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Codensity Games for Bisimilarity

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Abstract

Bisimilarity as an equivalence notion of systems has been central to process theory. Due to the recent rise of interest in quantitative systems (probabilistic, weighted, hybrid, etc.), bisimilarity has been extended in various ways, such as bisimulation metric between probabilistic systems. An important feature of bisimilarity is its game-theoretic characterization, where Spoiler and Duplicator play against each other; extension of bisimilarity games to quantitative settings has been actively pursued too. In this paper, we present a general framework that uniformly describes game characterizations of bisimilarity-like notions. Our framework is formalized categorically using fibrations and coalgebras. In particular, our characterization of bisimilarity in terms of fibrational predicate transformers allows us to derive what we call codensity bisimilarity games: a general categorical game characterization of bisimilarity. Our framework covers known bisimilarity-like notions (such as bisimulation metric and bisimulation seminorm) as well as new ones (including what we call bisimulation topology).

Keywords Coalgebra · Bisimulation · Safety game · Bisimulation metric · Fibration

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To Masami Hagiya on the occasion of his 2^6th birthday. Masami's research career has been a role model for us theoretical computer scientists who seek real-world impact through the power of logic. He has shown through his works that one good use of logical abstraction is to tame new computing paradigms. The current work draws inspiration from this, and uses logical (and categorical) abstraction to establish a uniform understanding of various bisimilarity-like notions that otherwise look very different from each other.

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1 Introduction

1.1 Bisimilarity Notions and Games

Since the seminal works by Park and Milner [1, 2], *bisimilarity* has played a central role in theoretical computer science. It is an equivalence notion between branching systems; it abstracts away internal states and stresses the black-box observation-oriented view on process semantics. Bisimilarity is usually defined as the largest *bisimulation*, which is a binary relation that satisfies a suitable mimicking condition. In fact, a bisimulation R can be characterized as a post-fixed point $R \subseteq \Phi(R)$ using a suitable relation transformer Φ; from this we obtain that bisimilarity is the greatest *fixed* point of Φ by the Knaster–Tarski theorem. This order-theoretic foundation is the basis of a variety of advanced techniques for reasoning about (or using) bisimilarity, such as bisimulation up to—see, e.g., [3].

Bisimilarity is conventionally defined for state-based systems with nondeterministic branching. However, as the applications of computer systems become increasingly pervasive and diverse (such as cyber-physical systems), extension of bisimilarity to systems with other branching types has been energetically sought in the literature. One notable example is the bisimulation notion for probabilistic systems in [4]: it is a relation that witnesses that two states are indistinguishable in their behaviors henceforth. This qualitative notion has also been made quantitative, as the notion of *bisimulation metric* [5]. It replaces a relation with a metric that is induced by the probabilistic transition structure.

There is a body of literature (including [6–12]) that aims to identify the mathematical essences that are shared by this variety of bisimilarity, and to express the identified essences in a rigorous manner using *category theory*. Our particular interest is in the correspondence between bisimilarity notions and *(safety) games*; three examples of the latter are given below. This interest in bisimilarity games is shared by the recent work [10], and the comparison is discussed in Sect. 1.4.

1.1.1 Bisimilarity Games

It is well-known that the following game (summarized in Table 1) characterizes the conventional notion of bisimilarity between Kripke frames. Let $(X, \to)$ be a Kripke frame, where $\to \subseteq X^2$; the game is played between Duplicator (D) and Spoiler (S). In a position (x, y), Spoiler challenges Duplicator's claim that x and y are bisimilar,

Table 1 Game for bisimilarity in a Kripke frame

Position	Player	Possible moves
$(x, y) \in X^2$	Spoiler	$(1, x', y)$ s.t. $x \to x'$ or $(2, x, y')$ s.t. $y \to y'$
$(1, x', y) \in \{1\} \times X^2$	Duplicator	(x', y') s.t. $y \to y'$
$(2, x, y') \in \{2\} \times X^2$	Duplicator	(x', y') s.t. $x \to x'$

by choosing one of the states (say x) and further choosing a transition $x \to x'$. Duplicator responds by choosing a transition $y \to y'$ from the other state, and the game is continued from (x', y'). Duplicator wins if Spoiler gets stuck, or the game continues infinitely long, and this witnesses that x and y are bisimilar.

1.1.2 Games for Probabilistic Bisimilarity

A recent step forward in the topic of bisimilarity and games is the characterization of probabilistic bisimulation introduced in [13]. For simplicity, here we describe its discrete version.

Let (X, c) be a Markov chain, where X is a countable set of states, and $c : X \to \mathcal{D}_{\le 1}X$ is a transition kernel that assigns to each state $x \in X$ a probability subdistribution $c(x) \in \mathcal{D}_{\le 1}X$. Here $\mathcal{D}_{\le 1}X = \{d : X \to [0,1] \mid \sum_{x \in X} d(x) \le 1\}$ denotes the set of probability subdistributions over X. For $Z \subseteq X$, let $c(x)(Z)$ denote the probability with which a successor of x is chosen from Z; that is, $c(x)(Z) = \sum_{x' \in Z} c(x)(x')$. Since $c(x)$ is only a *sub*-distribution over X, the probability $c(x)(X)$ is ≤ 1 rather than $= 1$. The remaining probability $1 - c(x)(X)$ can be thought of as the probability of x getting stuck.

Recall from [4] that an equivalence relation $R \subseteq X^2$ is a *(probabilistic) bisimulation* if, for any $(x, y) \in R$ and each R-closed subset $Z \subseteq X$, $c(x)(Z) = c(y)(Z)$ holds.

The game introduced in [13] is in Table 2. It is shown in [13] that Duplicator is winning in the game at (x, y) if and only if x and y are bisimilar, in the sense of [4] (recalled above). It is not hard to find an intuitive correspondence between the game in Table 2 and the definition of bisimulation [4]: Spoiler challenges the bisimilarity claim between x, y by exhibiting Z such that $c(x)(Z) = c(y)(Z)$ is violated; Duplicator makes a counterargument by claiming that Z is in fact not bisimilarity-closed, exhibiting a pair of states (x', y') that Duplicator claims are bisimilar.

1.1.3 Games for Probabilistic Bisimulation Metric

Our following observation marked the beginning of the current work: the game for (qualitative) bisimilarity for probabilistic systems (from [13], Table 2) can be almost literally adapted to (quantitative) *bisimulation metric* for probabilistic systems. This metric was first introduced in [5].

For simplicity we focus on the discrete setting; we also restrict to pseudometrics bounded by 1. Let (X, c) be a Markov chain with a countable state space X. The *bisimulation metric* $d_{(X,c)} : X^2 \to [0,1]$ is defined to be the smallest pseudometric (with respect to the pointwise order) that makes the transition kernel:

Table 2 Game for probabilistic bisimilarity from [13]

Position	Player	Possible moves
$(x, y) \in X^2$	Spoiler	$Z \subseteq X$ s.t. $c(x)(Z) \ne c(y)(Z)$
$Z \subseteq X$	Duplicator	$(x', y') \in X^2$ s.t. $x' \in Z \wedge y' \notin Z$

$$c : (X, d_{(X,c)}) \longrightarrow \big(\mathcal{D}_{\leq 1}X, \mathcal{K}(d_{(X,c)}) \big)$$

non-expansive with respect to the specified pseudometrics. Here $\mathcal{K}(d_{(X,c)})$ is the so-called *Kantorovich metric* over $\mathcal{D}_{\leq 1}X$ induced by the pseudometric $d_{(X,c)}$ over X. It is defined as follows. For $\mu, \nu \in \mathcal{D}_{\leq 1}X$:

$$\mathcal{K}(d_{(X,c)})(\mu, \nu) = \sup_f \left| E_\mu[f] - E_\nu[f] \right|, \tag{1}$$

where in the above sup,

- f ranges over all non-expansive functions from $(X, d_{(X,c)})$ to $\big([0,1], d_{[0,1]}\big)$,
- $d_{[0,1]}$ denotes the usual Euclidean metric, and
- $E_\mu[f]$ is the expectation $\sum_{x \in X} f(x) \cdot \mu(x)$ of f with respect to μ.

Our observation is that the bisimulation metric $d_{(X,c)}$ is characterized by the game in Table 3: Duplicator is winning at (x, y, ε) if and only if $d_{(X,c)}(x, y) \leq \varepsilon$.

The game seems to be new, although its intuition is similar to the one for Table 2. Note that the formula (1) appears in the condition of Spoiler's moves. Spoiler challenges by exhibiting a "predicate" f that suggests violation of the non-expansiveness of c; and Duplicator makes a counterargument that f is in fact not non-expansive and thus invalid.

1.1.4 Towards a Unifying Framework

The last two games (Table 2 from [13] and Table 3 that seems new) motivate a general framework that embraces both. There are some clear analogies: the games are about *indistinguishability* of states x, y under a class of *observations* (Z and f, respectively), and the *predicates* usable in those observations are subject to certain preservation properties (bisimilarity-closedness in the former, and non-expansiveness in the latter).

1.2 A Codensity-Based Framework for Bisimilarity and Games

The main contribution of the current paper is a categorical framework that derives a variety of bisimilarity notions and corresponding game notions. The correspondence is proved once and for all on the categorical level of generality. It covers the three examples introduced earlier in Sect. 1.1, much like the recent categorical framework

Table 3 Game for (probabilistic) bisimulation metric, adapting [13]

Position	Player	Possible moves
$(x, y, \varepsilon) \in X^2 \times [0,1]$	Spoiler	$f : X \to [0,1]$ Such that $\left|E_{c(x)}[f] - E_{c(y)}[f]\right| > \varepsilon$
$f : X \to [0,1]$	Duplicator	$(x', y', \varepsilon') \in X^2 \times [0,1]$ Such that $\left| f(x') - f(y') \right| > \varepsilon'$

in [10] does. However, our fibration-based formalization has another dimension of generality. For example, besides relations and metrics, our examples include an existing notion called *bisimulation seminorm* and a new one that we call *bisimulation topology*.

The overview of our categorical framework is in the left half of Fig. 1. We build on our previous works [14, 15]. In [14] a general construction called *codensity lifting* is introduced (see ③): given a fibration $\mathbb{E} \xrightarrow{p} \mathbb{C}$ and parameters $(\mathbf{\Omega}, \tau)$ that embody the kind of *observations* we can make, a functor $F : \mathbb{C} \to \mathbb{C}$ is lifted to $F^{\mathbf{\Omega},\tau} : \mathbb{E} \to \mathbb{E}$. In [15], codensity lifting is used to introduce a generic family of bisimulation notions called *codensity bisimilarity*—see ②. In this paper, we extend these previous results by

- introducing the notion of *codensity bisimilarity game* (①) that comes in two variants (*untrimmed* (Sect. 4) and *trimmed* (Sect. 5)),
- establishing the correspondence between codensity bisimulations (②) and games (①) on a fibrational level of generality, and
- working out several concrete examples (④, ⑤).

In general, devising a game notion (④) directly from a bisimilarity notion (⑤) is far from trivial. Indeed, doing so for an individual bisimilarity notion has itself been deemed a scientific novelty [13, 16]. Our codensity-based framework (in the left half of Fig. 1) can automate *part of* this process in the following precise sense.

We derive concrete notions of bisimilarity (⑤) and bisimilarity game (④) as instances; then the correspondence between the two is guaranteed by the categorical general result between ① and ②.

We note, however, that this is no panacea. When one starts with a given concrete notion of bisimilarity (⑤), their next task would be to identify the right choice of the parameters $\mathbb{E} \xrightarrow{p} \mathbb{C}, \mathbf{\Omega}, \tau$ for the codensity lifting (③). This task is not easy in general: we needed to get our hands dirty working out the examples in this paper, in [14], and in [15]. Nevertheless, we believe that the required passage from ⑤ to ③

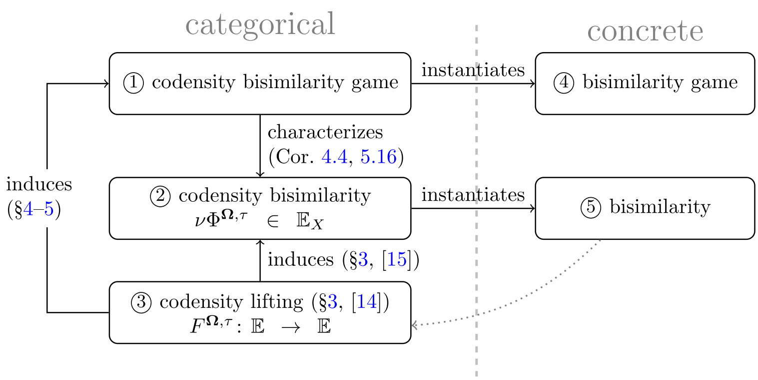


Fig. 1 Our codensity-based framework for bisimilarity and games

is much easier than the direct derivation from ⑤ to ④, with our categorical framework providing templates of bisimilarity games (see Tables 8, 10, 11). After all, our framework identifies which part of the path from ⑤ to ④ can be automated, and which part remains to be done individually. This is much like what many other categorical frameworks offer, as meta-level theories.

As an additional benefit, our categorical framework can be used to *discover* new bisimilarity notions (⑤), starting from (choices of parameters for) ③. We believe those derived new bisimilarity notions are useful, since our categorical theory embodies sound intuitions about observation, predicate transformation, and indistinguishability—see e.g., Sect. 2.2.

1.3 Contributions

Our main technical contributions are as follows.

- We introduce a categorical framework that uniformly describes various bisimulation notions (including metrics, preorders and topologies) and the corresponding game notions (Fig. 1). The framework is based on coalgebras, fibrations, and codensity liftings in particular [14]. Our general game notion comes in two variants.
  - The first (the *untrimmed* codensity game in Sect. 4) arises naturally in a fibration, using its objects and arrows as possible moves. The untrimmed game is theoretically clean, but it tends to have a huge arena.
  - We, therefore, introduce a method that restricts these arenas, leading to the (*trimmed*) codensity bisimilarity game (Sect. 5). The reduction method is also described in general fibrational terms, specifically using fibered separators and join-dense subsets.
- From the general framework, we derive several concrete examples of bisimilarity and its related notions (④ and ⑤ in Fig. 1). They are listed in Table 7 and elaborated in Sect. 8. Among them, a few bisimilarity notions seem new (especially *bisimulation topology* in Sect. 8.3), and several game notions also seem new (especially those for Λ-*bisimulation* in Sect. 8.2 and γ-*bisimulation seminorm* in Sect. 8.6).
- We discuss the *transfer of codensity bisimilarity* by suitable fibered functors (Sect. 7). As an example usage, we give an abstract proof of the fact that (usual) bisimilarity for Kripke frames is necessarily an equivalence (Example 7.4).

In addition, we give a direct proof of the equivalence between our game for bisimulation metric (Table 3), obtained from our general framework, and another game notion for probabilistic bisimilarity, previously introduced in [16]. In the proof, we exhibit a mutual translation of winning strategies (Appendix 1).

The current paper is an extended version of our previous paper [17]. The major additions are the following.

- We show a new transfer result in Sect. 7.2, which has a broader applicability than the result already presented in [17] (and in Sect. 7.1).
- In Sect. 8.1, we additionally present how to specialize codensity bisimilarity to recover another known notion of equivalence, namely, *behavioral equivalence* (see [18] for its relation to bisimilarity). Some examples (already presented in [17]) are reorganized using the result.
- In Sect. 8.2, we show a new connection between our codensity bisimulation and an existing notion of Λ-*bisimulation* [19]. We also derive a general game characterization of some special cases of it, where all the modalities are unary.
- In Sect. 8.6, we show how to represent γ-*bisimulation seminorm* [20] on weighted automata as a special case of codensity bisimilarity. We also derive a game characterization.

We included some proofs that were omitted in [17], too.

1.4 Related Work

Besides the one in [13], another game characterization of probabilistic bisimulation has been given in [16]. It is described later in Sect. 2 (Table 4). The latter game has a bigger arena than the one in [13]: in [16] both players have to play a subset $Z \subseteq X$, while in [13] only Spoiler does so.

The work that is the closest to ours is the recent work [10] that studies bisimilarity games in a categorical setting. Their formalization uses (co)algebras (following the (co)algebraic generalization of the Kantorovich metric introduced in [8]) and, therefore, embraces a variety of different branching types. The major differences between the two works are as follows.

- Our current work is fibration-based (in particular $\mathbf{CLat}_\sqcap$-fibrations), while [10] is not. As a consequence, ours accommodates an additional dimension of generality by changing fibrations, which correspond to different indistinguishability notions (relation, metric, topology, preorder, measurable structures, etc.). In contrast, the works [8, 10] deal exclusively with two settings: binary relations and pseudometrics.

Table 4 Game for probabilistic bisimilarity, from [16]

Position	Player	Possible moves
$(1, x, y) \in \{1\} \times X^2$	Spoiler	$(2, s, t, Z) \in \{2\} \times X^2 \times \mathcal{P}X$ s.t. $\{s, t\} = \{x, y\}$
$(2, s, t, Z) \in \{2\} \times X^2 \times \mathcal{P}X$	Duplicator	$(Z, Z') \in (\mathcal{P}X)^2$ s.t. $c(s)(Z) \le c(t)(Z')$
$(Z, Z') \in (\mathcal{P}X)^2$	Spoiler	$(Z, y') \in \mathcal{P}X \times X$ s.t. $y' \in Z'$ or $(Z', y) \in \mathcal{P}X \times X$ s.t. $y \in Z$
$(Z, y') \in \mathcal{P}X \times X$	Duplicator	$(x', y') \in X^2$ s.t. $x' \in Z$

- A relationship to *modal logic* is beautifully established in [10], while it is not done in this work. Some results connecting our codensity framework and modal logic are presented in [21].
- The categorical generalization [10] is based on the game notion in [16], while ours is based on that in [13]. Therefore, for some bisimulation notions (including the bisimulation metric), we obtain a game notion with a smaller arena. Compare Tables 3 (an instance of ours) and 5 (an instance of [10]).

There are a number of categorical studies of bisimilarity notions; notable mentions include open map-based approaches [22] and coalgebraic ones [23, 24]. The fibrational approach we adopt also uses coalgebras; it was initiated in [6] and pursued, e.g., in [7, 9, 11, 15]. For example, in the recent work [11], fibrational generality is exploited to study up-to techniques for bisimilarity metric. They use the *Wasserstein lifting* of functors introduced in [8] instead of the codensity lifting that we use (it generalizes the *Kantorovich lifting* in [8], see Example 3.5). It is known [8] that the Wasserstein and Kantorovich liftings can differ in general, while they coincide for some specific functors, such as the distribution functor.

Some of our new examples are topological: we derive what we call *bisimulation topology* and a game notion that characterizes it. The relation between these notions and the existing works on bisimulation and topology (including [25, 26]) is left as future work.

In Sect. 5, we reduce the game arena by focusing on a join-dense subset. A game notion proposed in [27] uses a similar method. A major difference is that they restrict themselves to *continuous lattices*, while we only require each fiber to be a complete lattice. This condition plays a critical role in their framework, but it is a future work to seek consequences of the continuity assumption in our setting.

