proof. The proof is by a two-step bootstrapping argument similar to the classical proof of the π-λ theorem. Define

$$\mathcal{M}_0 = \{Y \in M(\mathcal{E}) \mid \forall Z \in \mathcal{E}.\ Z \cup Y \in M(\mathcal{E}),\ Z \cap Y \in M(\mathcal{E})\}.$$

Claim 1: $\mathcal{M}_0$ is a monotone class.

To prove this, take any chain $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots$ with $Y_i \in \mathcal{M}_0$. Define $Y = \bigcup_i Y_i$ and pick any $Z \in \mathcal{E}$. Then calculate:

$$Y \cap Z = \bigcup_i (Y_i \cap Z) \qquad \text{and} \qquad Y \cup Z = \bigcup_i (Y_i \cup Z).$$

Since each Y_i is in $\mathcal{M}_0$, by definition each $Y_i \cap Z$ and $Y_i \cup Z$ is in $M(\mathcal{E})$. Since both unions on the right are unions of increasing chains, and since $M(\mathcal{E})$ is a monotone class, we get that $Y \cap Z$ and $Y \cup Z$ are in $M(\mathcal{E})$. Since $Z \in \mathcal{E}$ was chosen arbitrarily, we obtain $Y \in \mathcal{M}_0$.

An analogous argument shows that $\mathcal{M}_0$ is closed under intersections of decreasing chains, which proves Claim 1.

Claim 2: $\mathcal{E} \subseteq \mathcal{M}_0$.

This is obvious since $\mathcal{E}$ is a lattice of sets and $\mathcal{E} \subseteq M(\mathcal{E})$.

[5] A lattice of sets is sometimes called a *ring of sets*. However, in measure theory ring of sets means something else (a family closed under union and set difference), so we choose a different name.

Now define

$$\mathcal{M}_1 = \{Z \in M(\mathcal{E}) \mid \forall Y \in M(\mathcal{E}).\ Z \cup Y \in M(\mathcal{E}),\ Z \cap Y \in M(\mathcal{E})\}.$$

Claim 3: $\mathcal{M}_1$ is a monotone class.

This is proved entirely analogously to Claim 1.

Claim 4: $\mathcal{E} \subseteq \mathcal{M}_1$.

This is less obvious than Claim 2. For a proof, pick any $Z \in \mathcal{E}$ and any $Y \in M(\mathcal{E})$. Since $M(\mathcal{E})$ is the least monotone class that contains $\mathcal{E}$, from Claims 1 and 2 it follows that $M(\mathcal{E}) \subseteq \mathcal{M}_0$, hence $Y \in \mathcal{M}_0$. By definition of $\mathcal{M}_0$ we get that $Z \cup Y$ and $Z \cap Y$ are in $M(\mathcal{E})$. Since Y was chosen arbitrarily, we get $Z \in \mathcal{M}_1$.

Now, since $M(\mathcal{E})$ is the least monotone class that contains $\mathcal{E}$, from Claims 3 and 4 it follows that $M(\mathcal{E}) \subseteq \mathcal{M}_1$, which immediately implies that $M(\mathcal{E})$ is a lattice of sets. □

Lemma 6.3. Any lattice of sets $\mathcal{E}$ that is also a monotone class is a σ-lattice of sets.

Proof. For any countable family $Y_1, Y_2, \ldots \in \mathcal{E}$, check:

$$\bigcup_{i=1}^{\infty} Y_i = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{i} Y_j.$$

The outer union is increasing and each inner union is finite, therefore the entire union is in $\mathcal{E}$. An analogous argument works for countable intersections. □

Theorem 6.4 (Positive Monotone Class Theorem)**.** *For any lattice of sets $\mathcal{E}$ and any monotone class $\mathcal{M}$ on the same set X, if $\mathcal{E} \subseteq \mathcal{M}$ then $L(\mathcal{E}) \subseteq \mathcal{M}$.*

Proof. Assume $\mathcal{E} \subseteq \mathcal{M}$. By Lemmas 6.2 and 6.3, $M(\mathcal{E})$ is a σ-lattice of sets, therefore $L(M(\mathcal{E})) = M(\mathcal{E})$ and:

$$L(\mathcal{E}) \subseteq L(M(\mathcal{E})) = M(\mathcal{E}) \subseteq \mathcal{M}.$$

□

6.2. Positive Unique Structure Theorem

The following result strengthens Theorem 5.4. Its proof is also more involved, using ideas similar to the proof of Lusin's Separation Theorem for analytic sets (see [Kec95]). The proof was pointed out to us by Roman Pol.

Below, $\sqsubseteq_{\mathcal{E}}$ is the preorder determined by $\mathcal{E}$, i.e., $x \sqsubseteq_{\mathcal{E}} y$ if and only if, for every $Y \in \mathcal{E}$, $x \in Y$ implies $y \in Y$.

Theorem 6.5 (Positive Unique Structure Theorem)**.** *In any analytic space (X, Σ), for every countable family $\mathcal{E} \subseteq \Sigma$ such that $X \in \mathcal{E}$, every nonempty, measurable and $\sqsubseteq_{\mathcal{E}}$-closed subset of X is an element of $L(\mathcal{E})$.*

Proof. First, consider the special case where (X, Σ) is a Polish space. We need some basic terminology. First, for two topologies over the same set, the second one extends the first one if all sets which are open in the first are also open in the second. Two topologies are Borel-isomorphic if they induce the same σ-algebra of Borel sets. A clopen set is a set both closed and open.

By a result stated in [Kec95] as Exercise 13.5, the topology on X can be extended to another Polish space on X that is Borel-isomorphic to the original one, and such that every set in $\mathcal{E}$ is clopen. (That exercise is easy to prove from Theorem 13.1 and Lemma 13.3 in [Kec95].) It is enough to prove the theorem for $\mathcal{E}$ on that extended Polish space. Therefore, we shall safely assume that every set in $\mathcal{E}$ is clopen in X.

Assume towards a contradiction that there exists a nonempty, $\sqsubseteq_{\mathcal{E}}$-closed Borel set B such that $B \notin L(\mathcal{E})$. Since $X \in \mathcal{E}$, necessarily $B \neq X$. Therefore both B and $X \setminus B$ are nonempty subsets of a Polish space, so they, by definition, are analytic (see [Kec95], Defn 14.1). Hence they are images of some continuous maps from the Baire space $\mathbb{N}^{\mathbb{N}}$:

$$f : \mathbb{N}^{\mathbb{N}} \twoheadrightarrow B \qquad \text{and} \qquad g : \mathbb{N}^{\mathbb{N}} \twoheadrightarrow X \setminus B.$$

In the following, for $s \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$ and $\sigma \in \mathbb{N}^n$, denote:

$$s|_n = s_1 s_2 \dots s_n \in \mathbb{N}^n \qquad \text{and} \qquad \mathcal{C}(\sigma) = \{s \in \mathbb{N}^{\mathbb{N}} \mid s|_n = \sigma\}.$$

In particular, $s|_0 = \epsilon$ and $\mathcal{C}(\epsilon) = \mathbb{N}^{\mathbb{N}}$.