1.5 Organization

In Sect. 2, we present preliminaries on a general theory of games (we can restrict to *safety* games), and on fibrations. For the latter, we focus on a class called $\mathbf{CLat}_\sqcap$

Table 5 Game for bisimulation metric, from [10]

Position	Player	Possible moves
$(x, y, \varepsilon) \in X^2 \times [0,1]$	Spoiler	$(s, t, f, \varepsilon) \in X^2 \times [0,1]^X \times [0,1]$ s.t. $\{s, t\} = \{x, y\}$
$(s, t, f, \varepsilon) \in X^2 \times [0,1]^X \times [0,1]$	Duplicator	$(f, g, \varepsilon) \in ([0,1]^X)^2 \times [0,1]$ such that $\max\{0, E_{c(s)}[f] - E_{c(t)}[g]\} \le \varepsilon$
$(f, g, \varepsilon) \in ([0,1]^X)^2 \times [0,1]$	Spoiler	$(x', i, j, \varepsilon) \in X \times ([0,1]^X)^2 \times [0,1]$ such that $\{i, j\} = \{f, g\}$
$(x', i, j, \varepsilon) \in X \times ([0,1]^X)^2 \times [0,1]$	Duplicator	$(x', y', \varepsilon') \in X^2 \times [0,1]$ such that $i(x') \le j(y')$, and $\varepsilon' = j(y') - i(x')$

-*fibrations*, and argue that they offer an appropriate categorical abstraction of sets equipped with indistinguishability structures. In Sect. 3, we present codensity lifting and codensity bisimilarity ($\mathring{2}$, $\mathring{3}$ in Fig. 1). The material is based on [15], but we introduce some auxiliary notions needed for the correspondence with games. Our first game notion (the *untrimmed* one) is introduced in Sect. 4; in Sect. 5, we cut down the arenas and obtain *trimmed* codensity bisimilarity game. The theory is further extended in Sects. 6 and 7: in Sect. 6 we accommodate multiple observation domains, and in Sect. 7 we discuss the transfer of codensity bisimilarities by fibered functors preserving meets. These categorical observations give rise to the concrete examples in Sect. 8.

2 Preliminaries

Since bisimilarity is defined as the greatest fixed point of a certain map, we will often be manipulating such fixed points. We recall two well-known characterizations of fixed points:

Theorem 2.1 (Knaster–Tarski [28]) *Let L be a complete lattice and $f : L \to L$ be a monotone map. Then the greatest fixed point of f exists and it is the greatest prefixpoint of f, i.e., the greatest element $x \in L$ such that $f(x) \leq x$.* □

Theorem 2.2 (*Cousot–Cousot* [29]) *Let L be a complete lattice and $f : L \to L$ be a monotone map. Using transfinite induction, let us define a sequence $(f_\alpha)_\alpha$ (indexed by an ordinal α) by the following*:

$$f_\alpha = \sqcap_{\beta<\alpha} f(f_\beta).$$

Note that

1. $f_0 = \top$, *the greatest element of L, and*
2. $f_{\alpha+1} = f(f_\alpha)$ *for any ordinal α.*

Then there is an ordinal α such that $f_\alpha = f_{\alpha+1}$ and, for such α, f_α is the greatest fixed point of f. □

2.1 Safety Games

Here we recall some standard game-theoretic notions and results. In capturing bisimilarity-like notions, we can restrict ourselves to *safety games*—they have a simple winning condition, where every infinite play is won by the same player (namely, Duplicator). This winning condition reflects the characterization of bisimilarity-like notions by suitable *greatest* fixed points; the correspondence generalizes, for example, to the one between parity games and nested alternating fixed points—see [30]. The term “safety game” occurs, e.g., in [31, 32].

Safety games are played between two players; in this paper, they are called *Duplicator* (D) and *Spoiler* (S). We restrict to those games in which Duplicator and Spoiler alternate turns.

Definition 2.3 (safety game) A *(safety game) arena* is a triple $\mathcal{G} = (Q_D, Q_S, E)$ of a set Q_D of *Duplicator's positions*, a set Q_S of *Spoiler's positions*, and a *transition relation*

$E \subseteq (Q_D \times Q_S) \cup (Q_S \times Q_D)$. Hence $\mathcal{G}$ is a bipartite graph. We require that Q_D and Q_S are disjoint, and that $Q_D \cup Q_S \neq \emptyset$. We write $Q = Q_D \cup Q_S$.

For a position $q \in Q$, an element of the set $\{q' \in Q \mid (q, q') \in E\}$ is called a *possible move* at q. Unlike some works, we allow positions that have no possible moves at them.

A *play* in an arena $\mathcal{G} = (Q_D, Q_S, E)$ is a (finite or infinite) sequence of positions $q_0 q_1 \ldots$, such that $(q_{i-1}, q_i) \in E$ so long as q_i belongs to the sequence.

A play in $\mathcal{G}$ is *won* by either player, according to the following conditions: (1) a finite play $q_0 \ldots q_n$ is won by Spoiler (or by Duplicator) if $q_n \in Q_D$ (or $q_n \in Q_S$, respectively) and (2) every infinite play $q_0 q_1 \ldots$ is won by Duplicator.

Definition 2.4 (Strategy, winning position) In an arena $\mathcal{G} = (Q_D, Q_S, E)$, a *strategy* of Duplicator is a partial function $\sigma_D : Q^* \times Q_D \rightharpoonup Q_S$; we require that $\sigma_D(\vec{q}, q) = q'$ implies $(q, q') \in E$. A strategy of Duplicator σ_D is *positional* if $\sigma_D(\vec{q}, q)$ depends only on q. A *strategy* of Spoiler is defined similarly, as a partial function $\sigma_S : Q^* \times Q_S \rightharpoonup Q_D$ that returns a possible move at the last position in the history. It is *positional* if $\sigma_S(\vec{q}, q)$ does not depend on $\vec{q}$.

Given an initial position $q \in Q$ and two strategies σ_D and σ_S for Duplicator and Spoiler, respectively, the *play* from q induced by (σ_D, σ_S) is defined in a natural inductive manner. The induced play is denoted by $\pi^{\sigma_D, \sigma_S}(q)$.

A position $q \in Q$ is said to be *winning* for Duplicator if there exists a strategy σ_D of Duplicator such that, for any strategy σ_S of Spoiler, the induced play $\pi^{\sigma_D, \sigma_S}(q)$ is won by Duplicator.

In what follows, for simplicity, we restrict the initial position q of a play $\pi^{\sigma_D, \sigma_S}(q)$ to be in Q_S. (Note that Spoiler's position can be winning for Duplicator.)

Any position in a safety game is winning for one of the players. Moreover, the winning strategy can be taken to be positional one [30, Theorem 6]. Thus, we can focus on the winning positions of the players.

Winning positions of safety games are witnessed by *invariants* (Proposition 2.6). This is a well-known fact.

Definition 2.5 (invariant) Let $\mathcal{G} = (Q_D, Q_S, E)$ be an arena. A subset $P \subseteq Q_S$ is called an *invariant* for Duplicator if, for each $q \in P$ and any possible move $q' \in Q_D$ at q, there exists a possible move q'' at q' that is in P. That is,

$$\forall q \in P. \forall q' \in Q_D. \big((q, q') \in E \;\Rightarrow\; \exists q'' \in Q_S. \ (q', q'') \in E \land q'' \in P \big).$$

Proposition 2.6

1. *Any position $q \in P$ in an invariant P for Duplicator is winning for Duplicator.*
2. *Invariants are closed under arbitrary union. Therefore, there exists the largest invariant for Duplicator.*
3. *The largest invariant for Duplicator coincides with the set of winning positions for Duplicator in Q_S.*

Proof

1. Turn P into a positional strategy of Duplicator that forces a play back in P.
2. Obvious.
3. It suffices to show that every position $q \in Q_S$ winning for Duplicator lies in some invariant. Let $\sigma_D : Q^* \times Q_D \rightharpoonup Q_S$ be a strategy of Duplicator ensuring that q is winning. Define $P \subseteq Q_S$ as follows:

$$P = \{q' \in Q_S \mid \exists \sigma_S.\ q' \text{ is visited in } \pi^{\sigma_D,\sigma_S}(q)\}.$$

Then P is an invariant, because q is winning for Duplicator. □

Examples of safety games have been given in Tables 2 and 3. We present two other examples (Tables 4, 5).

Example 2.7 (Alternative games for probabilistic bisimilarity and bisimulation metric) In [16], the notion of *ε-bisimulation* and a game notion characterizing it are introduced. In the case where ε is 0, ε-bisimulation coincides with (qualitative) probabilistic bisimilarity and thus the game characterizes it. The game in $\varepsilon = 0$ case is in Table 4, presented in a slightly adapted form.

This game notion is categorically generalized in [10]; the generalization has freedom in the choice of coalgebra functors (i.e., branching types), as well as in the choice between relations and metrics. The instance of this general game notion for bisimulation metric is shown in Table 5.

The two games (Tables 4, 5) characterize the same bisimilarity-like notions as the games in Table 2 and 3, respectively; so they are equivalent. We can go further and give a direct equivalence proof by mutually translating winning strategies. Such a proof is not totally trivial; we do so for the pair for probabilistic bisimilarity.

See Appendix 1.

We note that the game in Table 3 (an instance of our current framework) is simpler than Table 5 (an instance of [10]). Table 3 is not only structurally simpler (it has fewer rows), but its set of moves are smaller too, asking for functions $X \to [0, 1]$ only at one place.

Our categorical framework based on codensity liftings (presented in later sections) covers Tables 2 and 3 but not Tables 4 and 5.

2.2 $\mathbf{CLat}_{\sqcap}$-Fibrations

2.2.1 Definition and Properties

Here we sketch a basic theory of fibrations—see, e.g., [33] for a comprehensive account. In particular, we focus on a class of poset fibrations called $\mathbf{CLat}_{\sqcap}$-*fibrations*. We observe that the simple axiomatics of the class adequately capture all the examples of interest—and hence the mathematical essences of the logical phenomena that we wish to model.

Our exposition here is largely based on that in [15]. However, in this paper we introduce new notation and terminology (such as *indistinguishability order* and *decent map*)—see Sect. 2.2.2. They help to further clarify the intuitions.

In Appendix 2, we include a rather gentle introduction to $\mathbf{CLat}_{\sqcap}$-fibration. In particular, Definition 9.1 gives a definition of $\mathbf{CLat}_{\sqcap}$-fibration that does not depend on the notion of (general) fibration.

Here we start with the following shorter definition, which does depend on the definition of fibration.

Definition 2.8 ($\mathbf{CLat}_{\sqcap}$-fibration) A $\mathbf{CLat}_{\sqcap}$-*fibration* is a fibration $\mathbb{E} \xrightarrow{p} \mathbb{C}$ such that each fiber $\mathbb{E}_X$ (for each $X \in \mathbb{C}$) is a complete lattice, and the pullback functor $f^* : \mathbb{E}_Y \to \mathbb{E}_X$ (for each $f : X \to Y$ in $\mathbb{C}$) preserves all meets $\sqcap$. The set of objects of a fiber $\mathbb{E}_X$ is denoted $|\mathbb{E}_X|$.

Via the Grothendieck construction, a $\mathbf{CLat}_{\sqcap}$-fibration is in a bijective correspondence with a functor $F_{\mathbb{E}} : \mathbb{C}^{\mathrm{op}} \to \mathbf{CLat}_{\sqcap}$, where $\mathbf{CLat}_{\sqcap}$ is the category of complete lattices and functions preserving all meets—see [33] and [7], as well as Appendix 2. The functor $F_{\mathbb{E}}$ assigns

- a complete lattice $\mathbb{E}_X$ (called the *fiber* over X) to each $X \in \mathbb{C}$, and
- a function $f^* : \mathbb{E}_Y \to \mathbb{E}_X$ preserving all meets to each $f : X \to Y$ in $\mathbb{C}$. The map f^* is called a *pullback* (or *reindexing*); it is also called a *pullback functor*, since, in the general theory of fibrations, a fiber $\mathbb{E}_X$ is a category rather than a poset.

Although the *indexed category* presentation $F_{\mathbb{E}} : \mathbb{C}^{\mathrm{op}} \to \mathbf{CLat}_{\sqcap}$ may be more intuitive at first, we shall stick to the *fibration* presentation $\mathbb{E} \xrightarrow{p} \mathbb{C}$, since we will eventually need some global structures in the *total category* $\mathbb{E}$. It turns out that $\mathbf{CLat}_{\sqcap}$-fibrations are special kinds of *topological functors* [34] in which each fiber category is a poset. Topological functors are a well-studied topic, and many examples and results are available; a good summary is found in [35].

The use of poset fibrations is common in categorical modeling of logics [7, 9]. $\mathbf{CLat}_{\sqcap}$-fibrations additionally require fibered small meets; this simple assumption turns out to be a mathematically powerful one.

Proposition 2.9 *Let* $\mathbb{E} \xrightarrow{p} \mathbb{C}$ *be a* $\mathbf{CLat}_{\sqcap}$*-fibration.*

1. *p is split, and faithful as a functor.*
2. *Each arrow* $f : X \to Y$ *has its pushforward* $f_* : \mathbb{E}_X \to \mathbb{E}_Y$*, so that an adjunction* $f_* \dashv f^*$ *is formed. This is a consequence of Freyd's adjoint functor theorem; it makes p a bifibration* [33].
3. $p^{\mathrm{op}} : \mathbb{E}^{\mathrm{op}} \to \mathbb{C}^{\mathrm{op}}$ *is also a* $\mathbf{CLat}_{\sqcap}$*-fibration.*
4. *The change-of-base* [33, *Lemma* 1.5.1] *of p along any functor* $H : \mathbb{D} \to \mathbb{C}$ *is also a* $\mathbf{CLat}_{\sqcap}$*-fibration.*
5. *If* $\mathbb{C}$ *is (co)complete, then the total category* $\mathbb{E}$ *is also (co)complete. This follows from* [33, *Proposition* 9.2.1]. □

2.2.2 Notation, Terminology and Intuitions

Our view of a $\mathbf{CLat}_{\sqcap}$-fibration $\mathbb{E} \xrightarrow{p} \mathbb{C}$ is that it equips objects of $\mathbb{C}$ with what we call *indistinguishability structures*. This suits our purpose, since various bisimilarity-like notions are all about degrees of indistinguishability between (the behaviors of) states of a system. We present examples later in Sect. 2.2.3.

Notation 2.10 (Indistinguishability predicate/order) Let $\mathbb{E} \xrightarrow{p} \mathbb{C}$ be a $\mathbf{CLat}_{\sqcap}$-fibration. An object $P \in \mathbb{E}_X$ in the fiber category $\mathbb{E}_X$ (i.e., an element of the complete lattice $\mathbb{E}_X$) is called an *indistinguishability predicate* over X. Our view is that P is an additional structure on X; therefore, as a convention, an object $P \in \mathbb{E}_X$ shall also be denoted by $(X, P) \in \mathbb{E}_X$.

Each fiber $\mathbb{E}_X$ is a complete lattice; its order is denoted by $\sqsubseteq$ and called the *indistinguishability order* over X. Intuitively, $P \sqsubseteq Q$ means that Q has a greater degree of indistinguishability than P—that is, Q is coarser than P, and P is more discriminating than Q.

The supremum and infimum with respect to the indistinguishability order $\sqsubseteq$ are denoted by $\bigsqcup$ and $\sqcap$, respectively.

Definition 2.11 (Decent map) Let $\mathbb{E} \xrightarrow{p} \mathbb{C}$ be a $\mathbf{CLat}_{\sqcap}$-fibration, $f : X \to Y$ be an arrow in $\mathbb{C}$, $(X, P) \in \mathbb{E}_X$ and $(Y, Q) \in \mathbb{E}_Y$ be objects in the fibers. We say that f is *decent* from P to Q if there exists a (necessarily unique) arrow $\dot{f} : P \to Q$ in $\mathbb{E}$ such that $p\dot{f} = f$. We write $f : (X, P) \dot{\to} (Y, Q)$ in this case. The following equivalences follow.

$$f : (X, P) \dot{\to} (Y, Q) \iff P \sqsubseteq f^* Q \iff f_* P \sqsubseteq Q$$

We write $f : (X, P) \dot{\not\to} (Y, Q)$ if f is *not* decent.

The notion of decency is a fibered generalization of continuity, non-expansiveness, relation-preservation, etc. Decency $f : (X, P) \dot{\to} (Y, Q)$ means f *respects* indistinguishability, carrying P-indistinguishable elements to Q-indistinguishable ones.

2.2.3 Examples

We first fix some notations.

Definition 2.12 We write $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ for the covariant powerset functor, and 2 for the two-point set $2 = \{\bot, \top\}$. We define the function $\Diamond : \mathcal{P}2 \to 2$ called the *may-modality* by $\Diamond S = \top$ if and only if $\top \in S$. We write Eq_I for the diagonal (equality) relation over a set I.

As shown in Table 6, various well-known categories can be seen as categories that equip sets with certain indistinguishability structures. The evident forgetful functors from the total categories (**Top**, **Meas**, etc.) to **Set** in Table 6 are all $\mathbf{CLat}_\sqcap$-fibrations.

Specifically, **Top** is the category of topological spaces and continuous maps; **Meas** is that of measurable spaces and measurable maps; $\mathbf{PMet}_1$ is that of 1-bounded pseudometric spaces (where a *pseudo*-metric is a metric without the condition $d(x, y) = 0 \Rightarrow x = y$) and non-expansive maps; **ERel** is that of sets with endorelations $(X, R \subseteq X^2)$ and relation-preserving maps; **Pre** is that of preordered sets and monotone maps; and **EqRel** is that of sets with equivalence relations and relation-preserving maps—see [15] for details.

Note that, in **Top** and **Meas**, the indistinguishability order is the opposite of the inclusion order. Therefore, the meet of a family of indistinguishability structures is computed as the one generated from the *union* of the family.

We also use a few $\mathbf{CLat}_\sqcap$-fibrations over categories other than **Set**. One is "the fibration of binary relations":

Definition 2.13 ($\mathbf{BRel} \to \mathbf{Set}^2$) We define the category **BRel** as follows:

- An object is a triple $(X, Y, R \subseteq X \times Y)$ of two sets and a relation between them.
- An arrow from (X, Y, R) to (Z, W, S) is a pair $(f : X \to Z, g : Y \to W)$ of functions such that $(x, y) \in R$ implies $(f(x), g(y)) \in S$.

Table 6 $\mathbf{CLat}_\sqcap$-fibrations over **Set**

Fibration	Indistinguishability structure	Decent map	$P \sqsubseteq Q$	$\sqcap P_i$
$\mathbf{Top} \to \mathbf{Set}$	Topology	Continuous func.	$P \supseteq Q$	Generated from $\bigcup P_i$
$\mathbf{Meas} \to \mathbf{Set}$	σ-algebra	Measurable func.	$P \supseteq Q$	Generated from $\bigcup P_i$
$\mathbf{PMet}_1 \to \mathbf{Set}$	Pseudometric	non-expansive func.	$\forall x, y.\, P(x, y) \geq Q(x, y)$	$(x, y) \mapsto \sup_i P_{i(x,y)}$
$\mathbf{ERel} \to \mathbf{Set}$	Endorelation	Relation preserving func.	$P \subseteq Q$	$\bigcap P_i$
$\mathbf{Pre} \to \mathbf{Set}$	Preorder	Monotone func.	$P \subseteq Q$	$\bigcap P_i$
$\mathbf{EqRel} \to \mathbf{Set}$	Equivalence relation	Relation preserving func.	$P \subseteq Q$	$\bigcap P_i$

The forgetful functor $\mathbf{BRel} \to \mathbf{Set}^2$ is then a $\mathbf{CLat}_\sqcap$-fibration.

This can be used for modeling bisimulations between two different systems. See Sect. 8.2.

Another one is "the fibration of (extended) seminorms."

Definition 2.14 ($\mathbf{ESemi}_\mathbb{R} \to \mathbf{Vect}_\mathbb{R}$) We define the category $\mathbf{ESemi}_\mathbb{R}$ as follows:

- An object is a pair (V, s) where
  - V is a real vector space and
  - $s : V \to \mathbb{R} \cup \{\infty\}$ is an *extended seminorm* on V, i.e., a seminorm that can take ∞ as a value.
- An arrow from (U, s_U) to (V, s_V) is a linear map $f : U \to V$ that is also nonexpansive, i.e., satisfying $s_U(u) \geq s_V(f(u))$ for all $u \in U$.

Let $\mathbf{Vect}_\mathbb{R}$ be the category of real vector spaces and linear maps. The forgetful functor $\mathbf{ESemi}_\mathbb{R} \to \mathbf{Vect}_\mathbb{R}$ is then a $\mathbf{CLat}_\sqcap$-fibration.

Note that including ∞ as a value is essential to make $\mathbf{ESemi}_\mathbb{R} \to \mathbf{Vect}_\mathbb{R}$ a $\mathbf{CLat}_\sqcap$ -fibration: without ∞ the fibers fail to be a complete lattice. This is used in analyzing real-weighted automata. See Sect. 8.6.

Yet another class of examples is given as follows: for any well-powered category $\mathbb{B}$ admitting small limits, the subobject fibration $\mathbf{Sub}(\mathbb{B}) \to \mathbb{B}$ of $\mathbb{B}$ is a $\mathbf{CLat}_\sqcap$-fibration. All the algebraic categories over $\mathbf{Set}$ and Grothendieck toposes satisfy these conditions of $\mathbb{B}$. We note, however, that the forgetful functors from algebraic categories over $\mathbf{Set}$ are rarely ($\mathbf{CLat}_\sqcap$-)fibrations.

3 Codensity Bisimilarity

We introduce *codensity lifting* ($\mathring{3}$ in Fig. 1) and *codensity bisimilarity* ($\mathring{2}$) based on [15]. These turn out to subsume many bisimilarity-like notions in the literature. The material in Sects. 3.1 and 3.2 is largely from [15]; Sect. 3.3 is new, paving the way to codensity bisimilarity games presented in later sections.

3.1 Codensity Lifting

Definition 3.1 [codensity lifting $F^{\mathbf{\Omega},\tau}$ [15]] Let $\mathbb{E} \xrightarrow{p} \mathbb{C}$ be a $\mathbf{CLat}_\sqcap$-fibration, and $F : \mathbb{C} \to \mathbb{C}$ be a functor. A *parameter of codensity lifting* of F along p is a pair of

- a $\mathbb{C}$-arrow $\tau : F\Omega \to \Omega$ (i.e., an F-algebra) called a *modality* [37, 38] and
- an $\mathbb{E}$-object $\mathbf{\Omega}$ above Ω called an *observation domain*.

The *codensity lifting* of $F : \mathbb{C} \to \mathbb{C}$ with parameter $(\mathbf{\Omega}, \tau)$ is the endofunctor $F^{\mathbf{\Omega},\tau} : \mathbb{E} \to \mathbb{E}$ defined as follows. On objects

$$F^{\mathbf{\Omega},\tau}P = \sqcap_{k\in\mathbb{E}(P,\mathbf{\Omega})}\big(\tau\circ F\big(p(k)\big)\big)^{*}\mathbf{\Omega}.$$

Its action on arrows is as follows. It is not hard to see that, for each arrow $l : P \to Q$ in $\mathbb{E}$, the arrow $F(p(l))$ is decent from $F^{\mathbf{\Omega},\tau}P$ to $F^{\mathbf{\Omega},\tau}Q$. Then we define $F^{\mathbf{\Omega},\tau}l : F^{\mathbf{\Omega},\tau}P \to F^{\mathbf{\Omega},\tau}Q$ to be the unique arrow in $\mathbb{E}$ above $F(p(l))$.

Let us elaborate on the above definition. Let $P \in \mathbb{E}_X$ and $X = pP$. The point is to regard an arrow $X \to \Omega$ in $\mathbb{C}$ as an "observation" on X and an object $P \in \mathbb{E}_X$ as "information" on X. Our goal is to obtain "information" on FX from that on X.

We begin with taking some $k : P \to \mathbf{\Omega}$ in $\mathbb{E}$. For such k, $p(k) : X \to \Omega$ can be seen as an "observation" on the space $X \in \mathbb{C}$. Here, $p(k)$ has to be decent from P to $\mathbf{\Omega}$. Intuitively, this means that the resulting "information" $(p(k))^*\mathbf{\Omega} \in \mathbb{E}_X$ of the "observation" $p(k)$ must be consistent with the information P on X we already have. For example, in Example 3.3, the arrow $p(k) : X \to 2$, intuitively an "observation," corresponds to a subset of X. The resulting "information" $(p(k))^*\mathrm{Eq}_2 \in \mathbf{EqRel}_X$ of $p(k)$ is the induced equivalence relation:

$$(p(k))^*\mathrm{Eq}_2 = \{(x, y) \in X^2 \mid p(k)(x) = p(k)(y)\},$$

and it must be "consistent" with the given equivalence relation $(X, R) \in \mathbf{EqRel}$, that is, each equivalence class of $(p(k))^*\mathrm{Eq}_2$ must be R-closed.

The "observation" $p(k) : X \to \Omega$ is simply an arrow, so we can apply the given functor $F : \mathbb{C} \to \mathbb{C}$ to it. The result is $F(p(k)) : FX \to F\Omega$. To obtain an "observation" on FX, we have to compose it with some modality $\tau : F\Omega \to \Omega$. In Example 3.3, this process gives an "observation" $\Diamond \circ \mathcal{P}(p(k)) : \mathcal{P}X \to 2$ on $\mathcal{P}X$, and it satisfies the following for each $S \in \mathcal{P}X$:

$$(\Diamond \circ \mathcal{P}(p(k)))(S) = \top \iff \exists x \in S.\ p(k)(x) = \top.$$

Note the existential quantification $\exists$ above. It is the part, where the modality $\Diamond$ comes up.

Now that we have an "observation" on FX, we obtain "information" on FX by pullback. The following diagram is the summary of this situation:

$$\begin{array}{ccccc}
\mathbb{E} & \big(\tau\circ F\big(p(k)\big)\big)^{*}\mathbf{\Omega} & & \cdots\cdots\cdots\cdots\rightarrow & \mathbf{\Omega} \\
\Big\downarrow p & & & & \\
\mathbb{C} & F(pP) & \xrightarrow[F(p(k))]{} & F\Omega & \xrightarrow[\tau]{} \Omega.
\end{array}$$

Finally, gathering all the "information" $(\tau\circ F(p(k)))^*\mathbf{\Omega}$ leads to the definition (Definition 3.1). In the setting of Example 3.3, the result of this process is the equivalence relation on $\mathcal{P}X$, defined for each $S, T \subseteq X$ by

$$\begin{aligned}
&\forall k : X \to 2.\ \big((\forall (x, y) \in R.\ k(x) = k(y)) \\
&\quad \Longrightarrow \big((\exists x \in S.\ k(x) = \top) \iff (\exists x \in T.\ k(x) = \top)\big)\big).
\end{aligned}$$

It is equivalent to another more familiar definition, as described in Example 3.3.

One might wonder how codensity lifting is related with *codensity monad* [39, Exercise X.7.3]. The following proposition exhibits the relationship.

Proposition 3.2 *Let $p : \mathbb{E} \to \mathbb{C}$ be a $\mathbf{CLat}_\sqcap$-fibration, $F : \mathbb{C} \to \mathbb{C}$ be a functor and $(\mathbf{\Omega}, \tau)$ be a parameter of codensity lifting of F along p. Moreover, we assume that $\mathbb{E}$ has powers [39, Section III.4] and p preserves them. For any $P \in \mathbb{E}$, $F^{\mathbf{\Omega},\tau}P$ coincides with the vertex of the following pullback*:

$$\begin{array}{ccc} \mathbb{E} & & F^{\mathbf{\Omega},\tau}P \cdots\cdots\to \mathbf{\Omega}^{\mathbb{E}(P,\mathbf{\Omega})} \\ \downarrow p & & \\ \mathbb{C} & & F(pP) \xrightarrow[\alpha_P]{} \Omega^{\mathbb{E}(P,\mathbf{\Omega})} \end{array}$$

where $\alpha_P = \langle \tau \circ F(p(k)) \rangle_{k \in \mathbb{E}(P,\mathbf{\Omega})}$ is the morphism obtained by the tupling of the power of Ω in $\mathbb{C}$.

In fact, codensity lifting of monads is first defined in terms of the above pullback [14]. The name "codensity lifting" comes from the fact that the above pullback involves the codensity monad $\mathbf{\Omega}^{\mathbb{E}(-,\mathbf{\Omega})}$.

Table 7 lists concrete examples of codensity liftings, with various fibrations p, functors F, and parameters $(\mathbf{\Omega}, \tau)$. Some of them coincide with known notions. For example, the entry 5 of the table says that the functor $(\mathcal{D}_{\leq 1})^{\mathbf{\Omega},\tau}$, with the designated $\mathbf{\Omega}$ and τ, carries a metric space (X, d) to the set $\mathcal{D}_{\leq 1}X$ equipped with the well-known Kantorovich metric $\mathcal{K}(d)$ induced by d. See (1).

Besides the functors listed in the table, there are some natural ways to systematically lift polynomial functors, by defining $\tau : F\Omega \to \Omega$ in an inductive manner—see, e.g., [11].

Example 3.3 Let us take a close look at the entry 4 of Table 7. There we codensity-lift the covariant powerset functor $\mathcal{P}$ along the $\mathbf{CLat}_\sqcap$-fibration $\mathbf{EqRel} \to \mathbf{Set}$. We use the parameter $((2, \mathrm{Eq}_2), \diamond)$, where $\diamond : \mathcal{P}2 \to 2$ is the modality given in Definition 2.12.

We shall abbreviate $(2, \mathrm{Eq}_2)$ by Eq_2—a notational convention that is used throughout the paper.

Then $\mathcal{P}^{\mathrm{Eq}_2,\diamond}(X, R)$ relates $S, T \in \mathcal{P}X$ if and only if

$$\begin{aligned} &\forall k : X \to 2. \Big((\forall (x, y) \in R.\ k(x) = k(y)) \\ &\quad \implies \big((\exists x \in S.\ k(x) = \top) \iff (\exists x \in T.\ k(x) = \top) \big) \Big). \end{aligned}$$

Straightforward calculation shows that this is equivalent to

$$(\forall x \in S.\ \exists y \in T.\ (x, y) \in R) \wedge (\forall y \in T.\ \exists x \in S.\ (x, y) \in R).$$

This lifting is the restriction (to **EqRel**) of the standard relational lifting of $\mathcal{P}$ along $\mathbf{ERel} \to \mathbf{Set}$, which is used for the usual bisimulation notion for Kripke frames [40].

Table 7 Codensity lifting of functors

	Fibration $\mathbb{E} \xrightarrow{p} \mathbb{C}$	Functor $F : \mathbb{C} \to \mathbb{C}$	obs. dom. $\mathbf{\Omega}$	Modality τ	Lifting $F^{\mathbf{\Omega},\tau}$ of F
1	$\mathbf{Pre} \to \mathbf{Set}$	Powerset $\mathcal{P}$	$(2, \leq)$	$\Diamond : \mathcal{P}2 \to 2$	Lower preorder [14]
2	$\mathbf{Pre} \to \mathbf{Set}$	Powerset $\mathcal{P}$	$(2, \geq)$	$\Diamond : \mathcal{P}2 \to 2$	Upper preorder [14]
3	$\mathbf{ERel} \to \mathbf{Set}$	Powerset $\mathcal{P}$	$(2, \mathrm{Eq}_2)$	$\Diamond : \mathcal{P}2 \to 2$	(See Ex. 3.4 & 7.4)
4	$\mathbf{EqRel} \to \mathbf{Set}$	Powerset $\mathcal{P}$	$(2, \mathrm{Eq}_2)$	$\Diamond : \mathcal{P}2 \to 2$	(See Ex. 3.3 & 7.4)
5	$\mathbf{PMet}_1 \to \mathbf{Set}$	Subdistrib. $\mathcal{D}_{\leq 1}$	$([0,1], d_{[0,1]})$	$e : \mathcal{D}_{\leq 1}[0,1] \to [0,1]$	Kantorovich metric [14]
6	$\mathbf{PMet}_1 \to \mathbf{Set}$	Powerset $\mathcal{P}$	$([0,1], d_{[0,1]})$	$\inf : \mathcal{P}[0,1] \to [0,1]$	Hausdorff metric (Appx. C)
7	$U^*(\mathbf{PMet}_1) \to \mathbf{Meas}$	Sub-Giry $\mathcal{G}_{\leq 1}$	$([0,1], d_{[0,1]})$	$e : \mathcal{G}_{\leq 1}[0,1] \to [0,1]$	Kantorovich metric [14]
$8^{\dagger}$	$\mathbf{Pre} \to \mathbf{Set}$	Powerset $\mathcal{P}$	$(2, \leq), (2, \geq)$	$\Diamond : \mathcal{P}2 \to 2$	Convex preorder [14]
$9^{\dagger}$	$\mathbf{EqRel} \to \mathbf{Set}$	Subdistrib. $\mathcal{D}_{\leq 1}$	$(2, \mathrm{Eq}_2)$	$(\tau_r : \mathcal{D}_{\leq 1}2 \to 2)_{r \in [0,1]}$	(For prob. bisim., see Ex. 8.15)
$10^{\dagger}$	$\mathbf{Top} \to \mathbf{Set}$	$2 \times (\_)^{\Sigma}$	*Sierpinski* sp.	(See Ex. 6.12)	(For bisim. top., see Ex. 6.12)
$11^{\dagger}$	$\mathbf{BRel} \to \mathbf{Set}^2$	Any functor	$((1,1), R_2)$	Any family	(For Λ-bisim., see Sect. 8.2)
$12^{\dagger}$	$\mathbf{ESemi}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{R}}$	$(\bigoplus_{a \in \Sigma}(\_)) \oplus \mathbb{R}$	$(\mathbb{R}, \lvert \cdot \rvert)$	(See Sect. 8.6)	(For bisim. seminorm, see Sect. 8.6)

The fibration $U^*(\mathbf{PMet}_1) \to \mathbf{Meas}$ is introduced in Sect. 8.5. $d_{[0,1]}$ denotes the Euclidean metric on the unit interval [0, 1]. The modality $\Diamond$ is introduced in Definition 2.12. The functions $e : \mathcal{D}_{\leq 1}[0,1] \to [0,1]$ and $e : \mathcal{G}_{\leq 1}[0,1] \to [0,1]$ both return expected values. The lower, upper and convex preorders are known for powerdomains; see e.g., [36]. The function $\tau_r : \mathcal{D}_{\leq 1}2 \to 2$ is introduced in Example 8.15. The examples marked with † involve multiple modalities and observation domains (Sect. 6)

Example 3.4 In the entry 3 of Table 7, we codensity-lift $\mathcal{P}$ along the $\mathbf{CLat}_{\sqcap}$-fibration $\mathbf{ERel}\to\mathbf{Set}$ (instead of $\mathbf{EqRel}\to\mathbf{Set}$) with the parameter $\big((2,\mathrm{Eq}_2),\Diamond\big)$.

The characterization of $\mathcal{P}^{\mathrm{Eq}_2,\Diamond}(X,R)$ is slightly involved. Its relation part relates $S,T\in\mathcal{P}X$ if and only if

$$(\forall x\in S.\ \exists y\in T.\ (x,y)\in R^{\mathrm{Eq}})\ \wedge\ (\forall y\in T.\ \exists x\in S.\ (x,y)\in R^{\mathrm{Eq}}),$$

where R^{Eq} denotes the equivalence closure of R.

It is not clear at this stage whether the codensity bisimilarities induced by the above liftings (Examples 3.3, 3.4, i.e., the entries 4 and 3 of Table 7) coincide with the usual bisimilarity notion for Kripke frames. This is because of the involvement of mandatory equivalence closures—specifically by the use of **EqRel** in Example 3.3, and by the occurrence of $(\_)^{\mathrm{Eq}}$ in Example 3.4. Later, in Example 7.4, we prove that both of the codensity bisimilarities indeed coincide with the usual bisimilarity notion. The proof relies crucially on transfer of codensity liftings via fibered functors.

Example 3.5 Here we follow [15, Example 3] and show that codensity lifting generalizes a categorical construction introduced in [8], namely, the *Kantorovich lifting* of functors.

Take $\mathbf{PMet}_1\to\mathbf{Set}$ as the $\mathbf{CLat}_{\sqcap}$-fibration p in Definition 3.1.

As $\mathbf{\Omega}$, we take $\Omega=[0,1]$ with the usual Euclidean metric $d_{[0,1]}$. There is freedom in the choice of a modality $\tau : F\Omega\to\Omega$—this corresponds to what is called an *evaluation function* in [8]. This way we recover the Kantorovich lifting in [8] as $F^{\mathbf{\Omega},\tau}$.

3.2 Codensity Bisimilarity

In [15], *codensity bisimulation* and *bisimilarity* are introduced. Recall that a *coalgebra* $c : X\to FX$ is a categorical presentation of state-based transition systems, such as automata, Markov chains, etc.—see, e.g., [23, 24], and also Sect. 8.

Definition 3.6 Assume the setting of Definition 3.1. Let $c : X\to FX$ be an F-coalgebra. An object $P\in\mathbb{E}_X$ is a $((\mathbf{\Omega},\tau)$-$)$ *codensity bisimulation* over c if $c : (X,P)\dot{\to}(FX,F^{\mathbf{\Omega},\tau}P)$; that is, c is decent with respect to the designated indistinguishability structures on X and FX.

We move on to the characterization of codensity bisimulations as post-fixpoints of suitable predicate transformers.

Definition 3.7 (Predicate transformer $\Phi^{\mathbf{\Omega},\tau}$) Assume the setting of Definition 3.6. We define a *predicate transformer* $\Phi^{\mathbf{\Omega},\tau}_c : \mathbb{E}_X\to\mathbb{E}_X$ with respect to c and $F^{\mathbf{\Omega},\tau}$ by

$$\Phi_c^{\mathbf{\Omega},\tau} P = c^*(F^{\mathbf{\Omega},\tau} P). \tag{2}$$

Since c^* is $\sqcap$-preserving, expanding the definition of $F^{\mathbf{\Omega},\tau}$ yields

$$\Phi_c^{\mathbf{\Omega},\tau} P = \sqcap_{k \in \mathbb{E}(P,\mathbf{\Omega})} \left(\tau \circ F(p(k)) \circ c\right)^* \mathbf{\Omega}.$$

Theorem 3.8 *Assume the setting of Definition 3.6. For any $P \in \mathbb{E}_X$, the following are equivalent.*

1. $c : (X, P) \dot{\to} (FX, F^{\mathbf{\Omega},\tau} P)$; *that is, P is a codensity bisimulation over c (Definition 3.6).*
2. $P \sqsubseteq \Phi_c^{\mathbf{\Omega},\tau} P$.
3. *For each $k \in \mathbb{C}(X, \Omega)$, $k : (X, P) \dot{\to} (\Omega, \mathbf{\Omega})$ implies $\tau \circ Fk \circ c : (X, P) \dot{\to} (\Omega, \mathbf{\Omega})$.*

□

Proof The equivalence between the conditions (1) and (2) can be seen from the definitions of $\Phi_c^{\mathbf{\Omega},\tau}$ (Definition 3.7) and decency (Definition 2.11). Now we show (2) $\iff$ (3).