We write $\prec$ for the prefix ordering on $\mathbb{N}^n$. We shall define, by induction on $n \in \mathbb{N}$, sequences $\sigma_n, \tau_n \in \mathbb{N}^n$ such that:

- $\sigma_0 = \epsilon \prec \sigma_1 \prec \sigma_2 \prec \cdots$,
- $\tau_0 = \epsilon \prec \tau_1 \prec \tau_2 \prec \cdots$, and
- there is no set C in $L(\mathcal{E})$ that contains $f(\mathcal{C}(\sigma_n))$ and is disjoint with $g(\mathcal{C}(\tau_n))$.

For $n = 0$, note that $f(\mathcal{C}(\sigma_0)) = B$ and $g(\mathcal{C}(\sigma_0)) = X \setminus B$, so the only candidate for C above is B, and $B \notin L(\mathcal{E})$ by assumption.

Assume that σ_n and τ_n have been chosen. Assume towards a contradiction that they cannot be extended to any σ_{n+1} and τ_{n+1}, i.e., that for every $i, j \in \mathbb{N}$ there is a set $C_{ij} \in L(\mathcal{E})$ that contains $f(\mathcal{C}(\sigma_n i))$ and is disjoint with $g(\mathcal{C}(\tau_n j))$. Then the set

$$C = \bigcup_i \bigcap_j C_{ij} \in L(\mathcal{E})$$

contains

$$f(\mathcal{C}(\sigma_n)) = \bigcup_i f(\mathcal{C}(\sigma_n i))$$

and is disjoint with

$$g(\mathcal{C}(\tau_n)) = \bigcup_i g(\mathcal{C}(\tau_n i))$$

which contradicts the choice of σ_n and τ_n. As a result, some C_{ij} as above does not exist, and we may take $\sigma_{n+1} = \sigma_n i$ and $\tau_{n+1} = \tau_n j$. This completes the inductive construction.

Let us now take $s, t \in \mathbb{N}^{\mathbb{N}}$ such that $s|_n = \sigma_n$ and $t|_n = \tau_n$ for $n \in \mathbb{N}$. Note that, by definition, $f(s) \in B$ and $g(t) \notin B$. Since B is $\sqsubseteq_{\mathcal{E}}$-closed, this means that $f(s) \not\sqsubseteq_{\mathcal{E}} g(t)$, hence there exists a set $C \in \mathcal{E}$ such that $f(s) \in C$ and $g(t) \notin C$. But C is clopen, and both f and g are continuous, so both inverse images $f^{-1}(C)$ and $g^{-1}(X \setminus C)$ are open in $\mathbb{N}^{\mathbb{N}}$. This implies that there must be some n such that

$$\mathcal{C}(\sigma_n) \subseteq f^{-1}(C) \qquad \text{and} \qquad \mathcal{C}(\tau_n) \subseteq g^{-1}(X \setminus C)$$

or, equivalently,

$$f(\mathcal{C}(\sigma_n)) \subseteq C \qquad \text{and} \qquad g(\mathcal{C}(\tau_n)) \cap C = \emptyset$$

which contradicts our choice of σ_n and τ_n. This contradiction completes the proof for the case where (X, Σ) is Polish.

Now consider an arbitrary analytic space (X, Σ), a countable family $\mathcal{E} \subseteq \Sigma$ with $X \in \mathcal{E}$, and a measurable, nonempty, $\sqsubseteq_{\mathcal{E}}$-closed subset $B \subseteq X$. We must prove that $B \in L(\mathcal{E})$.

By definition, (X, Σ) is the image of a measurable function f from a Polish space (Y, Δ). Since B is measurable in Σ, obviously $f^{-1}(B) \subseteq Y$ is measurable in Δ. Consider the (countable) family:

$$f^{-1}(\mathcal{E}) = \{f^{-1}(C) \mid C \in \mathcal{E}\} \subseteq \Delta.$$

Since f is surjective, obviously $Y \in f^{-1}(\mathcal{E})$.

It is easy to check that $f^{-1}(B)$ is $\sqsubseteq_{f^{-1}(\mathcal{E})}$-closed. Indeed, consider any $y \in f^{-1}(B)$ and $y' \in Y$ such that $y \sqsubseteq_{f^{-1}(\mathcal{E})} y'$. Then, for any $C \in \mathcal{E}$ such that $f(y) \in C$, we have $y \in f^{-1}(C)$, hence $y' \in f^{-1}(C)$ and $f(y') \in C$. Since B is $\sqsubseteq_{\mathcal{E}}$-closed, we obtain $f(y') \in B$, therefore $y' \in f^{-1}(B)$ as requested.

By our theorem applied to the Polish space (Y, Δ), the countable set $f^{-1}(\mathcal{E})$ and the nonempty measurable set $f^{-1}(B)$, we get

$$f^{-1}(B) \in L(f^{-1}(\mathcal{E})).$$

Since inverse images commute with unions and intersections, we have $L(f^{-1}(\mathcal{E})) = f^{-1}(L(\mathcal{E}))$, and, since f is surjective, $f(f^{-1}(C)) = C$ for every $C \subseteq X$. Altogether we get:

$$B = f(f^{-1}(B)) \in f(L(f^{-1}(\mathcal{E}))) = f(f^{-1}(L(\mathcal{E}))) = L(\mathcal{E})$$

as requested. $\square$

6.3. Logical Characterization

A proof of expressiveness of $\mathrm{PML}_{\vee\wedge}$ now becomes an minor variation of the proof of Theorem 5.5.

Theorem 6.6. *For any labelled Markov process (X, Σ, τ) where (X, Σ) is analytic and $\mathcal{A}$ is countable, and for any $x, y \in X$, if $x \leq_{\vee\wedge} y$ then $x \lesssim y$.*

Proof. We show that $\leq_{\vee\wedge}$ is a probabilistic simulation on (X, Σ, τ). Take some $x, y \in X$ and assume that there exists some $a \in A$ such that $\tau_a(x, C) > \tau_a(y, C)$ for some $\leq_{\vee\wedge}$-closed set $C \in \Sigma$. We need to prove that $x \not\leq_{\vee\wedge} y$.

Denote $\delta = \tau_a(x, -)$ and $\gamma = \tau_a(y, -)$. Pick any $\leq_{\vee\wedge}$-closed $C \in \Sigma$ such that $\delta(C) > \gamma(C)$. Then C cannot be empty, since $\delta(\emptyset) = \gamma(\emptyset) = 0$. Define

$$\mathcal{L} = \{[\![\phi]\!] \mid \phi \in \mathrm{PML}_{\vee\wedge}\} \qquad \text{and} \qquad \mathcal{M} = \{Y \in \Sigma \mid \delta(Y) \leq \gamma(Y)\}. \tag{4}$$

It is easy to see that $\mathcal{L}$ is a lattice of sets and (by continuity of measure) $\mathcal{M}$ is a monotone class. Clearly, $\sqsubseteq_{\mathcal{L}} = \leq_{\vee\wedge}$. Since $\top \in \mathrm{PML}_{\vee\wedge}$, we have $X \in \mathcal{L}$. Moreover, since there are only countably many formulas, $\mathcal{L}$ is countable hence, by Theorem 6.5, $C \in L(\mathcal{L})$. Since by assumption $C \notin \mathcal{M}$, we have $L(\mathcal{L}) \not\subseteq \mathcal{M}$, hence (by Theorem 6.4) $\mathcal{L} \not\subseteq \mathcal{M}$. In other words, there exists a formula ϕ such that $\delta([\![\phi]\!]) > \gamma([\![\phi]\!])$. Pick $p \in \mathbb{Q}$ such that $\delta([\![\phi]\!]) > p \geq \gamma([\![\phi]\!])$. We readily obtain $x \models \langle a \rangle_p \phi$ and $y \not\models \langle a \rangle_p \phi$, hence $x \not\leq_{\vee\wedge} y$ as requested. $\square$

7. Event simulation

In [DDLP06] it was argued that for probabilistic processes defined on continuous state spaces bisimulation should be understood as a σ-algebra of measurable sets rather than as an equivalence relation on the state space. The name *event bisimulation* was coined in order to emphasize the connection with the events of the sample space. From a categorical viewpoint, [DDLP06] used cospans of coalgebra morphisms rather than spans as in the original definition [DEP02].

In [DEP98, DEP02] a logical characterization of probabilistic bisimulation was given under the condition of the state space being analytic. By using event bisimulation rather than the Larsen-Skou definition [LS91] and its variants, [DDLP06, Cor. 5.6] extended the logical characterization result to arbitrary measurable spaces. When the state space is analytic this shows that event bisimulation induces the same equivalence relation as ordinary bisimulation. When the state space is not analytic the logical characterization may fail for the traditional definition of bisimulation [Ter11].

Until now it was not clear how to develop a suitable concept of simulation analogous to event bisimulation. In this section we use parts of the machinery introduced in Section 6 to define event simulation. We begin by recalling the definition of event bisimulation from [DDLP06].

7.1. Event bisimulation

Definition 7.1. An *event bisimulation* on an LMP (X, Σ, τ) is a sub-σ-algebra Λ of Σ such that (X, Λ, τ) is an LMP.

Intuitively, this definition tells us that Σ may be an unnecessarily fine σ-algebra for the LMP it defines. Indeed, even if we restrict to the coarser σ-algebra Λ, the function τ is still measurable. Thus there is no "loss of information" by using the coarser σ-algebra Λ. This should be compared with ordinary bisimulation: if R is a bisimulation according to Definition 5.2 then quotienting the state space by R does not lose any information.

The connection between event bisimulation and ordinary bisimulation is captured in the following theorems [DDLP06].

Theorem 7.2. *If R is a bisimulation on an LMP (X, Σ, τ), then the σ-algebra of R-closed Σ-measurable sets is an event bisimulation.*

For analytic spaces the two concepts coincide:

Theorem 7.3. *Given an LMP (X, Σ, τ) on an analytic space (X, Σ) and an event bisimulation $\Lambda = \sigma(\mathcal{C})$ for some countable $\mathcal{C}$ in Σ, then the relation $\equiv_\Lambda$ is a bisimulation.*

Here recall that $\equiv_\Lambda$ is the relation of equivalence up to Λ, i.e., $x \equiv_\Lambda y$ if and only if, for every $Y \in \Lambda$, $x \in Y$ iff $y \in Y$.

7.2. Event simulation

In the context of an LMP (X, Σ, τ), we shall slightly abuse the notation and denote

$$\langle a \rangle_r(A) ::= \tau_a(\cdot, A)^{-1}((r, 1]) = \{x \in X \mid \tau_a(x, A) > r\} \tag{5}$$

for an action a, a number $r \in [0, 1]$ and $A \subseteq X$. Comparing this with Definition 2.3, it is easy to check that then

$$\langle a \rangle_p [\![\phi]\!] = [\![\langle a \rangle_p \phi]\!]$$

for every PML formula ϕ and $p \in [0, 1] \cap \mathbb{Q}$.

The following definition is a minor variation of one from [DDLP06]:

Definition 7.4. A family $\mathcal{E} \subseteq \Sigma$ is *stable* for an LMP (X, Σ, τ) if for every action a, every $p \in [0,1] \cap \mathbb{Q}$ and every $A \in \mathcal{E}$,

$$\langle a \rangle_p(A) \in \mathcal{E}.$$

Note that $\langle a \rangle_1(A) = \emptyset$ for all A, so $\emptyset \in \mathcal{E}$ for every nonempty stable family $\mathcal{E}$.

It $\mathcal{E}$ is a σ-lattice (or, more generally, if it is closed under unions of increasing chains), then in the stability condition one may equivalently consider all $p \in [0,1]$, since

$$\langle a \rangle_p(A) = \bigcup_{q \in [p,1] \cap \mathbb{Q}} \langle a \rangle_q(A)$$

for every $p \in [0,1]$. This definition of stability is used in [DDLP06]. As observed there, event bisimulations are exactly those sub-σ-algebras of Σ that are stable. This motivates the following definition:

Definition 7.5. Given an LMP (X, Σ, τ), an *event simulation* is a stable σ-lattice $\mathcal{E} \subseteq \Sigma$ such that $X \in \mathcal{E}$.

Since a σ-algebra is a σ-lattice, every event bisimulation is an event simulation. The converse is not true in general, since a σ-lattice need not be closed under complements.

Working in the context of a fixed LMP, define:

$$\mathcal{L} = \{[\![\phi]\!] \mid \phi \in \mathrm{PML}_{\vee\wedge}\}$$

as in (4).

Lemma 7.6. *For any LMP (X, Σ, τ), $\mathcal{L}$ is the smallest stable lattice that contains X.*

Proof. Obviously $\mathcal{L}$ is a lattice: it is closed under finite union and intersection, and it contains $X = [\![\top]\!]$. Moreover, $\mathcal{L}$ is stable: for $[\![\phi]\!] \in \mathcal{L}$, by definition $\langle a \rangle_p [\![\phi]\!] = [\![\langle a \rangle_p \phi]\!] \in \mathcal{L}$.

Now, for a stable lattice $\mathcal{E}$ that contains X, let us prove that $\mathcal{L} \subseteq \mathcal{E}$:

- $[\![\top]\!] = X \in \mathcal{E}$ by definition,
- if $[\![\phi]\!]$ and $[\![\psi]\!]$ are in $\mathcal{E}$, then $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$ and $[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$ are in $\mathcal{E}$,
- if $[\![\phi]\!]$ is in $\mathcal{E}$, then $[\![\langle a \rangle_p \phi]\!] = \langle a \rangle_p [\![\phi]\!]$ is in $\mathcal{E}$ as it is stable.