Using Definition 3.7, the condition (1) is equivalent to

$$P \sqsubseteq \sqcap_{k \in \mathbb{E}(P,\mathbf{\Omega})} \left(\tau \circ F(p(k)) \circ c\right)^* \mathbf{\Omega}.$$

The definition of meet implies that the above inequality is equivalent to the following:

For each $k \in \mathbb{C}(X, \Omega)$, $k : (X, P) \dot{\to} (\Omega, \mathbf{\Omega})$ implies $P \sqsubseteq \left(\tau \circ F(p(k)) \circ c\right)^* \mathbf{\Omega}$.

This is, in turn, equivalent to the condition (3.8), as can be seen from the definition of decency (Definition 2.11). □

The predicate transformer $\Phi_c^{\mathbf{\Omega},\tau}$ is a monotone map from the complete lattice $\mathbb{E}_X$ to itself. Therefore, by the Knaster–Tarski theorem (Theorem 2.1), the greatest post-fixed point of $\Phi_c^{\mathbf{\Omega},\tau}$ exists and it is the greatest fixed point of $\Phi_c^{\mathbf{\Omega},\tau}$.

Definition 3.9 (Codensity bisimilarity $\nu\Phi_c^{\mathbf{\Omega},\tau}$) Assume the setting of Definition 3.6. The greatest codensity bisimulation, whose existence is guaranteed by the above arguments, is called the *codensity bisimilarity*. It is denoted by $\nu\Phi_c^{\mathbf{\Omega},\tau}$.

Some bisimilarity notions, including bisimilarity of deterministic automata (Example 8.11), are accommodated in the generalized framework with multiple observation domains—see Sect. 6.

Example 3.10 (Bisimulation metric) Consider the $\mathbf{CLat}_\sqcap$-fibration $\mathbf{PMet}_1 \to \mathbf{Set}$ and the subdistribution functor $\mathcal{D}_{\le 1} : \mathbf{Set} \to \mathbf{Set}$. Recall that $\mathcal{D}_{\le 1}(X) = \{p : X \to [0, 1] \mid \sum_{x \in X} p(x) \le 1\}$.

As a parameter of codensity lifting, we take $(\mathbf{\Omega}, \tau) = \big(([0,1], d_{[0,1]}),\ e : \mathcal{D}_{\leq 1}[0,1] \to [0,1] \big)$, where e is the *expectation function* $e(p) = \sum_{r \in [0,1]} r \cdot p(r)$ and $d_{[0,1]}$ is the Euclidean metric.

Let $c : X \to \mathcal{D}_{\leq 1}X$ be a coalgebra, identified with a Markov chain.

The codensity bisimilarity in this setting coincides with the bisimulation metric from [5] (see also Sect. 1.1.3). This fact is not hard to check directly; one can also derive the coincidence via Example 3.5 and the observations in [8].

3.3 Joint Codensity Bisimulation

We introduce the notion of *joint codensity bisimulation*. This minor variation of codensity bisimulation becomes useful in the proof of soundness and completeness of our game notion (Sect. 4).

Definition 3.11 Assume the setting of Definition 3.6. Let $\mathcal{V} \subseteq |\mathbb{E}_X|$; joins in $\mathbb{E}_X$ are denoted by $\bigsqcup$. We say that $\mathcal{V}$ is a *joint codensity bisimulation* over c if $\bigsqcup_{P \in \mathcal{V}} P$ is a codensity bisimulation over c.

For instance, the set of all codensity bisimulations is a joint codensity bisimulation, because the join of all codensity bisimulations is the largest codensity bisimulation $\nu\Phi_c^{\mathbf{\Omega},\tau}$, as discussed just before Definition 3.9.

Lemma 3.12 *In the setting of Definition* 3.6, *the downset* $\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})$ *is the largest joint codensity bisimulation* (*with respect to the inclusion order*).

Proof The downset $\downarrow(\nu\Phi^{\mathbf{\Omega},\tau})$ is a joint codensity bisimulation, because the union of all elements of $\downarrow(\nu\Phi^{\mathbf{\Omega},\tau})$ is equal to a codensity bisimulation $\nu\Phi^{\mathbf{\Omega},\tau}$.

Let $\mathcal{V}$ be a joint codensity bisimulation. Then for any $P \in \mathcal{V}$, we have $P \sqsubseteq \nu\Phi^{\mathbf{\Omega},\tau}$, because $P \sqsubseteq \bigsqcup_{Q \in \mathcal{V}} Q \sqsubseteq \nu\Phi^{\mathbf{\Omega},\tau}$. □

4 Untrimmed Games for Codensity Bisimilarity

As the first main technical contribution, we introduce what we call the *untrimmed* version of codensity bisimilarity game. It is mathematically simple but its game arenas can become much bigger than necessary. The *trimmed* version of games—with smaller arenas—will be introduced later in Sect. 5, after developing necessary categorical infrastructure.

Table 8 Untrimmed codensity bisimilarity game

Position	Player	Possible moves
$P \in \mathbb{E}_X$	Spoiler	$k \in \mathbb{C}(X, \Omega)$ s.t. $\tau \circ Fk \circ c : (X, P) \dot{\not\to} (\Omega, \mathbf{\Omega})$
$k \in \mathbb{C}(X, \Omega)$	Duplicator	$P' \in \mathbb{E}_X$ s.t. $k : (X, P') \dot{\not\to} (\Omega, \mathbf{\Omega})$

Definition 4.1 (Untrimmed codensity bisimilarity game) Assume the setting of Definition 3.6. The *untrimmed codensity bisimilarity game* is the safety game played by two players Duplicator and Spoiler, shown in Table 8.

Lemma 4.2 *Assume the setting of Definition 3.6. Let $\mathcal{V} \subseteq |\mathbb{E}_X|$. The following are equivalent.*

1. *$\mathcal{V}$ is an invariant for Duplicator (Definition 2.5) in the untrimmed codensity bisimilarity game (Table 8).*
2. *$\mathcal{V}$ is a joint codensity bisimulation over c.*

Proof We use the following logical equivalence:

$$\begin{aligned}
1) &\iff \left(\begin{array}{l} \forall P \in \mathcal{V}, k : X \to \Omega. \\ \tau \circ Fk \circ c : (X, P) \dot{\not\to} (\Omega, \mathbf{\Omega}) \implies \exists P' \in \mathcal{V}.\, k : (X, P') \dot{\not\to} (\Omega, \mathbf{\Omega}) \end{array} \right) \\
&\iff \left(\begin{array}{l} \forall P \in \mathcal{V}, k : X \to \Omega. \\ (\forall P' \in \mathcal{V}.\, k : (X, P') \dot{\to} (\Omega, \mathbf{\Omega})) \implies \tau \circ Fk \circ c : (X, P) \dot{\to} (\Omega, \mathbf{\Omega}) \end{array} \right) \\
&\iff \left(\begin{array}{l} \forall k : X \to \Omega. \\ (\forall P' \in \mathcal{V}.\, k : (X, P') \dot{\to} (\Omega, \mathbf{\Omega})) \\ \implies \forall P \in \mathcal{V}.\, \tau \circ Fk \circ c : (X, P) \dot{\to} (\Omega, \mathbf{\Omega}) \end{array} \right).
\end{aligned}$$

Here, since $k : (X, P') \dot{\to} (\Omega, \mathbf{\Omega})$ means $P' \sqsubseteq k^* \mathbf{\Omega}$, the condition

$$\forall P' \in \mathcal{V}.\, k : (X, P') \dot{\to} (\Omega, \mathbf{\Omega})$$

is equivalent to

$$k : (X, \textstyle\bigsqcup_{P' \in \mathcal{V}} P') \dot{\to} (\Omega, \mathbf{\Omega}).$$

Similarly, the condition

$$\forall P \in \mathcal{V}.\, \tau \circ Fk \circ c : (X, P) \dot{\to} (\Omega, \mathbf{\Omega})$$

is equivalent to

$$\tau \circ Fk \circ c : (X, \textstyle\bigsqcup_{P \in \mathcal{V}} P) \dot{\to} (\Omega, \mathbf{\Omega}).$$

These imply the following logical equivalence:

Table 9 Untrimmed codensity game for bisimulation metric

Position	Player	Possible moves
$d \in (\mathbf{PMet}_1)_X$	Spoiler	$k \in \mathbf{Set}(X, [0,1])$ s.t. $e \circ Fk \circ c \notin \mathbf{PMet}_1(d, d_{[0,1]})$
$k \in \mathbf{Set}(X, [0,1])$	Duplicator	$d' \in (\mathbf{PMet}_1)_X$ s.t. $k \notin \mathbf{PMet}_1(d', d_{[0,1]})$

$$1) \iff \left(\begin{array}{l} \forall k : X \to \Omega. \\ \left(k : \left(X, \bigsqcup_{P' \in \mathcal{V}} P'\right) \dot{\to} (\Omega, \mathbf{\Omega})\right) \\ \implies \tau \circ Fk \circ c : (X, \bigsqcup_{P \in \mathcal{V}} P) \dot{\to} (\Omega, \mathbf{\Omega}) \end{array} \right).$$

By Theorem 3.8, the condition in the right-hand side is equivalent to

$$\bigsqcup_{P \in \mathcal{V}} P \sqsubseteq \Phi_c^{\mathbf{\Omega},\tau}\left(\bigsqcup_{P \in \mathcal{V}} P\right).$$

□

Theorem 4.3 *Assume the setting of Definition 3.6. In the untrimmed codensity bisimilarity game (Table 10), the following coincide.*

1. *The set of all winning positions for Duplicator.*
2. *The downset $\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})$ of the codensity bisimilarity.*

Proof We use Lemma 4.2 to connect the game and the predicate transformer. By considering the largest set satisfying the condition in Lemma 4.2, it implies that the following two coincide if both exist:

1’ the largest invariant for Duplicator in the game in Table 10 and
2’ the largest joint codensity bisimulation over c.

By the general theory of safety games, in particular Proposition 2.6, the set (1’) is equal to (1). On the other hand, by Lemma 3.12, the set (2’) coincides with (2). Combining these proves the claim. □

We conclude that our game characterizes the codensity bisimilarity $\nu\Phi_c^{\mathbf{\Omega},\tau}$ (Definition 3.9).

Table 10 Trimmed codensity bisimilarity game

Position	Player	Possible moves
$P \in \mathcal{G}$	Spoiler	$k \in \mathbb{C}(X, \Omega)$ s.t. $\tau \circ Fk \circ c : (X, P) \dot{\nrightarrow} (\Omega, \mathbf{\Omega})$
$k \in \mathbb{C}(X, \Omega)$	Duplicator	$P' \in \mathcal{G}$ s.t. $k : (X, P') \dot{\nrightarrow} (\Omega, \mathbf{\Omega})$

Corollary 4.4 *In the untrimmed codensity bisimilarity game* (*Table* 10), $P \in \mathbb{E}_X$ *is a winning position for Duplicator if and only if* $P \sqsubseteq \nu\Phi_c^{\Omega,\tau}$. □

Example 4.5 Recall Example 3.10. Using the untrimmed codensity bisimilarity game, we can characterize the bisimulation metric from [5]. Our general definition (Definition 4.1) instantiates to the one in Table 9, which is, however, more complicated than the game we exhibited in the introduction (Table 3). For example, in Table 9, Duplicator's move is a pseudometric $d : X^2 \to [0,1]$ rather than a triple (x, y, ε).

5 Trimmed Codensity Games for Bisimilarity

Our previous untrimmed game (Table 8) is pleasantly simple from a theoretical point of view. However, as we saw in Example 4.5, its instances tend to have a much bigger arena than some known game notions.

Here we push our theory a step further, and present a fibrational construction that allows us to *trim* our games. We note that our construction still remains on the fibrational level of abstraction.

5.1 Join-Dense Subsets of Fibers and Fibered Separators

Our approach to trim down the game arena is to restrict Spoiler's position to *approximants* of elements in the fiber complete lattice. In lattice theory, the collection of such approximants is specified by a *join-dense subset* [41], which we recall below.

Definition 5.1 (Join-dense subset) A subset $\mathcal{G}$ of a complete lattice L is *join-dense* if for any $P \in L$, there exists $\mathcal{A} \subseteq \mathcal{G}$ such that $P = \bigsqcup \mathcal{A}$.

Example 5.2 Consider the $\mathbf{CLat}_\sqcap$-fibration $\mathbf{EqRel} \to \mathbf{Set}$ and $X \in \mathbf{Set}$. For any $x, y \in X$, we define the equivalence relation $E_{x,y}$ to be the least one equating x, y, that is, $(z, w) \in E_{x,y}$ if and only if $(z = w \vee \{z, w\} = \{x, y\})$. Then the set $\mathcal{G} = \{E_{x,y} \mid x, y \in X\}$ of all such equivalence relations is a join-dense subset of the fiber $\mathbf{EqRel}_X$.

Example 5.3 Recall Example 3.10. For $x, y \in X$ $(x \neq y)$ and $r \in [0,1]$, the pseudometric $d_{x,y,r}$ over X is defined by

$$d_{x,y,r}(z,w) = \begin{cases} 0 & z = w \\ r & \{z,w\} = \{x,y\} \\ 1 & \text{otherwise.} \end{cases}$$

Then the set of pseudometrics $\{d_{x,y,r} \mid x, y \in X, x \neq y, r \in [0,1]\}$ is a join-dense subset of the fiber $(\mathbf{PMet}_1)_X$.

We use the following characterization of a join-dense subset.

Lemma 5.4 *For a subset $\mathcal{G}$ of a complete lattice L, the following are equivalent.*

- *$\mathcal{G}$ is join-dense.*
- *For any $P, Q \in L$,*

$$(\forall G \in \mathcal{G}.\, G \sqsubseteq P \implies G \sqsubseteq Q) \implies P \sqsubseteq Q$$

holds.

Proof Assume that $\mathcal{G}$ is join-dense. For any $P, Q \in L$, we show $(\forall G \in \mathcal{G}.\, G \sqsubseteq P \implies G \sqsubseteq Q) \implies P \sqsubseteq Q$. Since $\mathcal{G}$ is join-dense, there exists a subset $\mathcal{A} \subseteq \mathcal{G}$ such that $P = \bigsqcup \mathcal{A}$. If $(\forall G \in \mathcal{G}.\, G \sqsubseteq P \implies G \sqsubseteq Q)$ holds, then, for each $A \in \mathcal{A}$, we have $A \sqsubseteq P$, and thus $A \sqsubseteq Q$. This implies $P \sqsubseteq Q$.

Conversely, assume that, for any $P, Q \in L$, $(\forall G \in \mathcal{G}.\, G \sqsubseteq P \implies G \sqsubseteq Q) \implies P \sqsubseteq Q$ holds. We show that $\mathcal{G}$ is join-dense, that is, for any $P \in L$, there exists $\mathcal{A} \subseteq \mathcal{G}$ such that $P = \bigsqcup \mathcal{A}$. More concretely, we define

$$A^{\mathcal{G}}(P) = \{P' \in \mathcal{G} \mid P' \sqsubseteq P\}$$

for each $P \in L$ and we show $P = \bigsqcup A^{\mathcal{G}}(P)$. It suffices to show the following for each $Q \in L$:

$$(\forall P' \in A^{\mathcal{G}}(P).\, P' \sqsubseteq Q) \implies P \sqsubseteq Q.$$

By the definition of $A^{\mathcal{G}}(P)$, it is equivalent to the following:

$$(\forall P' \in \mathcal{G}.\, P' \sqsubseteq P \implies P' \sqsubseteq Q) \implies P \sqsubseteq Q.$$

This is nothing but our assumption. □

We next consider the problem of equipping each fiber of a $\mathbf{CLat}_\sqcap$-fibration with a join-dense subset. One way to do so is to transfer a join-dense subset of the fiber over a special object called *fibered separator*, which we introduce below.

Definition 5.5 (Fibered separator) Let $\mathbb{E} \xrightarrow{p} \mathbb{C}$ be a $\mathbf{CLat}_\sqcap$-fibration. We say that $S \in \mathbb{C}$ is a *fibered separator* if, for any $X \in \mathbb{C}$ and $P, Q \in \mathbb{E}_X$, we have

$$(\forall f \in \mathbb{C}(S, X).\, f^*P = f^*Q) \implies P = Q.$$

Fibered separator can equivalently be defined using fiber order $\sqsubseteq$.

Lemma 5.6 *In the setting of Definition 5.5, the following are equivalent.*

- *$S \in \mathbb{C}$ is a fibered separator.*
- *For any $X \in \mathbb{C}$ and $P, Q \in \mathbb{E}_X$,*

$$(\forall f \in \mathbb{C}(S, X).\, f^*P \sqsubseteq f^*Q) \implies P \sqsubseteq Q$$

holds.

Proof Assume that $S \in \mathbb{C}$ is a fibered separator. For each $X \in \mathbb{C}$ and $P, Q \in \mathbb{E}_X$, we show $(\forall f \in \mathbb{C}(S, X).\, f^*P \sqsubseteq f^*Q) \implies P \sqsubseteq Q$. Assume $(\forall f \in \mathbb{C}(S, X).\, f^*P \sqsubseteq f^*Q)$. Then for each $f : S \to X$ we have $f^*P = f^*P \sqcap f^*Q = f^*(P \sqcap Q)$ and since S is a fibered separator, $P = P \sqcap Q$, that is, $P \sqsubseteq Q$. Thus we have $(\forall f \in \mathbb{C}(S, X).\, f^*P \sqsubseteq f^*Q) \implies P \sqsubseteq Q$.

Conversely, assume that $(\forall f \in \mathbb{C}(S, X).\, f^*P \sqsubseteq f^*Q) \implies P \sqsubseteq Q$ holds for any $X \in \mathbb{C}$ and $P, Q \in \mathbb{E}_X$. We show that $S \in \mathbb{C}$ is a fibered separator, that is, $(\forall f \in \mathbb{C}(S, X).\, f^*P = f^*Q) \implies P = Q$ for each $X \in \mathbb{C}$ and $P, Q \in \mathbb{E}_X$. Assume $(\forall f \in \mathbb{C}(S, X).\, f^*P = f^*Q)$. Then both $(\forall f \in \mathbb{C}(S, X).\, f^*P \sqsubseteq f^*Q)$ and $(\forall f \in \mathbb{C}(S, X).\, f^*P \sqsupseteq f^*Q)$ hold. By the assumption, we have both $P \sqsubseteq Q$ and $P \sqsupseteq Q$. Thus $P = Q$. □

A join-dense subset of the fiber over a fibered separator induces one over any other fiber by the following theorem.