The conclusions follows by induction on the structure of formulas in $\mathrm{PML}_{\vee\wedge}$. □

Recall from Section 6.1 that for a family $\mathcal{E}$, $L(\mathcal{E})$ denotes the least σ-lattice that contains $\mathcal{E}$.

Lemma 7.7. *For any LMP (X, Σ, τ), if $\mathcal{E}$ is a stable lattice then $L(\mathcal{E})$ is stable.*

Proof. We will first prove that the set

$$\mathcal{D} = \{A \in \Sigma \mid \forall a \in \mathcal{A}\ \forall p \in [0,1] \cap \mathbb{Q}.\ \langle a \rangle_p(A) \in L(\mathcal{E})\}$$

is a monotone class (see Section 6.1). Note that $L(\mathcal{E})$ is stable if and only if $L(\mathcal{E}) \subseteq \mathcal{D}$.

Let $(A_n)_{n \in \mathbb{N}}$ be an increasing chain with $A_n \in \mathcal{D}$ for all $n \in \mathbb{N}$. We want to prove that $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{D}$. To do that, we shall prove that $\langle a \rangle_p \left(\bigcup_{n \in \mathbb{N}} A_n\right) = \bigcup_{n \in \mathbb{N}} \langle a \rangle_p (A_n)$.

Let $x \in \langle a\rangle_p \left(\bigcup_{n\in\mathbb{N}} A_n\right)$, i.e. $\tau_a \left(x, \left(\bigcup_{n\in\mathbb{N}} A_n\right)\right) > p$. By continuity of measure we know that $\tau_a \left(x, \left(\bigcup_{n\in\mathbb{N}} A_n\right)\right) = \lim_{n\to\infty} \tau_a(x, A_n)$. This means in particular that $\lim_{n\to\infty} \tau_a(x, A_n) > p$ and therefore there exists N such that for all $n \geq N$, $\tau_a(x, A_n) > p$, which means that $x \in \bigcup_{n\in\mathbb{N}} \langle a\rangle_p (A_n)$.

Now let $x \in \bigcup_{n\in\mathbb{N}} \langle a\rangle_p (A_n)$, i.e. there exists $N \in \mathbb{N}$ such that $\tau_a(x, A_N) > p$. Then, since $A_N \subseteq \bigcup_{n\in\mathbb{N}} A_n$, also $\tau_a \left(x, \left(\bigcup_{n\in\mathbb{N}} A_n\right)\right) > p$. Hence, $x \in \langle a\rangle_p \left(\bigcup_{n\in N} A_n\right)$.

We have just proven that $\langle a\rangle_p \left(\bigcup_{n\in\mathbb{N}} A_n\right) = \bigcup_{n\in\mathbb{N}} \langle a\rangle_p (A_n)$. By our assumption, $\langle a\rangle_p A_n$ is in $L(\mathcal{E})$ for all $n \in \mathbb{N}$. As $L(\mathcal{E})$ is a σ-lattice, it is closed under countable unions, and therefore $\langle a\rangle_p \left(\bigcup_{n\in\mathbb{N}} A_n\right)$ is in $L(\mathcal{E})$, which proves that $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{D}$.

Now let $(B_n)_{n\in\mathbb{N}}$ be a decreasing chain with $B_n \in \mathcal{D}$ for all $n \in \mathbb{N}$. We want to prove that $\bigcap_{n\in\mathbb{N}} B_n \in \mathcal{D}$. As before, it is enough to prove that $\langle a\rangle_p \left(\bigcap_{n\in\mathbb{N}} B_n\right) = \bigcap_{n\in\mathbb{N}} \langle a\rangle_p (B_n)$.

Let x in $\langle a\rangle_p \left(\bigcap_{n\in\mathbb{N}} B_n\right)$, i.e. $p < \tau_a \left(x, \bigcap_{n\in\mathbb{N}} B_n\right)$. For all k, $\bigcap_{n\in\mathbb{N}} B_n \subseteq B_k$, which means that $\tau_a \left(x, \bigcap_{n\in\mathbb{N}} B_n\right) < \tau_a(x, B_k)$. In particular, for all n in $\mathbb{N}$, $p < \tau_a(x, B_n)$, i.e. $x \in \langle a\rangle_p(B_n)$ and therefore $x \in \bigcap_{n\in\mathbb{N}} \langle a\rangle_p (B_n)$.

Conversely, let x be in $\bigcap_{n\in\mathbb{N}} \langle a\rangle_p (B_n)$, i.e. for all n in $\mathbb{N}$, x is in $\langle a\rangle_p (B_n)$, i.e. $\tau_a(x, B_n) > p$. This means in particular that for all $n \in \mathbb{N}$, $\tau_a(x, \overline{B_n}) \leq 1 - p$. But as (B_n) is a decreasing chain, $(\overline{B_n})$ is an increasing chain, therefore by monotone convergence theorem, $\tau_a(x, \bigcup_{n\in\mathbb{N}} \overline{B_n}) = \lim_{n\to\infty} \tau_a(x, \overline{B_n})$. Since $\tau_a(x, \overline{B_n}) \leq p$ for all n, the limit is less than or equal to $1 - p$, i.e. $\tau_a(x, \bigcup_{n\in\mathbb{N}} \overline{B_n}) \leq 1 - p$. Furthermore, note that $\bigcup_{n\in\mathbb{N}} \overline{B_n} = \overline{\bigcap_{n\in\mathbb{N}} B_n}$. This allows us to conclude that $\tau_a(x, \bigcap_{n\in\mathbb{N}} B_n) > p$, i.e. $x \in \langle a\rangle_p(B_n)$.

We have just proven that $\langle a\rangle_p \left(\bigcap_{n\in\mathbb{N}} B_n\right) = \bigcap_{n\in\mathbb{N}} \langle a\rangle_p (B_n)$. By our assumption $\langle a\rangle_p B_n$ is in $L(\mathcal{E})$. As $L(\mathcal{E})$ is a σ-lattice, it is closed under countable intersections, and therefore $\langle a\rangle_p \left(\bigcap_{n\in\mathbb{N}} B_n\right)$ is in $L(\mathcal{E})$, which proves that $\bigcap_{n\in\mathbb{N}} B_n \in \mathcal{D}$.