Theorem 5.7 *Let $S \in \mathbb{C}$ be a fibered separator of a $\mathbf{CLat}_\sqcap$-fibration $\mathbb{E} \xrightarrow{p} \mathbb{C}$, and $\mathcal{G}$ be a join-dense subset of $\mathbb{E}_S$. For any $X \in \mathbb{C}$, the following is a join-dense subset of $\mathbb{E}_X$ (below f_* denotes the pushforward along f; see Sect. 2.2):*

$$\{f_* G \mid G \in \mathcal{G}, f \in \mathbb{C}(S, X)\}.$$

Proof Let $P, Q \in \mathbb{E}_X$. By Lemma 5.4, it suffices to show

$$(\forall G \in \mathcal{G}, f \in \mathbb{C}(S, X).\, f_* G \sqsubseteq P \implies f_* G \sqsubseteq Q) \implies P \sqsubseteq Q.$$

Since f_* is the left adjoint of f^* (Proposition 2.9), it is equivalent to

$$(\forall G \in \mathcal{G}, f \in \mathbb{C}(S, X).\ G \sqsubseteq f^*P \implies G \sqsubseteq f^*Q) \implies P \sqsubseteq Q.$$

Assume $(\forall G \in \mathcal{G}, f \in \mathbb{C}(S, X).\ G \sqsubseteq f^*P \implies G \sqsubseteq f^*Q)$. Since $\mathcal{G}$ is join-dense in $\mathbb{E}_S$, Lemma 5.4 implies $(\forall f \in \mathbb{C}(S, X).\, f^*P \sqsubseteq f^*Q)$. It in turn implies $P \sqsubseteq Q$ by Lemma 5.6. □

In fact, it is Theorem 5.7 that is behind Examples 5.2 and 5.3: in both cases, $2 \in \mathbf{Set}$ turns out to be a fibered separator for the fibrations in question ($\mathbf{EqRel} \to \mathbf{Set}$ and $\mathbf{PMet}_1 \to \mathbf{Set}$), and the presented generating sets are obtained via pushforward.

We next relate fibered separators and separators in a category $\mathbb{C}$. Recall that an object S in a category $\mathbb{C}$ is a *separator* [39, Section V.7] if for any parallel pair of morphisms $f, g : X \to Y$, if $f \circ x = g \circ x$ holds for any $x : S \to X$, then $f = g$.

Proposition 5.8 (Fibered separator and separator) *Let* $\mathbb{E} \xrightarrow{p} \mathbb{C}$ *be a* $\mathbf{CLat}_{\sqcap}$*-fibration. Let* $V \in \mathbb{C}$ *be an object such that there is a family of injections* $\iota_X : |\mathbb{E}_X| \rightarrowtail \mathbb{C}(X, V)$ *natural in* $X \in \mathbb{C}$*. If* $S \in \mathbb{C}$ *is a separator of* $\mathbb{C}$*, then it is also a fibered separator of* p*.*

Note that here we regard $|\mathbb{E}_{(\_)}|$ as a contravariant functor $\mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ by the pull-back operation.

Proof Assume that $S \in \mathbb{C}$ is a separator of $\mathbb{C}$. Expanding the definition for $V \in \mathbb{C}$ yields the following:

$$\forall X \in \mathbb{C}, p, q \in \mathbb{C}(X, V).\ \big((\forall f \in \mathbb{C}(S, X).\ p \circ f = q \circ f) \implies p = q\big). \tag{3}$$

Now, let $X \in \mathbb{C}$ and $P, Q \in \mathbb{E}_X$. We show the implication in Definition 5.5, as follows.

$$\begin{aligned}
&(\forall f \in \mathbb{C}(S, X).\ f^*P = f^*Q) \\
&\implies (\forall f \in \mathbb{C}(S, X).\ \iota_S(f^*P) = \iota_S(f^*Q)) \\
&\implies (\forall f \in \mathbb{C}(S, X).\ \iota_S(P) \circ f = \iota_S(Q) \circ f) && \text{(by naturality)} \\
&\implies \iota_S(P) = \iota_S(Q) && \text{(by(3))} \\
&\implies P = Q. && \text{(since } \iota_X \text{ is injective)}
\end{aligned}$$

Thus $S \in \mathbb{C}$ is fibered separator. □

Any example of this is "unary," as can be seen in the following one.

Example 5.9 (fibered separator of $\mathbf{Pred} \to \mathbf{Set}$) We define the category $\mathbf{Pred}$ as follows:

- An object is a triple $(X, R \subseteq X)$ of a set and a predicate on it.
- An arrow from (X, R) to (Y, S) is a function $f : X \to Y$ such that $x \in R$ implies $f(x) \in S$.

The forgetful functor $\mathbf{Pred} \to \mathbf{Set}$ is then a $\mathbf{CLat}_{\sqcap}$-fibration.

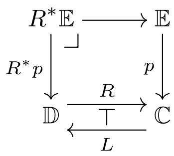
There exists a natural family of injections $\iota_X : |\mathbf{Pred}_X| \rightarrowtail \mathbf{Set}(X, 2)$, which sends $R \subseteq X$ to $\iota_X(R) : X \to 2$ defined as follows:

$$\iota_X(R)(x) = \begin{cases} \top & \text{if } x \in R \\ \bot & \text{otherwise} \end{cases}$$

Since 1 is a separator of $\mathbf{Set}$, we can conclude that it is also a fibered separator of $\mathbf{Pred} \to \mathbf{Set}$ by Proposition 5.8.

The following result is useful in finding fibered separators—see Sect. 8.5.

Proposition 5.10 (Change-of-base and fibered separators) *Let* $\mathbb{E} \xrightarrow{p} \mathbb{C}$ *be a* $\mathbf{CLat}_{\sqcap}$ *-fibration,* $R : \mathbb{D} \to \mathbb{C}$ *be a functor with a left adjoint* $L : \mathbb{C} \to \mathbb{D}$*, and* $S \in \mathbb{C}$ *be a fibered separator for* p*. Then* $LS \in \mathbb{D}$ *is a fibered separator for the change-of-base fibration* R^*p:



Proof For any $X \in \mathbb{D}$, the mapping $f \mapsto Rf \circ \eta_S$ is a bijection of type $\mathbb{D}(LS, X) \to \mathbb{C}(S, RX)$. Thus, naturally identifying $(R^*\mathbb{E})_X$ and $\mathbb{E}_{RX}$, we have the following for any $P, Q \in (R^*\mathbb{E})_X$.

$$\begin{aligned} \forall f \in \mathbb{D}(LS, X).\, f^*P = f^*Q &\implies \forall f \in \mathbb{D}(LS, X).\ (Rf)^*P = (Rf)^*Q \\ &\implies \forall f \in \mathbb{D}(LS, X).\ (Rf \circ \eta_S)^*P = (Rf \circ \eta_S)^*Q \\ &\iff \forall g \in \mathbb{C}(S, RX).\ g^*P = g^*Q \\ &\iff P = Q \end{aligned}$$

□

5.2 $\mathcal{G}$-Joint Codensity Bisimulation

We use join-dense subsets to restrict moves in codensity games.

Definition 5.11 In the setting of Definition 3.6, let $\mathcal{G}$ be a join-dense subset of $\mathbb{E}_X$. A *$\mathcal{G}$-joint codensity bisimulation* over $c : X \to FX$ is a joint codensity bisimulation $\mathcal{V}$ over c such that $\mathcal{V} \subseteq \mathcal{G}$.

Lemma 5.12 *Assume the setting of Definition* 3.6*, and let* $\mathcal{G}$ *be a join-dense subset of* $\mathbb{E}_X$*. The intersection* $\left(\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})\right) \cap \mathcal{G}$ *of the downset* $\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})$ *and the join-dense subset* $\mathcal{G}$ *is the largest* $\mathcal{G}$*-joint codensity bisimulation.*

Proof Since $\mathcal{G}$ is join-dense, the union of all elements of $\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau}) \cap \mathcal{G}$ is equal to $\nu\Phi_c^{\mathbf{\Omega},\tau}$. Thus, $\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau}) \cap \mathcal{G}$ is a $\mathcal{G}$-joint codensity bisimulation.

For any $\mathcal{G}$-joint codensity bisimulation $\mathcal{V}$, it is a joint codensity bisimulation, and we have already shown $\mathcal{V} \subseteq \downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})$ in the proof of Lemma 3.12. We also have $\mathcal{V} \subseteq \mathcal{G}$ by definition. These imply $\mathcal{V} \subseteq \downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau}) \cap \mathcal{G}$. □

5.3 Trimmed Codensity Bisimilarity Games

The above structural results lead to our second game notion.

Definition 5.13 Assume the setting of Definition 3.6, and that $\mathcal{G} \subseteq \mathbb{E}_X$ is a join-dense subset. The *(trimmed) codensity bisimilarity game* is the safety game played by two players Duplicator and Spoiler, shown in Table 10.

Lemma 5.14 *Assume the setting of Definition 5.13. Let $\mathcal{V} \subseteq |\mathbb{E}_X|$. The following are equivalent.*

1. *$\mathcal{V}$ is an invariant for Duplicator (Definition 2.5) in the trimmed codensity bisimilarity game (Table 10).*
2. *$\mathcal{V}$ is a $\mathcal{G}$-joint codensity bisimulation over c.*

Proof For a subset $\mathcal{V} \subseteq |\mathbb{E}_X|$, the condition (1) is equivalent to the condition that the following both holds:

(a) For any $P \in \mathcal{V}$ and $k : X \to \Omega$ satisfying $\tau \circ Fk \circ c : (X, P) \dot{\nrightarrow} (\Omega, \mathbf{\Omega})$, there exists $P' \in \mathcal{V}$ such that $k : (X, P') \dot{\nrightarrow} (\Omega, \mathbf{\Omega})$ holds.

(b) $\mathcal{V} \subseteq \mathcal{G}$.

The above condition (a) is equivalent to "$\mathcal{V}$ is an invariant for Duplicator in the (untrimmed) codensity bisimilarity game (Table 8)." By Lemma 4.2, it is equivalent to "$\mathcal{V}$ is a joint codensity bisimulation over c."

Thus, the condition (1) is equivalent to "$\mathcal{V}$ is a joint codensity bisimulation c and it is a subset of $\mathcal{G}$." This is, by definition, equivalent to the condition (2). □

Theorem 5.15 *Assume the setting of Definition 5.13. The following sets coincide.*

1. *The set of winning positions for Duplicator in the trimmed codensity bisimilarity game (Table 10).*
2. *The intersection $\left(\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})\right) \cap \mathcal{G}$ of the downset of the codensity bisimilarity over c and the join-dense subset $\mathcal{G}$.*

Proof By Proposition 2.6, (1) is the largest invariant for Duplicator in the trimmed codensity bisimilarity game (Table 10). In turn, by Lemma 5.14, it is the largest $\mathcal{G}$ -joint codensity bisimulation over c. By Lemma 5.12, it coincides with (2). □

We conclude that our second game characterizes the codensity bisimilarity $\nu\Phi_c^{\mathbf{\Omega},\tau}$ (Definition 3.9) too.

Corollary 5.16 *In Definition 5.13, $P \in \mathcal{G}$ is a winning position for Duplicator if and only if $P \sqsubseteq \nu\Phi_c^{\mathbf{\Omega},\tau}$.*

□

6 Multiple Observation Domains

We extend the theory so far and accommodate multiple observation domains and modalities. This extension is needed for some examples, such as those marked with † in Table 7.

We consider the class $\mathbf{Lift}(F, p)$ of liftings of an endofunctor $F : \mathbb{C} \to \mathbb{C}$ along a $\mathbf{CLat}_{\sqcap}$-fibration $\mathbb{E} \xrightarrow{p} \mathbb{C}$. It comes with a natural pointwise partial order:

$$G \sqsubseteq H \iff \forall X \in \mathbb{E}.\ GX \sqsubseteq HX \quad (G, H \in \mathbf{Lift}(F, p)), \tag{4}$$

and the partially ordered class $\mathbf{Lift}(F, p)$ admits meets of arbitrary size. As done in the original codensity lifting of endofunctors in [15] (and that of monads in [14]), we extend the codensity lifting so that it takes a family of parameters $\{(\mathbf{\Omega}_A, \tau_A)\}_{A \in \mathbb{A}}$, and returns the *intersection* of the codensity liftings of F with these parameters.

Definition 6.1 [Codensity lifting of a functor with multiple parameters [15]]
Let $F : \mathbb{C} \to \mathbb{C}$ be a functor, $\mathbb{E} \xrightarrow{p} \mathbb{C}$ be a $\mathbf{CLat}_{\sqcap}$-fibration, $\mathbb{A}$ be a class, and $\{(\mathbf{\Omega}_A, \tau_A)\}_{A \in \mathbb{A}}$ be an $\mathbb{A}$-indexed family of parameters (of the codensity lifting of F along p), which is denoted simply by $(\mathbf{\Omega}, \tau)$. The (multiple-parameter) *codensity lifting* of F with $(\mathbf{\Omega}, \tau)$ is the endofunctor $F^{\mathbf{\Omega}, \tau} : \mathbb{E} \to \mathbb{E}$ defined by the intersection of the codensity liftings:

$$F^{\mathbf{\Omega}, \tau} P = \sqcap_{A \in \mathbb{A}} F^{\mathbf{\Omega}_A, \tau_A} P, \quad \text{that is,} \quad \sqcap_{A \in \mathbb{A}, k \in \mathbb{E}(P, \mathbf{\Omega}_A)} \left(\tau_A \circ F(p(k))\right)^* (\mathbf{\Omega}_A).$$

The rest of the theoretical development is completely parallel to the one in Sects. 3, 4 and 5. The difference is that we have to replace a single-parameter codensity lifting (Definition 3.1) by a multi-parameter one (Definition 6.1).

Definition 6.2 (Codensity bisimulation and codensity bisimilarity) Assume the setting of Definition 6.1. Let $c : X \to FX$ be an F-coalgebra. An object $P \in \mathbb{E}_X$ is a *codensity bisimulation* over c if $c : (X, P) \dot{\to} (FX, F^{\mathbf{\Omega}, \tau} P)$; that is, $c : X \to FX$ is decent with respect to the designated indistinguishability structures.

The largest codensity bisimulation is called the *codensity bisimilarity* and denoted by $\nu \Phi_c^{\mathbf{\Omega}, \tau}$.

Definition 6.3 (Predicate transformer $\Phi^{\mathbf{\Omega}, \tau}$) Assume the setting of Definition 6.2. We define a *predicate transformer* $\Phi_c^{\mathbf{\Omega}, \tau} : \mathbb{E}_X \to \mathbb{E}_X$ with respect to c and $F^{\mathbf{\Omega}, \tau}$ by

$$\Phi_c^{\mathbf{\Omega}, \tau} P = c^* (F^{\mathbf{\Omega}, \tau} P).$$

Since c^* is $\sqcap$-preserving, expanding the definition of $F^{\mathbf{\Omega}, \tau}$ yields

$$\Phi_c^{\mathbf{\Omega}, \tau} P = \sqcap_{A \in \mathbb{A}, k \in \mathbb{E}(P, \mathbf{\Omega}_A)} \left(\tau_A \circ F(p(k)) \circ c\right)^* \mathbf{\Omega}_A.$$

Theorem 6.4 *Assume the setting of Definition 6.2. For any $P \in \mathbb{E}_X$, the following are equivalent.*

Table 11 Trimmed codensity bisimilarity game with multiple observations

Position	Player	Possible moves
$P \in \mathcal{G}$	Spoiler	$A \in \mathbb{A}$ and $k \in \mathbb{C}(X, \Omega_A)$ s.t. $\tau_A \circ Fk \circ c : (X, P) \not\dot\rightarrow (\Omega_A, \mathbf{\Omega}_A)$
$A \in \mathbb{A}$ and $k \in \mathbb{C}(X, \Omega_A)$	Duplicator	$P' \in \mathcal{G}$ s.t. $k : (X, P') \not\dot\rightarrow (\Omega_A, \mathbf{\Omega}_A)$

1. $c : (X, P) \dot\rightarrow (FX, F^{\mathbf{\Omega},\tau} P)$; *that is, P is a codensity bisimulation over c* (*Definition* 6.2).
2. $P \sqsubseteq \Phi_c^{\mathbf{\Omega},\tau} P$.
3. *For each* $A \in \mathbb{A}$ *and* $k \in \mathbb{C}(X, \Omega_A)$, $k : (X, P) \dot\rightarrow (\Omega_A, \mathbf{\Omega}_A)$ *implies* $\tau_A \circ Fk \circ c : (X, P) \dot\rightarrow (\Omega_A, \mathbf{\Omega}_A)$.

Proof The same as Theorem 3.8, except that we have multiple parameters here. □

Definition 6.5 Assume the setting of Definition 6.2. We say that $\mathcal{V} \subseteq |\mathbb{E}_X|$ is a *joint codensity bisimulation* over c if $\bigsqcup_{P \in \mathcal{V}} P$ is a codensity bisimulation over c.

Definition 6.6 In the setting of Definition 6.2, let $\mathcal{G}$ be a join-dense subset of $\mathbb{E}_X$. A *$\mathcal{G}$-joint codensity bisimulation* over $c : X \to FX$ is a joint codensity bisimulation $\mathcal{V}$ over c such that $\mathcal{V} \subseteq \mathcal{G}$.

Lemma 6.7 *Assume the setting of Definition* 6.6. *The intersection* $\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau}) \cap \mathcal{G}$ *of the join-dense subset* $\mathcal{G}$ *and the downset* $\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})$ *is the largest* $\mathcal{G}$*-joint codensity bisimulation.*

Proof The same as Lemma 5.12, except that we have multiple parameters here. □

Definition 6.8 (Codensity bisimilarity game) In the setting of Definition 6.2, let $\mathcal{G}$ be a join-dense subset of $\mathbb{E}_X$. The *(trimmed) codensity bisimilarity game (with multiple observations)* is the safety game, played by two players D and S, shown in Table 11.