From this we conclude that $\mathcal{D}$ is a monotone class. Since $\mathcal{E}$ is stable and $\mathcal{E} \subseteq L(\mathcal{E})$, we have $\mathcal{E} \subseteq \mathcal{D}$. By Theorem 6.4 we get that $L(\mathcal{E}) \subseteq \mathcal{D}$, i.e. $L(\mathcal{E})$ is stable. $\square$

We arrive at the main result of this section (compare with [DDLP06, Prop. 5.5], an analogous result for event bisimulation):

Theorem 7.8. *For any LMP (X, Σ, τ), $L(\mathcal{L})$ is the smallest event simulation.*

Proof. $L(\mathcal{L})$ is a stable σ-lattice that contains X by Lemmas 7.6 and 7.7. On the other hand, take any stable σ-lattice $\mathcal{E}$ that contains X. By Lemma 7.6 we get $\mathcal{L} \subseteq \mathcal{E}$, so (since $\mathcal{E}$ is a σ-lattice) $L(\mathcal{L}) \subseteq \mathcal{E}$. $\square$

Note that we made no assumptions on the state space of the LMP in question.

7.3. Comparison to simulation

To link event simulation with the relational notion of simulation, we now aim for simulation counterparts of Theorems 7.2 and 7.3.

Given a preorder R on a measurable space, let $\Lambda(R)$ be the family of all R-closed measurable sets. Also recall that $\sqsubseteq_\Lambda$ is the preorder induced by a family $\Lambda \subseteq \Sigma$, i.e., $x \sqsubseteq_\Lambda y$ if and only if, for every $Y \in \Lambda$, $x \in Y$ implies $y \in Y$.

Lemma 7.9. *For any preorder R, $\Lambda(R)$ is a σ-lattice.*

Proof. Consider $(A_n)_{n\in\mathbb{N}}$ a sequence of R-closed measurable sets. We need to prove that their countable union and intersection are also R-closed (they are obviously measurable).

Consider $x \in \bigcup_{n\in\mathbb{N}} A_n$. This means that there exists $k \in \mathbb{N}$ such that $x \in A_k$. If xRy, then $y \in A_k$ since A_k is R-closed, and therefore $y \in \bigcup_{n\in\mathbb{N}} A_n$.

Consider $x \in \bigcap_{n\in\mathbb{N}} A_n$ and assume xRy. This means that for all $n \in \mathbb{N}$, $x \in A_n$. But since all sets A_n are R-closed, then we also have that $y \in A_n$ for all $n \in \mathbb{N}$, i.e. $y \in \bigcap_{n\in\mathbb{N}} A_n$. □

Theorem 7.10. *For any LMP (X, Σ, τ), a preorder R on X is a simulation if and only if $\Lambda(R)$ is an event simulation.*

Proof. By definition, R is a simulation if and only if

$$xRy \quad \text{and} \quad B \in \Lambda(R) \quad \text{implies} \quad \tau_a(x, B) \leq \tau_a(y, B) \tag{6}$$

for every $x, y \in X$ and $B \in \Sigma$, and for every action a. The conclusion of this implication can be equivalently rewritten as

$$\tau_a(x, B) > p \quad \text{implies} \quad \tau_a(y, B) > p \qquad \text{for all } p \in [0, 1] \cap \mathbb{Q}$$

which, for every p, is equivalent to saying

$$x \in \langle a \rangle_p(B) \quad \text{implies} \quad y \in \langle a \rangle_p(B).$$

The implication (6) is therefore equivalent to saying that the set $\langle a \rangle_p(B)$ is R-closed whenever B is; in other words, that $\Lambda(R)$ is stable. Since $\Lambda(R)$ is always a σ-lattice by Lemma 7.9, and it always contains X, the conclusion follows. □

Lemma 7.11. *For any family $\mathcal{E} \subseteq \Sigma$, $\sqsubseteq_{L(\mathcal{E})} = \sqsubseteq_{\mathcal{E}}$.*

Proof. The $\subseteq$ containment is obvious since $\mathcal{E} \subseteq L(\mathcal{E})$. For $\supseteq$, consider any $x \sqsubseteq_{\mathcal{E}} y$ and let $\Lambda \subseteq \Sigma$ be the family of all subsets $Y \in \Sigma$ such that $x \in Y$ implies $y \in Y$. It is easy to check that Λ is a σ-lattice and obviously $\mathcal{E} \subseteq \Lambda$, therefore $L(\mathcal{E}) \subseteq \Lambda$. Since by definition $x \sqsubseteq_\Lambda y$, we get $x \sqsubseteq_{L(\mathcal{E})} y$ as required. □

Lemma 7.12. *If (X, Σ) is an analytic space and $\Lambda = L(\mathcal{E})$ for some countable family $\mathcal{E} \subseteq \Sigma$ with $X \in \mathcal{E}$, if $\emptyset \in \Lambda$ then $\Lambda(\sqsubseteq_\Lambda) = \Lambda$.*

Proof. First, consider $B \in \Lambda$. Then $B \in \Sigma$. To prove that B is $\sqsubseteq_\Lambda$-closed, let $x \in B$ and assume $x \sqsubseteq_\Lambda y$, i.e. for all $C \in \Lambda$, $x \in C \Rightarrow y \in C$. This is true in particular for $C = B \in \Lambda$, and hence $y \in B$. This proves that $B \in \Lambda(\sqsubseteq_\Lambda)$.

Now consider $B \in \Lambda(\sqsubseteq_\Lambda)$. This means that B is $\sqsubseteq_\Lambda$-closed, therefore by Lemma 7.11 it is also $\sqsubseteq_{\mathcal{E}}$-closed. If $B = \emptyset$ then $B \in \Lambda$ by assumption. For nonempty B, apply Theorem 6.5 to obtain $B \in L(\mathcal{E}) = \Lambda$. □

From all this we deduce:

Theorem 7.13. *Assume an LMP (X, Σ, τ) where (X, Σ) is an analytic space, and $\Lambda = L(\mathcal{E})$ for some countable family $\mathcal{E}$ with $X \in \mathcal{E}$. If Λ is an event simulation, then $\sqsubseteq_\Lambda$ is a simulation.*

Proof. If Λ is an event simulation then it is stable, hence in particular $\emptyset \in \Lambda$. By Lemma 7.12 we get that $\Lambda(\sqsubseteq_\Lambda)$ is an event simulation, hence by Theorem 7.10, $\sqsubseteq_\Lambda$ is a simulation. □

One could combine Theorems 7.8, 7.10 and 7.13 to give an alternative proof of Theorem 6.6. Note that in such a proof our two key measure-theoretic ingredients, Theorems 6.4 and 6.5, would still be present: the former in the proof of Lemma 7.7, the latter in Lemma 7.12.

8. The case of uncountably many labels

Our proofs of the logical characterizations for simulation and bisimulation rely on the assumption that the set of formulas (and, equivalently, the set of transition labels) is countable. In this section we investigate the necessity of this assumption. We first observe that indeed if there are uncountably many labels, then the logical characterization fails in general. However, we show that if the transition structure is continuous, then the logical characterization holds again, without any assumption on the set of labels.

An alternative approach to uncountable sets of labels was recently developed in [Gbu18]. There, the set of labels is assumed to carry the structure of a measurable space itself, which allows a characterization of (bi)similarity by a countable set of formulas.