Lemma 6.9 *Assume the setting of Definition* 6.6. *Let* $\mathcal{V} \subseteq |\mathbb{E}_X|$ *be a set of objects. The following are equivalent.*

1. $\mathcal{V}$ *is an invariant for Duplicator in the trimmed codensity bisimilarity game with multiple observations* (*Table* 11).
2. $\mathcal{V}$ *is a* $\mathcal{G}$*-joint codensity bisimulation over c.*

Proof We use the following logical equivalence:

$$1) \iff \left(\begin{array}{l} \mathcal{V} \subseteq \mathcal{G} \text{ and} \\ \forall P \in \mathcal{V}, A \in \mathbb{A}, k : X \to \Omega_A. \\ \tau_A \circ Fk \circ c : (X,P) \not\dot\to (\Omega_A, \boldsymbol{\Omega}_A) \\ \implies \exists P' \in \mathcal{V}.\ k : (X,P') \not\dot\to (\Omega_A, \boldsymbol{\Omega}_A) \end{array}\right)$$

$$\iff \left(\begin{array}{l} \mathcal{V} \subseteq \mathcal{G} \text{ and} \\ \forall P \in \mathcal{V}, A \in \mathbb{A}, k : X \to \Omega_A. \\ (\forall P' \in \mathcal{V}.\ k : (X,P') \dot\to (\Omega_A, \boldsymbol{\Omega}_A)) \\ \implies \tau_A \circ Fk \circ c : (X,P) \dot\to (\Omega_A, \boldsymbol{\Omega}_A) \end{array}\right)$$

$$\iff \left(\begin{array}{l} \mathcal{V} \subseteq \mathcal{G} \text{ and} \\ \forall A \in \mathbb{A}, k : X \to \Omega_A. \\ (\forall P' \in \mathcal{V}.\ k : (X,P') \dot\to (\Omega_A, \boldsymbol{\Omega}_A)) \\ \implies \forall P \in \mathcal{V}.\ \tau_A \circ Fk \circ c : (X,P) \dot\to (\Omega_A, \boldsymbol{\Omega}_A) \end{array}\right).$$

Here, since $k : (X,P') \dot\to (\Omega_A, \boldsymbol{\Omega}_A)$ means $P' \sqsubseteq k^*\boldsymbol{\Omega}_A$, the condition

$$\forall P' \in \mathcal{V}.\ k : (X,P') \dot\to (\Omega_A, \boldsymbol{\Omega}_A)$$

is equivalent to

$$k : (X, \textstyle\bigsqcup_{P' \in \mathcal{V}} P') \dot\to (\Omega_A, \boldsymbol{\Omega}_A).$$

Similarly, the condition

$$\forall P \in \mathcal{V}.\ \tau_A \circ Fk \circ c : (X,P) \dot\to (\Omega_A, \boldsymbol{\Omega}_A)$$

is equivalent to

$$\tau_A \circ Fk \circ c : (X, \textstyle\bigsqcup_{P \in \mathcal{V}} P) \dot\to (\Omega_A, \boldsymbol{\Omega}_A).$$

These imply the following logical equivalence:

$$1) \iff \left(\begin{array}{l} \mathcal{V} \subseteq \mathcal{G} \text{ and} \\ \forall A \in \mathbb{A}, k : X \to \Omega_A. \\ \left(k : \left(X, \bigsqcup_{P' \in \mathcal{V}} P'\right) \dot\to (\Omega_A, \boldsymbol{\Omega}_A)\right) \\ \implies \tau_A \circ Fk \circ c : (X, \bigsqcup_{P \in \mathcal{V}} P) \dot\to (\Omega_A, \boldsymbol{\Omega}_A) \end{array}\right).$$

By Theorem 6.4, the condition in the right-hand side is equivalent to the conjunction of $\mathcal{V} \subseteq \mathcal{G}$ and

$$\bigsqcup_{P \in \mathcal{V}} P \sqsubseteq \Phi_c^{\boldsymbol{\Omega},\tau}\left(\bigsqcup_{P \in \mathcal{V}} P\right).$$

□

Theorem 6.10 *Assume the setting of Definition 6.6. Let $\mathcal{G} \subseteq |\mathbb{E}_X|$ be a join-dense subset. The following sets coincide.*

1. *The set of winning positions for Duplicator in the game in Table 11.*
2. *The intersection* $(\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})) \cap \mathcal{G}$ *of the downset* $\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})$ *of the codensity bisimilarity over c and the join-dense subset* $\mathcal{G}$.

Proof By Proposition 2.6, (1) is the largest invariant for Duplicator in the game in Table 10. In turn, by Lemma 6.9, it is the largest $\mathcal{G}$-joint codensity bisimulation over c. By Lemma 6.7, it coincides with (2). □

Corollary 6.11 (*Soundness and completeness of codensity games*) *Assume the setting of Definition 6.8. In particular, let* $\mathcal{G}$ *be a join-dense subset of* $\mathbb{E}_X$. $P \in \mathbb{E}_X$ *is a winning position for Duplicator if and only if* $P \sqsubseteq \nu\Phi_c^{\mathbf{\Omega},\tau}$.

□

Example 6.12 (Bisimulation topology for deterministic automata) Here we describe the topological example in Table 6. Consider the $\mathbf{CLat}_\sqcap$-fibration $\mathbf{Top} \to \mathbf{Set}$ and the functor $A_\Sigma = 2 \times (\_)^\Sigma : \mathbf{Set} \to \mathbf{Set}$, where Σ is a fixed alphabet. Coalgebras for this functor are deterministic automata over Σ; see, e.g., [23, 24].

We take the following data as a parameter of codensity lifting (cf. Definition 6.1): $\mathbb{A} = \{\varepsilon\} \cup \Sigma$, $\mathbf{\Omega}_\alpha$ is the Sierpinski space for each $\alpha \in \mathbb{A}$, and the modalities $\tau_\varepsilon, \tau_A : A_\Sigma 2 \to 2$ (where $a \in \Sigma$) are defined by

$$\tau_\varepsilon(t, \rho) = t \quad \text{and} \quad \tau_A(t, \rho) = \rho(a).$$

Recall that the Sierpinski space is the set $2 = \{\bot, \top\}$ with the topology $\{\emptyset, \{\top\}, 2\}$. Based on the slogan "Open sets are semi-decidable properties," which is explained in, e.g., [42], this observation domain models the situation, where acceptance of a word is only *semi*-decidable, not decidable, in the sense of computability theory.

Let $c : X \to A_\Sigma X$ be a deterministic automata. The above choice of parameters leads to the following codensity bisimilarity:

the state space X is equipped with the topology generated by the following family of open sets:

$$\{x \in X \mid \text{ w is accepted from x}\} \subseteq X, \quad \text{for each } w \in \Sigma^*$$

One can extract various information from this *bisimulation topology* via standard topological constructs. For example, the specialization order (see, e.g., [42, Chapter 7]) of this topology coincides with the language inclusion order.

For illustration by comparison, consider changing the observation domain from the Sierpinski space to the discrete 2-point set.

The bisimulation topology over X is now generated by

$$\begin{aligned}&\{x \in X \mid \text{ w is accepted from x}\} \text{ and}\\&\{x \in X \mid \text{ w is not accepted from x}\}, \quad \text{for each } w \in \Sigma^*.\end{aligned}$$

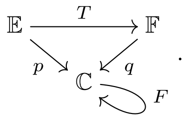
We can now observe rejection of a word, too, because $\{\bot\} \subseteq 2$ is open. The specialization order of this topology is the language equivalence, and it satisfies the R0 separation axiom (while the last Sierpinski example does not).

We take these examples of bisimulation topology as a process-semantical incarnation of the "observability via topology, computability via continuity" paradigm from domain theory. The definition of codensity bisimulation (cf. Definition 3.1) fits well with this intuition, too: a continuous map $k : (X, P) \dot{\to} \mathbf{\Omega}$ in Definition 3.1 is a "computable observation"; accordingly, an open set of the bisimulation topology is a property that is decided by finitely many of those computable observations.

7 Transfer of Codensity Bisimilarities

In our formulation, for the same endofunctor $F : \mathbb{C} \to \mathbb{C}$, we can use various $\mathbf{CLat}_{\sqcap}$-fibrations p and parameters $(\mathbf{\Omega}, \tau)$ to equip F-coalgebras with different bisimilarity-like notions. Some relations among those codensity bisimilarity notions can be categorically captured by general results. In this section we show two such results.

Definition 7.1 In this section, we consider the following situation:



Here, $p : \mathbb{E} \to \mathbb{C}$ and $q : \mathbb{F} \to \mathbb{C}$ are $\mathbf{CLat}_{\sqcap}$-fibrations. We assume that $q \circ T = p$ holds on the nose, and that T is "fibered": for $f : X \to Y$ in $\mathbb{C}$ and $E \in \mathbb{E}_Y$, $f^*(TE) = T(f^*E)$ holds.

7.1 Transfer Result for One Shared Family of Parameters

First, we consider the case, where the families of parameters are "shared" among two fibrations.

We use the following lemma.

Lemma 7.2 ([15, Lemma 20]) *In the setting of Definition 7.1, assume also that T preserves fiberwise meets. Let $\dot{F} : \mathbb{E} \to \mathbb{E}$ and $\ddot{F} : \mathbb{F} \to \mathbb{F}$ be liftings of F along p and q, respectively. Let $c : X \to FX$ be an F-coalgebra. If $T\dot{F}P = \ddot{F}TP$ holds for each $P \in \mathbb{E}$, then $T\nu(c^* \circ \dot{F}) = \nu(c^* \circ \ddot{F})$ holds.*

Proof For each ordinal α, we define $\nu_\alpha(c^* \circ \dot{F})$ by

$$\nu_\alpha(c^* \circ \dot{F}) = \sqcap_{\beta<\alpha} c^* \dot{F}(\nu_\beta(c^* \circ \dot{F}))$$

using induction on α. We define $\nu_\alpha(c^* \circ \ddot{F})$ in the same way. By Theorem 2.2, these converge to $\nu(c^* \circ \dot{F})$ and $\nu(c^* \circ \ddot{F})$, respectively. It suffices to show $T\nu_\alpha(c^* \circ \dot{F}) = \nu_\alpha(c^* \circ \ddot{F})$ by induction on α.

Assume that the above inequality holds for all ordinals smaller than α. Then we have

$$\begin{aligned}
T\nu_\alpha(c^* \circ \dot{F}) &= T \sqcap_{\beta<\alpha} c^*\dot{F}(\nu_\beta(c^* \circ \dot{F})) \\
&= \sqcap_{\beta<\alpha} Tc^*\dot{F}(\nu_\beta(c^* \circ \dot{F})) && \text{(since } T \text{ preserves meets)} \\
&= \sqcap_{\beta<\alpha} c^*T\dot{F}(\nu_\beta(c^* \circ \dot{F})) && \text{(since } T \text{ is fibered)} \\
&= \sqcap_{\beta<\alpha} c^*\ddot{F}T(\nu_\beta(c^* \circ \dot{F})) && \text{(by the assumption } T\dot{F} = \ddot{F}T\text{)} \\
&= \sqcap_{\beta<\alpha} c^*\ddot{F}(\nu_\beta(c^* \circ \ddot{F})) && \text{(by induction hypothesis)} \\
&= \nu_\alpha(c^* \circ \ddot{F}).
\end{aligned}$$

□

The following is the main result of Sect. 7.1. Note that the parameters $\{(T\mathbf{\Omega}_A, \tau_A)\}_{A\in\mathbb{A}}$ for $q : \mathbb{F} \to \mathbb{C}$ are "induced" from $\{(\mathbf{\Omega}_A, \tau_A)\}_{A\in\mathbb{A}}$ for $p : \mathbb{E} \to \mathbb{C}$.

Theorem 7.3 (Transfer of codensity bisimilarity) *In the setting of Definition 7.1, let $c : X \to FX$ be an F-coalgebra and $\{(\mathbf{\Omega}_A, \tau_A)\}_{A\in\mathbb{A}}$ be an $\mathbb{A}$-indexed family of parameters for codensity lifting of F along p (Definition 6.1). Assume that $T : \mathbb{E} \to \mathbb{F}$ is full and faithful, and that it preserves fiberwise meets. In this setting, $\{(T\mathbf{\Omega}_A, \tau_A)\}_{A\in\mathbb{A}}$ is an $\mathbb{A}$-indexed family of parameters for codensity lifting of F along q, and we have $\nu\Phi_c^{T\mathbf{\Omega},\tau} = T(\nu\Phi_c^{\mathbf{\Omega},\tau})$.*

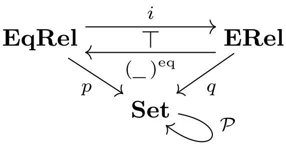
Proof For any $P \in \mathbb{E}_X$, we have $TF^{\mathbf{\Omega},\tau}P = F^{T\mathbf{\Omega},\tau}TP$, because the following hold:

$$\begin{aligned}
&TF^{\mathbf{\Omega},\tau}P \\
&= T\left(\sqcap_{A\in\mathbb{A}} \sqcap_{k\in\mathbb{E}(P,\mathbf{\Omega}_A)} (\tau_A \circ F(pk))^*\mathbf{\Omega}_A\right) \\
&= \sqcap_{A\in\mathbb{A}} \sqcap_{k\in\mathbb{E}(P,\mathbf{\Omega}_A)} T(\tau_A \circ F(pk))^*\mathbf{\Omega}_A && \text{(since } T \text{ preserves meets)} \\
&= \sqcap_{A\in\mathbb{A}} \sqcap_{k\in\mathbb{E}(P,\mathbf{\Omega}_A)} (\tau_A \circ F(pk))^*T\mathbf{\Omega}_A && \text{(since } T \text{ is fibered)} \\
&= \sqcap_{A\in\mathbb{A}} \sqcap_{k\in\mathbb{E}(P,\mathbf{\Omega}_A)} (\tau_A \circ F(q(Tk)))^*T\mathbf{\Omega}_A && \text{(since } q \circ T = p\text{)} \\
&= \sqcap_{A\in\mathbb{A}} \sqcap_{l\in\mathbb{F}(TP,T\mathbf{\Omega}_A)} (\tau_A \circ F(ql))^*T\mathbf{\Omega}_A && \text{(since } T \text{ is full)} \\
&= F^{T\mathbf{\Omega},\tau}TP
\end{aligned}$$

Considering this and the fact that T preserves meets, Lemma 7.2 implies $T(\nu\Phi^{\mathbf{\Omega},\tau}) = \nu\Phi^{T\mathbf{\Omega},\tau}$. □

Example 7.4 We show that the codensity bisimilarities in Examples 3.3 and 3.4 are indeed the usual bisimilarity notions for Kripke frames. Recall that they are built on the two $\mathbf{CLat}_\sqcap$-fibrations **EqRel**→ **Set** and **ERel**→ **Set**.

We first note that the inclusion functor $i : \mathbf{EqRel} \to \mathbf{ERel}$ is a reflection, having the equivalence closure $(\_)^{\mathrm{Eq}} : \mathbf{ERel} \to \mathbf{EqRel}$ as the left adjoint. It follows that i is meet-preserving. Moreover, i is fibered.



We introduce shorthands $\dot{\mathcal{P}}_2, \dot{\mathcal{P}}_3$ for the liftings in Examples 3.3 and 3.4:

$$\begin{aligned} \dot{\mathcal{P}}_2 &= \mathcal{P}^{\mathrm{Eq}_2,\diamond} &: \mathbf{EqRel} \to \mathbf{EqRel} \quad &\text{(Example 3.3)},\\ \dot{\mathcal{P}}_3 &= \mathcal{P}^{\mathrm{Eq}_2,\diamond} &: \mathbf{ERel} \to \mathbf{ERel} \quad &\text{(Example 3.4)}. \end{aligned}$$

Now, for the sake of our proof, let us introduce a relational lifting $\dot{\mathcal{P}}_1 : \mathbf{ERel} \to \mathbf{ERel}$ of $\mathcal{P}$ along $\mathbf{ERel} \to \mathbf{Set}$, for which it is obvious that the corresponding bisimilarity notion is the usual bisimilarity for Kripke frames. We do so in concrete terms, instead of as a codensity lifting:

$$(S, T) \in \dot{\mathcal{P}}_1(R) \iff (\forall x \in S.\ \exists y \in T.\ (x, y) \in R) \ \wedge (\forall y \in T.\ \exists x \in S.\ (x, y) \in R).$$

We note that $\dot{\mathcal{P}}_2$ is the restriction of $\dot{\mathcal{P}}_1$ from **ERel** to **EqRel** along i. This means $i \circ \dot{\mathcal{P}}_2 = \dot{\mathcal{P}}_1 \circ i$. Note also that $\dot{\mathcal{P}}_3 = \dot{\mathcal{P}}_1 \circ i \circ (\_)^{\mathrm{Eq}}$.

Let $c : X \to \mathcal{P}X$ be a Kripke frame and $\Phi_i = c^* \circ \dot{\mathcal{P}}_i$ $(i = 1, 2, 3)$ be the predicate transformer corresponding to each lifting. Proposition 7.3 yields that $\nu\Phi_3 = i(\nu\Phi_2)$.

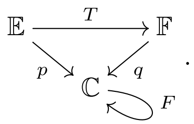
Furthermore, by $\dot{\mathcal{P}}_1 \sqsubseteq \dot{\mathcal{P}}_3$ (where $\sqsubseteq$ is the order in (4)), we have $\nu\Phi_1 \sqsubseteq \nu\Phi_3$. From $i \circ \dot{\mathcal{P}}_2 = \dot{\mathcal{P}}_1 \circ i$ and fiberedness of c, we can see that $i(\nu\Phi_2)$ is a fixed point of Φ_1:

$$\begin{aligned} \Phi_1(i(\nu\Phi_2)) &= c^*(\dot{\mathcal{P}}_1(i(\nu\Phi_2))) = c^*(i(\dot{\mathcal{P}}_2(\nu\Phi_2)))\\ &= i(c^*(\dot{\mathcal{P}}_2(\nu\Phi_2))) = i(\Phi_2(\nu\Phi_2)) = i(\nu\Phi_2). \end{aligned}$$

By this fact and the definition of $\nu\Phi_1$, $i(\nu\Phi_2) \sqsubseteq \nu\Phi_1$ holds. The three (in)equalities so far allow us to conclude $\nu\Phi_3 = i(\nu\Phi_2) = \nu\Phi_1$, stating that the conventional bisimilarity $\nu\Phi_1$ is equal to the codensity bisimilarities in Examples 3.3 and 3.4. As a consequence, the conventional bisimilarity $\nu\Phi_1$ is necessarily an equivalence relation.

7.2 Transfer Result for Two Different Families of Parameters

Consider the following situation again (Definition 7.1):



Now consider two families of parameters, $(\boldsymbol{\Omega}, \tau) = \{(\boldsymbol{\Omega}_A, \tau_A)\}_{A \in \mathbb{A}}$ for lifting F along p and $(\boldsymbol{\Psi}, \rho) = \{(\boldsymbol{\Psi}_B, \rho_B)\}_{B \in \mathbb{B}}$ for lifting F along q. Let $c : X \to FX$ be an F-coalgebra. In Sect. 7.2 we compare $T\nu\Phi_c^{\boldsymbol{\Omega},\tau}$ and $\nu\Phi_c^{\boldsymbol{\Psi},\rho}$ (both in $\mathbb{F}_X$).

First, we show an "order-version" of Lemma 7.2. It reduces the comparison of $T\nu\Phi_c^{\boldsymbol{\Omega},\tau}$ and $\nu\Phi_c^{\boldsymbol{\Psi},\rho}$ to that of $TF^{\boldsymbol{\Omega},\tau}$ and $F^{\boldsymbol{\Psi},\rho}T$:

Proposition 7.5 *In the setting of Definition 7.1, assume also that T preserves fiberwise meets. Let $\dot{F} : \mathbb{E} \to \mathbb{E}$ and $\ddot{F} : \mathbb{F} \to \mathbb{F}$ be liftings of F along p and q, respectively. Let $c : X \to FX$ be an F-coalgebra. If $T\dot{F}P \sqsupseteq \ddot{F}TP$ holds for each $P \in \mathbb{E}$, then $T\nu(c^* \circ \dot{F}) \sqsupseteq \nu(c^* \circ \ddot{F})$ holds.*

Proof For each ordinal α, we define $\nu_\alpha(c^* \circ \dot{F})$ by

$$\nu_\alpha(c^* \circ \dot{F}) = \sqcap_{\beta<\alpha} c^* \dot{F}(\nu_\beta(c^* \circ \dot{F}))$$

using induction on α. We define $\nu_\alpha(c^* \circ \ddot{F})$ in the same way. By Theorem 2.2, these converge to $\nu(c^* \circ \dot{F})$ and $\nu(c^* \circ \ddot{F})$, respectively. It suffices to show $T\nu_\alpha(c^* \circ \dot{F}) \sqsupseteq \nu_\alpha(c^* \circ \ddot{F})$ by induction on α.

Assume that the above inequality holds for all ordinals smaller than α. Then we have

$$\begin{aligned} T\nu_\alpha(c^* \circ \dot{F}) &= T \sqcap_{\beta<\alpha} c^* \dot{F}(\nu_\beta(c^* \circ \dot{F})) \\ &= \sqcap_{\beta<\alpha} Tc^* \dot{F}(\nu_\beta(c^* \circ \dot{F})) && \text{(since } T \text{ preserves meets)} \\ &= \sqcap_{\beta<\alpha} c^* T\dot{F}(\nu_\beta(c^* \circ \dot{F})) && \text{(since } T \text{ is fibered)} \\ &\sqsupseteq \sqcap_{\beta<\alpha} c^* \ddot{F}T(\nu_\beta(c^* \circ \dot{F})) && \text{(by the assumption } T\dot{F} \sqsupseteq \ddot{F}T) \\ &\sqsupseteq \sqcap_{\beta<\alpha} c^* \ddot{F}(\nu_\beta(c^* \circ \ddot{F})) && \text{(by induction hypothesis)} \\ &= \nu_\alpha(c^* \circ \ddot{F}). \end{aligned}$$

□

The following is the main result of Sect. 7.2. It says that, if we have a certain data connecting two families of parameters $\{(\mathbf{\Omega}_A, \tau_A)\}_{A\in\mathbb{A}}$ and $\{(\mathbf{\Psi}_B, \rho_B)\}_{B\in\mathbb{B}}$, then the inequality $TF^{\mathbf{\Omega},\tau} \sqsupseteq F^{\mathbf{\Psi},\rho}T$ holds:

Proposition 7.6 *In the setting of Definition 7.1, assume also that T preserves fiberwise meets. Let $(I_{A,B})_{A\in\mathbb{A},B\in\mathbb{B}}$ be some family of sets and $(t_{A,B,i} : T\mathbf{\Omega}_A \to \mathbf{\Psi}_B)_{A\in\mathbb{A},B\in\mathbb{B},i\in I_{A,B}}$ be a family of $\mathbb{F}$-arrows such that*

$$\tau_A^* T\mathbf{\Omega}_A \sqsupseteq \sqcap_{B\in\mathbb{B},i\in I_{A,B}} (F(q(t_{A,B,i})))^* \rho_B^* \mathbf{\Psi}_B$$

holds for each $A \in \mathbb{A}$. Then $TF^{\mathbf{\Omega},\tau}P \sqsupseteq F^{\mathbf{\Psi},\rho}TP$ holds for each $P \in \mathbb{E}$.

Proof Let $X = pP$. Since

$$\begin{aligned} TF^{\mathbf{\Omega},\tau}P &= T\left(\sqcap_{A\in\mathbb{A},f : P\to\mathbf{\Omega}_A} (F(pf))^* \tau_A^* \mathbf{\Omega}_A\right) \\ &= \sqcap_{A\in\mathbb{A},f : P\to\mathbf{\Omega}_A} (F(pf))^* \tau_A^* T\mathbf{\Omega}_A \end{aligned}$$

and

$$F^{\mathbf{\Psi},\rho}TP = \sqcap_{B\in\mathbb{B},g : TP\to\mathbf{\Psi}_B} (F(qg))^* \rho_B^* \mathbf{\Psi}_B,$$

it suffices to show that, for each $f : P \to \mathbf{\Omega}$ and $A \in \mathbb{A}$,

$$(F(pf))^* \tau_A^* T\mathbf{\Omega}_A \sqsupseteq \sqcap_{B \in \mathbb{B}, g: TP \to \mathbf{\Psi}_B} (F(qg))^* \rho_B^* \mathbf{\Psi}_B$$

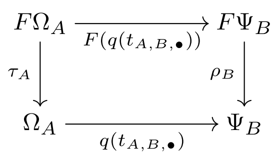
holds.

Let $A \in \mathbb{A}$ and $f : P \to \mathbf{\Omega}_A$. For each $B \in \mathbb{B}$ and $i \in I_{A,B}$, consider $g = TP \xrightarrow{Tf} T\mathbf{\Omega}_A \xrightarrow{t_{A,B,i}} \mathbf{\Psi}_B$. Then $F(qg) = FX \xrightarrow{F(pf)} F\Omega_A \xrightarrow{Fqt_{A,B,i}} Fq\Psi_B$. Thus $(F(qg))^* \rho_B^* \mathbf{\Psi}_B = (F(pf))^* (F(q(t_{A,B,i})))^* \rho_B^* \mathbf{\Psi}_B$ holds. Restricting the range of g to the class considered above, it suffices to show

$$\tau_A^* T\mathbf{\Omega}_A \sqsupseteq \sqcap_{B \in \mathbb{B}, i \in I_{A,B}} (F(q(t_{A,B,i})))^* \rho_B^* \mathbf{\Psi}_B.$$

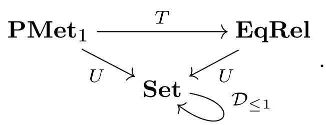
This is nothing but our assumption. □

Remark 7.7 As a special case, if each $I_{A,B}$ is the singleton $\{\bullet\}$ and the diagram:



commutes, then the condition in Proposition 7.6 holds. However, it seems that such cases are rather special. The condition in Proposition 7.6 can be regarded as a weakening of it: we use multiple $t_{A,B,i}$ to obtain as much information as given by one arrow making the above diagram commute.

Example 7.8 Consider the following situation:



Here, T is defined by $T(X, d) = (X, R_d)$ and

$$(x, y) \in R_d \iff d(x, y) = 0.$$

This is a fibered lifting of $\mathrm{Id}_{\mathbb{C}}$ and preserves fibered meets.

We describe the parameter for $\mathbb{E} = \mathbf{PMet}_1$: $\mathbb{A} = \{\bullet\}$ and $\mathbf{\Omega}_\bullet = ([0,1], d_e)$, where d_e is the Euclidean metric. The modality $\tau_\bullet : \mathcal{D}_{\le 1}[0,1] \to [0,1]$ is given by the expected value function. In this setting, for each coalgebra $c : X \to \mathcal{D}_{\le 1} X$, the codensity bisimilarity $\nu\Phi_c^{\mathbf{\Omega},\tau}$ coincides with the bisimulation metric (Examples 3.10, 4.5 and 5.3).

We move on to the parameter for $\mathbb{F} = \mathbf{EqRel}$: $\mathbb{B} = [0,1]$ and $\mathbf{\Psi}_r = (2, \mathrm{Eq}_2)$ for all $r \in [0,1]$. The modality $\rho_r : \mathcal{D}_{\le 1} 2 \to 2$ is the threshold modality defined by

$$\rho_r(p) = \top \iff p(\top) \ge r.$$

For each coalgebra $c : X \to \mathcal{D}_{\leq 1}X$, the codensity bisimilarity $\nu\Phi^{\mathbf{\Psi},\rho}$ coincides with the probabilistic bisimilarity (Example 8.15).

For each $r \in [0,1]$, let $I_{\bullet,r} = [0,1]$. For each $r, s \in [0,1]$, we define an **EqRel**-arrow $t_{\bullet,r,s} : T([0,1], d_e) \to (2, \mathrm{Eq}_2)$ by

$$t_{\bullet,r,s}(u) = \top \iff u \geq s.$$

In this setting the condition in Proposition 7.6 is satisfied. Let $\mu, \nu \in \mathcal{D}_{\leq 1}[0,1]$; if $\mu(\{x \mid x \geq s\}) \geq r \iff \nu(\{x \mid x \geq s\}) \geq r$ holds for all $r, s \in [0,1]$, then their expected values coincide.

Using Propositions 7.5 and 7.6, we can conclude that, for any $\mathcal{D}_{\leq 1}$-coalgebra $c : X \to \mathcal{D}_{\leq 1}X$, $T(\nu\Phi^{\mathbf{\Omega},\tau}) \sqsupseteq \nu\Phi^{\mathbf{\Psi},\rho}$ holds. This means that, if two states are bisimilar, then the bisimulation metric between them is 0.

On the other hand, the converse inequality $T(\nu\Phi^{\mathbf{\Omega},\tau}) \sqsubseteq \nu\Phi^{\mathbf{\Psi},\rho}$ cannot be derived from the above general theory. It is known to hold [5, Theorem 5.2], but the proof involves a real-valued modal logic. Purely fibrational proof of this fact is a future work.

Note that this example does not make the diagram in Remark 7.7 commute.

8 Examples

In this section, we list examples of our framework. We group them by the fibrations they rely upon: **EqRel** $\to$ **Set** in Sect. 8.1, **BRel** $\to$ $\mathbf{Set}^2$ in Sect. 8.2, **Top** $\to$ **Set** in Sect. 8.3, $\mathbf{PMet}_1$ $\to$ **Set** in Sect. 8.4, and $\mathbf{ESemi}_{\mathbb{R}}$ $\to$ $\mathbf{Vect}_{\mathbb{R}}$ in Sect. 8.6. In Sect. 8.5, we use a fibration $U^*(\mathbf{PMet}_1) \to \mathbf{Meas}$ that is newly defined there.

8.1 Set-Coalgebras and Behavioral Equivalence

Behavioral equivalence is an equivalence notion for coalgebras. While (relational) bisimilarity is based on spans of coalgebra morphisms, behavioral equivalence is defined by *cospans* of coalgebra morphisms. For their detailed comparison, see [18] (where behavioral equivalence is often referred to as *kernel-bisimulation*).

In Sect. 8.1, we show that behavioral equivalence for coalgebras in **Set** can also be defined in terms of fibrations (Proposition 8.4), and that they can be characterized by codensity games (Theorem 8.8) in the cases, where the functor admits a *separating family* (Definition 8.6).

We start with the standard definition of behavioral equivalence. The intuition here is that a coalgebra morphism is "behavior preserving." See [24].

Definition 8.1 (Behavioral equivalence [43, Definition 1]) Let $F : \mathbf{Set} \to \mathbf{Set}$ be a functor and $c : X \to FX$ be an F-coalgebra. The states $x, x' \in X$ are *behaviorally equivalent* if there is another F-coalgebra $d : Y \to FY$ and a coalgebra morphism $f : X \to Y$ such that $f(x) = f(x')$.

This can be modeled fibrationally by the fibration **EqRel** $\to$ **Set**. We use a functor lifting, which is essentially the same as the one defined in [44, Section 4].

Definition 8.2 (The lifting $\overline{F}_{\mathrm{BE}} : \mathbf{EqRel} \to \mathbf{EqRel}$) Let $F : \mathbf{Set} \to \mathbf{Set}$ be a functor. We define a lifting $\overline{F}_{\mathrm{BE}} : \mathbf{EqRel} \to \mathbf{EqRel}$ by the following: for $(X, R) \in \mathbf{EqRel}$, let $q : X \twoheadrightarrow X/R$ be the canonical surjection. Then $\overline{F}_{\mathrm{BE}}(X, R)$ is defined as the kernel of $Fq : FX \to F(X/R)$, that is,

$$\overline{F}_{\mathrm{BE}}(X, R) = (FX, \{(z, z') \in (FX)^2 \mid (Fq)(z) = (Fq)(z')\}).$$

Proposition 8.3 *The assignment $\overline{F}_{\mathrm{BE}}$ above indeed specifies a functor, i.e., for any decent morphism $f : (X, R) \to (Y, S)$, Ff is decent from $\overline{F}_{\mathrm{BE}}(X, R)$ to $\overline{F}_{\mathrm{BE}}(Y, S)$.*

Proof Let $q : X \twoheadrightarrow X/R$ and $r : Y \twoheadrightarrow Y/S$ be the canonical surjections. Let us fix $z, z' \in FX$ and assume $(Fq)(z) = (Fq)(z')$. It suffices to show $(Fr)((Ff)(z)) = (Fr)((Ff)(z'))$.

Since $f : (X, R) \to (Y, S)$ is decent, $R \sqsubseteq f^*S$ holds. Therefore, there exists a map $g : X/R \to Y/S$ which makes the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle r} \\ X/R & \xrightarrow[\ g\ ]{} & Y/S \end{array}$$

commute. Using this we see

$$(Fr)((Ff)(z)) = (F(r \circ f))(z) = (F(g \circ q))(z) = (Fg)((Fq)(z)).$$

For the same reason $(Fr)((Ff)(z')) = (Fg)((Fq)(z'))$ holds, and the assumption $(Fq)(z) = (Fq)(z')$ now implies $(Fr)((Ff)(z)) = (Fr)((Ff)(z'))$. $\square$

The lifting $\overline{F}_{\mathrm{BE}}$ indeed captures behavioral equivalence, provided that F preserves monos.

Proposition 8.4 *Let $F : \mathbf{Set} \to \mathbf{Set}$ be a functor and $c : X \to FX$ be an F-coalgebra. Assume that F preserves monos. The states $x, x' \in X$ are behaviorally equivalent if and only if there is an equivalence relation R on X such that $(X, R) \sqsubseteq c^*\overline{F}_{\mathrm{BE}}(X, R)$.*

Proof Let $q : X \twoheadrightarrow X/R$ be the canonical surjection. Then $c^*\overline{F}_{\mathrm{BE}}(X, R)$ can be concretely presented by

$$c^*\overline{F}_{\mathrm{BE}}(X, R) = (X, \{(x, x') \in X^2 \mid (Fq)(c(x)) = (Fq)(c(x'))\}).$$

Let $x, x' \in X$. First, we show that if x and x' are behaviorally equivalent, there exists some R such that $(x, x') \in R$ and $(X, R) \sqsubseteq c^*\overline{F}_{\mathrm{BE}}(X, R)$ hold. Assume x and x' are behaviorally equivalent. There is another F-coalgebra $d : Y \to FY$ and a coalgebra morphism $f : X \to Y$ such that $f(x) = f(x')$. Let $R \subseteq X \times X$ be

$$R = \{(x_1, x_2) \in X^2 \mid f(x_1) = f(x_2)\}.$$

Then $(X, R) \in \mathbf{EqRel}$ and, by the definition, $(x, x') \in R$. Let $q : X \twoheadrightarrow X/R$ be the canonical surjection. By the definition of R, there exists a monomorphism $m : X/R \rightarrowtail Y$ such that $f = m \circ q$. Since f is a coalgebra morphism, the outer square of the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow[q]{} & X/R & \xrightarrow[m]{} & Y \\ {\scriptstyle c}\downarrow & & {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle d} \\ FX & \xrightarrow[Fq]{} & F(X/R) & \xrightarrow[Fm]{} & FY. \end{array}$$

In this diagram, q is epic and, since m is monic, Fm is also monic. Therefore, there exists a unique $e : X/R \to F(X/R)$ making the two squares in the above diagram commute (the *diagonalization property* of a factorization system—see [35]). Now we prove $(X, R) \sqsubseteq c^* \overline{F}_{\mathrm{BE}}(X, R)$. Assume $(x_1, x_2) \in R$. Since $(Fq)(c(x_1)) = e(q(x_1)) = e(q(x_2)) = (Fq)(c(x_2))$, $(x_1, x_2) \in c^* \overline{F}_{\mathrm{BE}}(X, R)$ holds.

Second, for R satisfying $(X, R) \sqsubseteq c^* \overline{F}_{\mathrm{BE}}(X, R)$, we show that any pair $(x, x') \in R$ is behaviorally equivalent. Assume that there exists R such that $(x, x') \in R$ and $(X, R) \sqsubseteq c^* \overline{F}_{\mathrm{BE}}(X, R)$ hold. The second condition means that, for each $(x_1, x_2) \in R$, $(Fq \circ c)(x_1) = (Fq \circ c)(x_2)$ holds. Thus there is a (unique) $d : X/R \to F(X/R)$ making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow[q]{} & X/R \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle d} \\ FX & \xrightarrow[Fq]{} & F(X/R). \end{array}$$

Now q is a coalgebra morphism from $c : X \to FX$ to $d : X/R \to F(X/R)$. Since $q(x) = q(x')$, x and x' are behaviorally equivalent. □

Remark 8.5 (On preservation of monomorphisms) In Proposition 8.4, F is assumed to preserve monos. However, this is not very restricting: If $X \in \mathbf{Set}$ is nonempty, then any monomorphism $f : X \rightarrowtail Y$ splits, and Ff is also a split mono. Therefore, we only have to check that, for $f : 0 \to Y$, Ff is injective. See [45] for details.

Now we move on to representing $\overline{F}_{\mathrm{BE}}$ as a codensity lifting. The key notion here is *separation*. It is mainly used in coalgebraic modal logic literature like [43, 46]. While it is standard to define it for *predicate liftings* like in [43, Definition 7], we adapt it for F-algebras.

Definition 8.6 (Separating family of F-algebras) Let $X \in \mathbf{Set}$ and $F : \mathbf{Set} \to \mathbf{Set}$. An Å-indexed family $(\tau_A : F2 \to 2)_{A \in Å}$ of F-algebras is *separating for* X if each $z \in FX$ is uniquely determined by the values of $\tau_A((Ff)(z))$ for $A \in Å$ and

$f : X \to 2$, that is, for each pair $z, z' \in FX$, if $\tau_A((Ff)(z)) = \tau_A((Ff)(z'))$ holds for all $A \in$ Å and $f : X \to 2$, then $z = z'$.

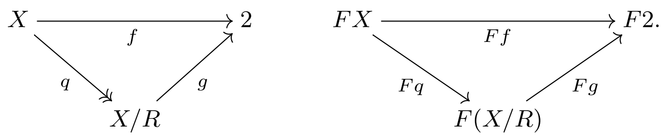
For an Å-indexed family $(\tau_A : F2 \to 2)_{A \in Å}$ of F-algebras, note that $\{(\mathrm{Eq}_2, \tau_A)\}_{A \in Å}$ is an $\mathbb{A}$-indexed family of lifting parameters and we can define the codensity lifting $F^{\mathrm{Eq}_2, \tau}$. This turns out to coincide with $\overline{F}_{\mathrm{BE}}$ if the family is separating.

Proposition 8.7 *Let* (X, R) *be an object in* **EqRel**, F : **Set** $\to$ **Set** *be a functor, and* $(\tau_A : F2 \to 2)_{A \in Å}$ *be an* Å*-indexed family of F-algebras. If* $(\tau_A : F2 \to 2)_{A \in Å}$ *is separating for X/R, then*

$$F^{\mathrm{Eq}_2, \tau}(X, R) = \overline{F}_{\mathrm{BE}}(X, R)$$

holds.

Proof First, we show $F^{\mathrm{Eq}_2, \tau}(X, R) \sqsupseteq \overline{F}_{\mathrm{BE}}(X, R)$. Let $(z, z') \in (FX)^2$, $f : (X, R) \to (2, \mathrm{Eq}_2)$ and $A \in$ Å. Let $q : X \twoheadrightarrow X/R$ be the canonical surjection and assume that $(Fq)(z) = (Fq)(z')$. It suffices to show $\tau_A((Ff)(z)) = \tau_A((Ff)(z'))$. Since $f : (X, R) \to (2, \mathrm{Eq}_2)$ is decent, there is a (unique) map $g : X/R \to 2$ making the left one of the following diagrams commute:



By the functoriality of F, the right one also commutes. Thus, we have $(Ff)(z) = (Fg)((Fq)(z)) = (Fg)((Fq)(z')) = (Ff)(z')$. This implies $\tau_A((Ff)(z)) = \tau_A((Ff)(z'))$.