8.1. A counterexample

In the classical logical characterization of (bi)similarity for nondeterministic labelled transition systems [HM80], one can restrict to a logic with finite conjunction and disjunction only if the systems in question satisfy a finite branching property called image finiteness: each state can have only finitely many successors for any given transition label. Since [DEP98, DEP02] it has been known that this restriction does not apply to probabilistic systems, where a finitary logic is enough to characterize bisimilarity on systems with arbitrary (probabilistic) branching.

On the other hand, in the classical nondeterministic setting, once image finiteness is ensured, the size of the set of transition labels matters very little. Even if infinitely many, or even uncountably many labels are permitted, a finitary logic (with a correspondingly large set of modal operators) is enough to characterize (bi)similarity for nondeterministic transition systems labelled with them.

We now show that this is not the case for labelled Markov processes with continuous state spaces. Specifically, we show an example where the set of labels is uncountable and the logical characterization fails, even though the space of states is a particularly simple, compact Polish space.

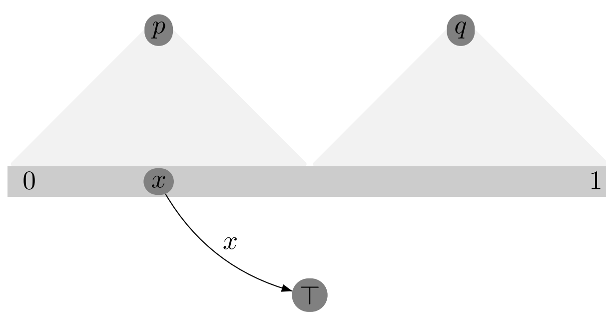

Figure 1: The two states p and q do not simulate each other, but they satisfy the same formulas of $\mathrm{PML}_{\vee\wedge}$.

Denote $X = \{p, q, \top\} \cup [0, 1]$. We equip X with the smallest σ-algebra that makes all Borel sets of $[0, 1]$ as well as the singletons $\{p\}, \{q\}$ and $\{\top\}$ measurable. Denote by μ the Lebesgue[6]

[6]We mean the usual measure on $[0, 1]$ which assigns to intervals their length. However this is usually extended to

probability measure on $[0,1]$.

Consider a set of actions $\mathcal{A} = [0,1]$. Define functions $\tau_a : X \times \Sigma \to [0,1]$ for each $a \in \mathcal{A}$ as follows:

$$\begin{array}{rcl} \tau_a(p, C) & = & \mu(C \cap [0, \frac{1}{2}]) \\ \tau_a(q, C) & = & \mu(C \cap [\frac{1}{2}, 1]) \\ \tau_a(x, \top) & = & \begin{cases} 1 & \text{if } x = a \\ 0 & \text{otherwise} \end{cases} \end{array}$$

The following proposition easily implies that logical characterizations both for bisimulation and for simulation fail for this labelled Markov process.

Proposition 8.1. *Neither p nor q simulates the other, but they satisfy the same formulas of $PML_{\bigvee\!\!\bigwedge}$.*

Proof. We prove that neither p nor q simulates the other. First, for any x, y in $[0,1]$, if $x \neq y$ then neither of these simulates the other. Indeed, from x, the action $a = x$ leads to $\top$ with probability 1 and leads nowhere from y. It follows that every subset of $[0,1]$ is $\lesssim$-closed; in particular this applies to $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. This implies that neither p nor q simulates the other, because $\tau_a(p, [0, \frac{1}{2}]) = 1$ and $\tau_a(q, [0, \frac{1}{2}]) = 0$, and vice-versa $\tau_a(p, [\frac{1}{2}, 1]) = 0$ and $\tau_a(q, [\frac{1}{2}, 1]) = 1$.

To see that p and q satisfy the same formulas, we observe that for every finite subset $\mathcal{B} \subseteq \mathcal{A}$, p and q do simulate each other (indeed, they are even bisimilar) in the system restricted to labels from $\mathcal{B}$. The claim easily follows from this, since every formula of $\mathrm{PML}_{\bigvee\!\!\bigwedge}$ uses finitely many labels.

So for a finite $\mathcal{B} \subseteq \mathcal{A}$, define a relation R on X to be the least equivalence relation such that pRq and xRy for each $x, y \in [0,1] \setminus \mathcal{B}$. We claim that R is a bisimulation on the system restricted to labels with $\mathcal{B}$. The only nontrivial case is the pair pRq: every R-closed set $C \subseteq [0,1]$ is either finite or co-finite, from which it easily follows that $\tau_a(p, C) = \tau_a(q, C)$. □

Intuitively, the core of the problem here is the highly non-continuous nature of transitions from $[0,1]$, allowing one to observe specific states from that uncountable space. Indeed, as we show in the following section, the problem disappears and the logical characterizations hold if we assume that the transition function $\tau_a(\cdot, C)$ is continuous for each a and C.

8.2. Logical characterizations for continuous transition functions

Given a labelled Markov process (X, Σ, τ) with labels from a set $\mathcal{A}$, we denote by $(X, \Sigma, \tau_{|\mathcal{B}})$ the same system restricted to labels from $\mathcal{B} \subseteq \mathcal{A}$.

Theorem 8.2. *For any labelled Markov process (X, Σ, τ) where (X, Σ) is Polish and such that for all $a \in \mathcal{A}, C \in \Sigma$, the function $\tau_a(\cdot, C)$ is continuous, there exists a countable set $\mathcal{B}$ such that the bisimilarity relation $\approx$ on $(X, \Sigma, \tau_{|\mathcal{B}})$ coincides with that on (X, Σ, τ).*

Proof. We will use the fact that, under the above assumptions, X^2 is also a Polish space for the product topology, hence it satisfies the *hereditary Lindelöf property*: any open cover of a subset of X^2 has a countable subcover.

By definition, the bisimilarity relation $\approx$ on (X, Σ, τ) is the largest bisimulation. It is standard to define it as the greatest fixpoint of a certain operator on binary relations on X. For us it will

the Lebesgue σ-algebra, i.e. the one obtained by completing the Borel σ-algebra with respect to this measure. We are just using this measure on the Borel sets.

be convenient to speak in terms of complements, and we consider the *complement* of $\approx$ as the *least* fixpoint of the monotone operator:

$$\Phi(R) = \left\{ (x,y) \in X^2 \,\middle|\, \exists a \in \mathcal{A}, \exists C \in \Sigma\ (X^2 \setminus R)\text{-closed, s.t. } \tau_a(x,C) \neq \tau_a(y,C) \right\}$$

Thanks to Tarski's fixed point theorem, this is obtained by defining a sequence $(W_\alpha)_\alpha$ of subsets of X^2 indexed by ordinals α: for $\alpha+1$ a successor ordinal and β a limit ordinal, define:

$$\begin{array}{lcl} W_0 & = & \emptyset \\ W_{\alpha+1} & = & \left\{ (x,y) \in X^2 \,\middle|\, \exists a \in \mathcal{A}, \exists C \in \Sigma\ (X^2 \setminus W_\alpha)\text{-closed, s.t. } \tau_a(x,C) \neq \tau_a(y,C) \right\} \\ W_\beta & = & \bigcup_{\alpha<\beta} W_\alpha. \end{array}$$

The complement of $\approx$ on (X,Σ,τ) is the union of all W_α for all ordinals α. More specifically, $(W_\alpha)_\alpha$ form an increasing sequence that reaches a fixpoint at some ordinal γ not larger than the cardinality of $\mathcal{P}(X^2)$.

Note that all W_α are open sets in X^2. This is proved by ordinal induction: for a successor ordinal, $W_{\alpha+1}$ is a union of sets of the form

$$\left\{ (x,y) \in X^2 \mid \tau_a(x,C) \neq \tau_a(y,C) \right\}$$

for some as and Cs. Such a set is open, since it is the inverse image of the (open) inequality relation on $[0,1]$ along the continuous function $\tau_a(\cdot,C)$.

For each ordinal α we construct a countable subset $\mathcal{B}_\alpha \subseteq \mathcal{A}$ such that W_α calculated on $(X,\Sigma,\tau_{|\mathcal{B}_\alpha})$ coincides with W_α calculated on (X,Σ,τ). We let $\mathcal{B}_0 = \emptyset$.

For successor ordinals, rewrite the definition of $W_{\alpha+1}$ as:

$$W_{\alpha+1} = \bigcup_{a\in\mathcal{A}} \left\{ (x,y) \in X^2 \mid \exists C \in \Sigma\ (X^2 \setminus W_\alpha)\text{-closed, s.t. } \tau_a(x,C) \neq \tau_a(y,C) \right\}.$$

This is a union of open sets. Since X^2 is hereditary Lindelöf, one can extract a countable subcover of this union, indexed by some set $\mathcal{B} \subseteq \mathcal{A}$. It is then enough to take $\mathcal{B}_{\alpha+1} = \mathcal{B}_\alpha \cup \mathcal{B}$.

For limit ordinals, extract a countable subcover of the union $W_\beta = \bigcup_{\alpha<\beta} W_\alpha$ and take $\mathcal{B}_\beta$ to be the union of the $\mathcal{B}_\alpha$'s defined for α's from that subcover.

Now the countable set $\mathcal{B}_\gamma$, where γ is the ordinal for which W_γ reaches the least fixpoint of Φ, satisfies the desired property. □

The same result holds for simulation:

Theorem 8.3. *For any labelled Markov process (X,Σ,τ) where (X,Σ) is Polish and such that for all $a \in \mathcal{A}, C \in \Sigma$, the function $\tau_a(\cdot,C)$ is continuous, there exists a countable set $\mathcal{B}$ such that the similarity preorder $\lesssim$ on $(X,\Sigma,\tau_{|\mathcal{B}})$ coincides with that on (X,Σ,τ).*

Proof. Completely analogous to the proof of Theorem 8.2, but with the operator

$$\Phi(R) = \left\{ (x,y) \in X^2 \,\middle|\, \exists a \in \mathcal{A}, \exists C \in \Sigma\ (X^2 \setminus R)\text{-closed, s.t. } \tau_a(x,C) > \tau_a(y,C) \right\}$$

instead. In particular the fact that each W_α is open, still holds. □

The following immediately follows from Theorems 8.2 and 8.3 in the light of Theorems 5.5 and 6.6.

Corollary 8.4. *For any labelled Markov process (X, Σ, τ) where (X, Σ) is Polish and such that for all $a \in \mathcal{A}, C \in \Sigma$, the function $\tau_a(\cdot, C)$ is continuous, for any $x, y \in X$,*

- *$x \equiv_\wedge y$ if and only if $x \approx y$,*
- *$x \leq_{\vee\wedge} y$ if and only if $x \lesssim y$.*

9. Probabilistic (bi)simulation games

The classical notion of bisimulation and simulation for nondeterministic processes has a simple and elegant characterization in terms of games. These games, played between two players named Spoiler (who tries to prove that some two states in a process are not bisimilar) and Duplicator (who claims the opposite), provide convenient intuitions about the essence of bisimilarity.

For probabilistic systems, a similar game was proposed in [DLT08], both for simulation and bisimulation. It was later generalised to deal with a pseudometric version of probabilistic bisimulation in [KM18]; a rather different game for the same purpose was studied in [vBW14]. Here we propose a new game for probabilistic simulation and bisimulation. Our games have a pleasantly simple structure, even in the setting of continuous space processes. Each round consists of a single move by Spoiler and a single response by Duplicator, unlike the game from [DLT08] where each round involves multiple interleaved actions from both players.

We begin with the case of **bisimulation game**. As in the classical case, we consider a spoiler/duplicator game with two players. Duplicator's plays are pairs of states that she claims are bisimilar. Spoiler tries to show that a given pair of states proffered by Duplicator are not bisimilar. Let $\mathcal{S} = (X, \Sigma, \tau)$ be a labelled Markov process, and $x, y \in X$. The bisimulation game starting from the position (x, y) alternates between moves of the following kinds:

- Spoiler chooses $a \in A$ and $C \in \Sigma$ such that $\tau_a(x, C) \neq \tau_a(y, C)$,
- Duplicator answers by choosing $x' \in C$ and $y' \notin C$ and the game continues from (x', y').

A player who cannot make a move at any point loses. Duplicator wins if the game goes on forever.

Note that the only way for Spoiler to win is to choose $C = X$ at some point; otherwise Duplicator can always choose some x' and y' as prescribed. (The only other situation where Duplicator cannot proceed would be $C = \emptyset$, but that is not a legal move for Spoiler since always $\tau_a(x, \emptyset) = \tau_a(y, \emptyset) = 0$.) On the other hand, Duplicator can win either by forcing an infinite play or by reaching a position (x, y) where $\tau_a(x, C) = \tau_a(y, C)$ for every $C \in \Sigma$.

The intuition behind the game should be clear. Spoiler tries to prove that states x and y are not bisimilar by showing a label a and a set C, purportedly closed under bisimilarity, such that the probabilities of a-labelled transitions to C are different for x and y. This difference of probabilities is checked but not disputed by Duplicator, who instead claims that C, in fact, is not closed under bisimilarity. She does that by choosing $x' \in C$ and $y' \notin C$ and claiming that these two are bisimilar; the game then proceeds in the same fashion.

Before we formally prove the correctness of this game, let us spend a moment to consider what makes a game-theoretic characterization "elegant". In our opinion, the classical bisimulation game for nondeterministic processes is elegant because it allows one to characterize a global property of behaviours (bisimilarity) in terms of a game whose rules only depend on local considerations. Indeed, whether a move in the game is legal or not does not depend on bisimilarity or other long-range properties, but merely on local observations about transition capabilities that cannot be disputed by either player.

We argue that this criterion of elegance is satisfied by our probabilistic game. One can imagine the players engaging in a brief experiment with the given Markov process after each move by Spoiler, to determine that the two transition probabilities involved are indeed different. By performing random a-transitions from x and y sufficiently many times, Spoiler can demonstrate to Duplicator, with an arbitrarily high confidence level, that the probabilities of getting to C are different and so that the move to C is legal for Spoiler. It is important to note, comparing the game to the definition of probabilistic bisimulation itself, that the legality of a Spoiler's move does not depend on the set C being actually closed under bisimilarity; a game with such a condition would not be "elegant".

The question of *how many* random transitions are enough to convince Duplicator that a Spoiler's move is legal, and hence how much time it takes for Spoiler to win the game if x and y are not bisimilar, suggests a potentially interesting connection of the bisimulation game to the quantitative framework of metrics on labelled Markov processes [DGJP04]. We leave this for future work.

Back to formal development. Since all infinite plays are won by the same player (Duplicator), standard game-theoretic arguments prove that:

Fact 9.1. *The bisimulation game is determined, i.e., from every position (x, y) either Spoiler has a winning strategy or Duplicator does.* □

From this we infer:

Theorem 9.2. *The states x and y are bisimilar if and only if Duplicator has a winning strategy from (x, y).*

Proof. For the left-to-right implication, for bisimilar x and y, we construct a winning strategy from (x, y) for Duplicator. In this strategy, for any move a and C by Spoiler, Duplicator chooses some arbitrary $x' \in C$ and $y' \notin C$ such that x' and y' are bisimilar. This is always possible: since Spoiler's move was legal, and it originated from a pair of bisimilar states, C cannot be closed under bisimilarity. This strategy is winning for Duplicator since it allows her response to any move by Spoiler, and Duplicator wins all infinite plays.

For the right-to-left implication, we shall show that the set R of all pairs (x, y) whence Duplicator has a winning strategy, is a bisimulation. To this end, first we need to show that R is an equivalence relation. Reflexivity is immediate, since from a position (x, x) Spoiler has no legal moves. For symmetry, given a winning strategy from (x, y) Duplicator builds a strategy from (y, x) trivially: she simply replies to any first move by Spoiler as if she would reply to a move from (x, y), and then she follows the given strategy. For transitivity, assume winning strategies for Duplicator from (x, y) and (y, z). A winning strategy for (x, z) works as follows: for a legal move a and C from Spoiler, it must be that $\tau_a(x, C) \neq \tau_a(y, C)$ or $\tau_a(y, C) \neq \tau_a(z, C)$. Depending on which of these inequalities holds, reply according to the strategy from (x, y) or from (y, z), and then follow that winning strategy.

Now assume towards contradiction that R is not a bisimulation. This means that for some x, y such that xRy, there exists a label a in A and an R-closed subset C of X such that $\tau_a(x, C) \neq \tau_a(y, C)$. Consider a and C as a Spoiler's move from (x, y). No matter what Duplicator chooses as $x' \in C$ and $y' \notin C$, since C is R-closed we have that not $(x'Ry')$ and, by Fact 9.1, Spoiler has a winning strategy from (x', y'). This forms a winning strategy for Spoiler from (x, y), contradicting the assumption that xRy. □

Simulation game is defined in a very similar fashion, alternating the following moves:

- Spoiler chooses $a \in A$ and $C \in \Sigma$ such that $\tau_a(x, C) > \tau_a(y, C)$,
- Duplicator answers by choosing $x' \in C$ and $y' \notin C$ and the game continues from (x', y').

Again, a player who cannot make a move at any point loses, and Duplicator wins all infinite plays.

The intuition behind the game is as before, except now Spoiler maintains that his chosen sets C are $\lesssim$-closed, and Duplicator contradicts that by choosing $x' \in C$ and $y' \notin C$ and maintaining that $x' \lesssim y'$. All other considerations remain virtually the same, up to and including:

Theorem 9.3. *$x \lesssim y$ if and only if Duplicator has a winning strategy from (x, y).* □

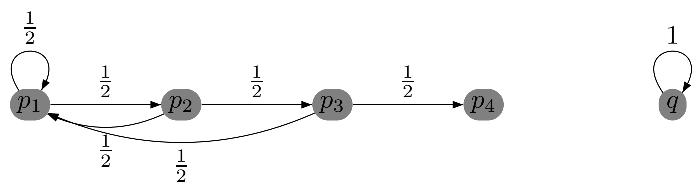

Figure 2: It takes four steps for Spoiler to convince Duplicator that the state p_1 does not simulate q.

Example 9.4. We illustrate the simulation game on an example (see Fig. 2). In this Markov process there is only one label. From the state q, the process loops forever. On the other hand, from the state p_1, one can reach the deadlock state p_4 through the path to p_2 and p_3.

We examine the simulation game and how Spoiler can successfully prove to Duplicator that the state p_1 does not simulate q. We start the simulation game from (q, p_1). A possible first move is $C = \{q, p_2\}$ since $\tau(q, C) = 1 > \tau(p_1, C) = \frac{1}{2}$, but it allows Duplicator to play (q, p_1), back to the original position. A smarter move is $C = \{q, p_1\}$, to which Duplicator has several possible answers, all losing. For instance, if Duplicator plays (q, p_4), Spoiler wins immediately by choosing $C = X$. Duplicator may survive more steps by playing (q, p_2), then (q, p_3), before the fatal (q, p_4).

10. Conclusions

In this paper we have given a unified presentation of several proofs of bisimulation for probabilistic transition systems of increasing complexity: finite, discrete and continuous. The last of these uses the same tools as the proof in [DEP98, DEP02] but the logical organization is different and allows one to see that all three proofs have the same pattern. The second contribution is to give analogous proofs for the logical characterization of simulation. Previous proofs were limited to the case where one of the systems was discrete and relied on approximation theory. The present proofs are more general and follow the same pattern as the bisimulation proofs. This required new descriptive set theory results which are positive versions of the toolkit used for the bisimulation proofs. We have also developed an "event" version of simulation analogous to event bisimulation [DDLP06]. Finally we developed the theory for uncountably many labels. The crude logical characterization result fails but is restored with additional continuity assumptions. All these developments were illuminated by a game-theoretic characterization of bisimulation and simulation.

One can ask how these techniques apply to other situations. There are a number of possible cases to consider. First one can contemplate the combination of probability and nondeterminism. Note first of all that the pure nondeterministic case embeds in this and hence there is no hope of obtaining a simulation result. Essentially the same remark applies to weak simulation. For bisimulation and weak bisimulation the techniques used here could apply more or less in the same way, but we have not looked at it and there may be subtleties that are not apparent at first sight. Certainly, these cases involve issues that are orthogonal to anything we have discussed. For real-time systems, more precisely CTMCs, a logical characterization has been available [DP03]. The proof techniques of the present paper should apply to that case straightforwardly once the subtleties of the real-time case (such as the presence of Zeno paths) are taken into account.

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