Second, we show $F^{\mathrm{Eq}_2, \tau}(X, R) \sqsubseteq \overline{F}_{\mathrm{BE}}(X, R)$. Let $(z, z') \in (FX)^2$ and assume that, for each $A \in$ Å and $f : (X, R) \to (2, \mathrm{Eq}_2)$, $\tau_A((Ff)(z)) = \tau_A((Ff)(z'))$ holds. Let $q : X \twoheadrightarrow X/R$ be the canonical surjection. It suffices to show $(Fq)(z) = (Fq)(z')$. Let $g : X/R \to 2$ be any arrow. Then $g \circ q$ is decent from (X, R) to $(2, \mathrm{Eq}_2)$. By the assumption, $\tau_A((Fg)((Fq)(z))) = \tau_A((Ff)(z)) = \tau_A((Ff)(z')) = \tau_A((Fg)((Fq)(z')))$ holds for each $A \in$ Å. Since g is arbitrary and $(\tau_A : F2 \to 2)_{A \in Å}$ is separating for X/R, $(Fq)(z) = (Fq)(z')$ holds. □

In such case the codensity bisimilarity (Definition 6.2) coincides with the behavioral equivalence (Definition 8.1).

Theorem 8.8 *Let* F : **Set** $\to$ **Set** *be a functor,* $(\tau_A : F2 \to 2)_{A \in Å}$ *be an* Å*-indexed family of F-algebras, and* $c : X \to FX$ *be an F-coalgebra. Assume that F preserves monos. If* $(\tau_A : F2 \to 2)_{A \in Å}$ *is separating for every set Y, then the behavioral equivalence of c coincides with the codensity bisimilarity* $\nu\Phi_c^{\mathrm{Eq}_2, \tau}$.

Proof By Proposition 8.4, the behavioral equivalence is the greatest fixed point of $c^* \circ \overline{F}_{\mathrm{BE}}$. Moreover, this coincides with $\nu\Phi_c^{\mathrm{Eq}_2,\tau}$ by Proposition 8.7. □

Theorem 8.8 characterizes the behavioral equivalence of F-coalgebras by codensity games, when F preserves monos and has separating family of F-algebras. In the following, we use the join-dense subset described in Example 5.2 to trim games.

Example 8.9 (Kripke frames) Consider the powerset functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$. Since $\mathcal{P}0 \simeq 1$, for any $f : 0 \to Y$ in $\mathbf{Set}$, $\mathcal{P}f : \mathcal{P}0 \to \mathcal{P}Y$ is monic. Thus it preserves monos by Remark 8.5. A $\mathcal{P}$-coalgebra $c : X \to \mathcal{P}X$ is nothing but a Kripke frame.

The one-member family $(\Diamond : \mathcal{P}2 \to 2)$ (used in Example 3.3) is separating for any set X. Indeed, if we define $f_x : X \to 2$ by $f_x(x') = \top \iff x = x'$, then for $S \in \mathcal{P}X$, $x \in S$ if and only if $\Diamond((\mathcal{P}f_x)(S)) = \top$.

By Theorem 8.8, the behavioral equivalence (Definition 8.1) for a Kripke frame $c : X \to \mathcal{P}X$ coincides with the codensity bisimilarity $\nu\Phi_c^{\mathrm{Eq}_2,\Diamond}$. Thus, by Corollary 5.16, it is characterized by the codensity game (Table 10) specialized to this situation. The game in this case is shown in Table 12. It is trimmed by the join-dense subset in Example 5.2.

Theorem 8.10 *Let $c : X \to \mathcal{P}X$ be a Kripke frame. The position $(x, y) \in X \times X$ in the game in Table 12 is winning for Duplicator if and only if $(x, y) \in \nu\Phi_c^{\mathrm{Eq}_2,\Diamond}$, if and only if x and y are behaviorally equivalent.* □

As shown in Example 7.4, the codensity bisimilarity $\nu\Phi_c^{\mathrm{Eq}_2,\Diamond}$ (which is $\nu\Phi_2$ in Example 7.4) also coincides with the conventional bisimilarity on the Kripke frame c. Therefore, we also see that the conventional bisimilarity and the behavioral equivalence are equal for Kripke frames.

Example 8.11 (Deterministic automata) Consider the functor $A_\Sigma : \mathbf{Set} \to \mathbf{Set}$ from Example 6.12, for which a coalgebra is a deterministic automaton. Since $A_\Sigma 0 \simeq 0$, for any $f : 0 \to Y$ in $\mathbf{Set}$, $A_\Sigma f : A_\Sigma 0 \to A_\Sigma Y$ is monic. Thus it preserves monos by Remark 8.5.

The family $\{\tau_\varepsilon\} \cup \{\tau_A \mid a \in \Sigma\}$ introduced in Example 6.12 is separating for every set X. Indeed, if we define $f_x : X \to 2$ by $f_x(x') = \top \iff x = x'$, then for

Table 12 Codensity bisimilarity game for conventional bisimilarity

Position	Player	Possible moves
$(x, y) \in X \times X$	Spoiler	$k \in \mathbf{Set}(X, 2)$ such that exactly one of $\exists x' \in c(x).\ k(x') = \top$ and $\exists y' \in c(y).\ k(y') = \top$ holds
$k \in \mathbf{Set}(X, 2)$	Duplicator	(x'', y'') s.t. $k(x'') \neq k(y'')$

$y = (t, \rho) \in A_\Sigma X$ (where $t \in 2$ and $\rho : \Sigma \to X$), $t = \top$ if and only if $\tau_\varepsilon((A_\Sigma f_x)(y)) = \top$, and $\rho(a) = x$ if and only if $\tau_A((A_\Sigma f_x)(y)) = \top$.

By Theorem 8.8, the behavioral equivalence (Definition 8.1) for a deterministic automaton $c : X \to A_\Sigma X$ coincides with the codensity bisimilarity $\nu\Phi_c^{\mathrm{Eq}_2,\tau}$. Thus, by Corollary 6.11, it is characterized by the codensity game (Table 11) specialized to this situation. The game in this case is shown in Table 13. It is trimmed by the join-dense subset in Example 5.2. It is also simplified in the case, where the position $(x, y) \in X \times X$ satisfies $c_1(x) \neq c_1(y)$: strictly in such case, Spoiler can play any constant map from X to 2 and any $a \in \Sigma$, and then Duplicator cannot play any longer.

Theorem 8.12 *Let $c : X \to A_\Sigma X$ be a deterministic automaton. The position $(x, y) \in X \times X$ in the game in Table 13 is winning for Duplicator if and only if $(x, y) \in \nu\Phi_c^{\mathrm{Eq}_2,\tau}$, if and only if x and y are behaviorally equivalent.* □

Since we are considering deterministic automata here, the language equivalence coincides with the behavioral equivalence. Thus the game in Table 13 also characterizes the language equivalence.

Example 8.13 (Nondeterministic automata) Let us now turn to nondeterministic automata, that is, N_Σ-coalgebras for the functor $N_\Sigma = 2 \times (\mathcal{P}\_)^\Sigma$. Since $N_\Sigma = A_\Sigma \circ \mathcal{P}$ and both A_Σ and $\mathcal{P}$ preserve monos (Examples 8.9, 8.11), N_Σ preserves monos.

Consider the family $\{\tau_\varepsilon\} \cup \{\tau_A \mid a \in \Sigma\}$ of maps from $N_\Sigma 2$ to 2 defined as follows:

$$\tau_\varepsilon(t, \rho) = t, \;\; \tau_A(t, \rho) = \Diamond(\rho(a)).$$

This family is separating for every set X. Indeed, if we define $f_x : X \to 2$ by $f_x(x') = \top \iff x = x'$, then for $y = (t, \rho) \in N_\Sigma X$ (where $t \in 2$ and $\rho : \Sigma \to \mathcal{P}X$), $t = \top$ if and only if $\tau_\varepsilon((N_\Sigma f_x)(y)) = \top$, and $x \in \rho(a)$ if and only if $\tau_A((N_\Sigma f_x)(y)) = \top$.

By Theorem 8.8, the behavioral equivalence (Definition 8.1) for a nondeterministic automaton $c : X \to N_\Sigma X$ coincides with the codensity bisimilarity $\nu\Phi_c^{\mathrm{Eq}_2,\tau}$. Thus, by Corollary 6.11, it is characterized by the codensity game (Table 11) specialized

Table 13 Codensity bisimilarity game for deterministic automata and their language equivalence

Position	Player	Possible moves
$(x, y) \in X \times X$	Spoiler	If $c_1(x) \neq c_1(y)$ then Spoiler wins If $c_1(x) = c_1(y)$ then $a \in \Sigma$ and $k \in \mathbf{Set}(X, 2)$ such that $k(c_2(x)(a)) \neq k(c_2(y)(a))$
$a \in \Sigma$ and $k \in \mathbf{Set}(X, 2)$	Duplicator	$(x'', y'') \in X \times X$ s.t. $k(x'') \neq k(y'')$

The arrows $c_1 : X \to 2$ and $c_2 : X \to X^\Sigma$ are the first and second projections of $c : X \to A_\Sigma X = 2 \times X^\Sigma$, respectively

Table 14 Codensity bisimilarity game for nondeterministic automata and their behavioral equivalence

Position	Player	Possible moves
$(x, y) \in X \times X$	Spoiler	If $c_1(x) \neq c_1(y)$ then Spoiler wins If $c_1(x) = c_1(y)$ then $a \in \Sigma$ and $k \in \mathbf{Set}(X, 2)$ such that $\exists x' \in c_2(x)(a).\ k(x') = \top$ $\not\Leftrightarrow \exists y' \in c_2(y)(a).\ k(y') = \top$
$a \in \Sigma$ and $k \in \mathbf{Set}(X, 2)$	Duplicator	$(x'', y'') \in X \times X$ s.t. $k(x'') \neq k(y'')$

The arrows $c_1 : X \to 2$ and $c_2 : X \to (\mathcal{P}X)^\Sigma$ are the first and second projections of $c : X \to N_\Sigma X = 2 \times (\mathcal{P}X)^\Sigma$, respectively

Table 15 Codensity bisimilarity game for probabilistic bisimilarity

Position	Player	Possible moves
$(x, y) \in X \times X$	Spoiler	$r \in [0, 1]$ and $k \in \mathbf{Set}(X, 2)$ s.t. $c(x)(k^{-1}(\top)) \geq r > c(y)(k^{-1}(\top))$, or $c(y)(k^{-1}(\top)) \geq r > c(x)(k^{-1}(\top))$
$r \in [0, 1]$ and $k \in \mathbf{Set}(X, 2)$	Duplicator	(x'', y'') s.t. $k(x'') \neq k(y'')$

to this situation. The game in this case is shown in Table 14. It is trimmed by the join-dense subset in Example 5.2. It is also simplified in the case, where the position $(x, y) \in X \times X$ satisfies $c_1(x) \neq c_1(y)$: strictly in such case, Spoiler can play any constant map from X to 2 and any $a \in \Sigma$, and then Duplicator cannot play any longer.

Theorem 8.14 *Let $c : X \to N_\Sigma X$ be a nondeterministic automaton. The position $(x, y) \in X \times X$ in the game in Table 14 is winning for Duplicator if and only if $(x, y) \in \nu\Phi_c^{\mathrm{Eq}_2,\tau}$, if and only if x and y are behaviorally equivalent.* □

Example 8.15 (Markov chains) Consider the functor $\mathcal{D}_{\leq 1} : \mathbf{Set} \to \mathbf{Set}$ (introduced in Sect. 1.1.2), for which a coalgebra is a Markov chain. Since $\mathcal{D}_{\leq 1}0 \simeq 1$, for any $f : 0 \to Y$ in $\mathbf{Set}$, $\mathcal{D}_{\leq 1}f : \mathcal{D}_{\leq 1}0 \to \mathcal{D}_{\leq 1}Y$ is monic. Thus it preserves monos by Remark 8.5.

For each real number $r \in [0, 1]$, define a *threshold modality* $\tau_r : \mathcal{D}_{\leq 1}2 \to 2$ by $\tau_r(p) = \top$ if and only if $p(\top) \geq r$. Then the family $\{\tau_r \mid r \in [0, 1]\}$ is separating for every set X. Indeed, if we define $f_x : X \to 2$ by $f_x(x') = \top \iff x = x'$, then for $d \in \mathcal{D}_{\leq 1}X$, holds.

By Theorem 8.8, the behavioral equivalence (Definition 8.1) for a Markov chain $c : X \to N_\Sigma X$ coincides with the codensity bisimilarity $\nu\Phi_c^{\mathrm{Eq}_2,\tau}$. Thus, by Corollary 6.11, it is characterized by the codensity game (Table 11) specialized to this situation. The game in this case is shown in Table 15. It is trimmed by the join-dense subset in Example 5.2. It is essentially the same as Table 2 (arising from [13]). The difference is that r is additionally present in Table 15; it is easy to realize that r plays no role in the game.

Theorem 8.16 *Let $c : X \to \mathcal{D}_{\leq 1}X$ be a Markov chain. The position $(x, y) \in X \times X$ in the game in Table 15 is winning for Duplicator if and only if $(x, y) \in \nu\Phi_c^{\mathrm{Eq}_2,\tau}$, if and only if x and y are behaviorally equivalent.* □

Concretely, for any $R \in \mathbf{EqRel}_X$, the relation part of the codensity lifting $\mathcal{D}_{\leq 1}^{\mathbf{\Omega},\tau}(X, R)$ relates $p, q \in \mathcal{D}_{\leq 1}(X)$ if and only if the following holds:

$$\forall r \in [0,1]. \forall k : X \to 2. \big((\forall (x,y) \in R.\ k(x) = k(y)) \\ \implies \big(\textstyle\sum_{x \in k^{-1}(\top)} p(x) \geq r \Leftrightarrow \sum_{x \in k^{-1}(\top)} q(x) \geq r\big)\big).$$

From this, it is not hard to see that the resulting codensity bisimilarity also coincides with probabilistic bisimilarity in [4]. Note, for example, that a relation-preserving map $k : (X, R) \dot{\to} (2, \mathrm{Eq}_2)$ coincides with an R-closed subset of X.

8.2 Set-Coalgebras and Λ-Bisimulation

In [19], a bisimulation notion called Λ-*bisimulation* is introduced. Their intention is to start from a behavior functor and a modal logic, and construct a corresponding notion of bisimulation. The special cases include *precocongruence* for neighborhood frames, *rel*-Δ-*bisimulation* for Kripke frames, and *nbh*-Δ-*bisimulation* for neighborhood frames [19, Examples 14–16], and the latter two examples are related to *contingency logic*.

In Sect. 8.2 we see how their definition and our codensity bisimilarity overlap. Specifically, when all of the given modalities are unary, the induced Λ-bisimulation turns out to be a special case of codensity bisimulation (Proposition 8.21). Using this overlap, we also derive a game characterization of such Λ-bisimulations (Corollary 8.24).

Definition 8.17 (From [19, Section 2]) A *similarity type* is a set of modal operators with finite arities. For a similarity type Λ, a Λ-*structure* $(F, (\llbracket \heartsuit \rrbracket)_{\heartsuit \in \Lambda})$ is a pair of a functor $F : \mathbf{Set} \to \mathbf{Set}$ and a family of *predicate liftings* $\llbracket \heartsuit \rrbracket : \mathbf{Set}(\_, 2)^n \Rightarrow \mathbf{Set}(F\_, 2)$, where n is the arity of the modal operator $\heartsuit \in \Lambda$.

Note that, by the Yoneda lemma, a predicate lifting $\llbracket \heartsuit \rrbracket : \mathbf{Set}(\_, 2)^n \Rightarrow \mathbf{Set}(F\_, 2)$ can be equivalently represented by an arrow $\tau_\heartsuit : F(2^n) \to 2$. Concretely, from $\llbracket \heartsuit \rrbracket$, we can obtain $\tau_\heartsuit$ by $\llbracket \heartsuit \rrbracket_{2^n}(\pi_1, \dots, \pi_n)$; and from $\tau_\heartsuit$, we can recover $\llbracket \heartsuit \rrbracket$ by $\llbracket \heartsuit \rrbracket_X(f_1, \dots, f_n) = \tau_\heartsuit \circ F(\langle f_1, \dots, f_n \rangle)$.

Since Λ-bisimulations include not only endorelations but also binary relations between two different sets, we use the $\mathbf{CLat}_\sqcap$-fibration $\mathbf{BRel} \to \mathbf{Set}^2$ (Definition 2.13) here. One key notion in [19] is *R-coherence*.

Definition 8.18 (R-coherent pairs [19, Definition 8, Lemma 9 (b)]) Let $(X, Y, R) \in \mathbf{BRel}$, $U \subseteq X$, and $V \subseteq Y$. The pair (U, V) is *R-coherent* if both of the following hold:

- $(x, y) \in R \land x \in U \implies y \in V$.
- $(x, y) \in R \land y \in V \implies x \in U$.

Equivalently, the pair (U, V) is R-coherent if and only if, for each $(x, y) \in R$, $x \in U \iff y \in V$ holds.

The notion of R-coherence turns out to be expressible in terms of the fibration $\mathbf{BRel} \to \mathbf{Set}^2$.

Proposition 8.19 (Coherence as decency) *Let* $(X, Y, R) \in \mathbf{BRel}$, $f : X \to 2$, *and* $g : Y \to 2$. *Let* $\mathrm{Eq}_2 \subseteq 2 \times 2$ *be the diagonal relation (Definition 2.12). Then the pair* $(f^{-1}(\top), g^{-1}(\top))$ *is R-coherent if and only if the arrow* (f, g) *in* $\mathbf{Set}^2$ *is decent from* (X, Y, R) *to* $(2, 2, \mathrm{Eq}_2)$.

Proof By Definition 8.18, the pair $(f^{-1}(\top), g^{-1}(\top))$ is R-coherent if and only if, for each $(x, y) \in R$, $f(x) = \top \iff g(y) = \top$ holds. Here, the condition $f(x) = \top \iff g(y) = \top$ is equivalent to $(f(x), g(y)) \in \mathrm{Eq}_2$. The claim follows from Definition 2.13. □

From now on, we consider a similarity type Λ with only unary modal operators. It turns out that, in such cases, a Λ-bisimulation is the same thing as a codensity bisimulation with an appropriate family of lifting parameters.

Let us fix a Λ-structure $(F, (\llbracket\heartsuit\rrbracket)_{\heartsuit\in\Lambda})$. For each $\heartsuit \in \Lambda$, let $\tau_\heartsuit : F2 \to 2$ be the arrow corresponding to $\llbracket\heartsuit\rrbracket : \mathbf{Set}(\_, 2) \Rightarrow \mathbf{Set}(F\_, 2)$.

Definition 8.20 (Λ-bisimulation [19, Definition 11]) Let $c : X \to FX$ and $d : Y \to FY$ be F-coalgebras. A relation $Z \subseteq X \times Y$ is a Λ-*bisimulation* if, for every pair $(x, y) \in Z$, modal operator $\heartsuit \in \Lambda$, and Z-coherent pair (U, V),

$$c(x) \in \llbracket\heartsuit\rrbracket_X(U) \iff d(y) \in \llbracket\heartsuit\rrbracket_X(V)$$

holds.

This definition can be characterized using codensity lifting. We use the lifting of $F^2 : \mathbf{Set}^2 \to \mathbf{Set}^2$ by the family of parameters $\{((2, 2, \mathrm{Eq}_2), \tau_\heartsuit)_{\heartsuit\in\Lambda}\}$.

Proposition 8.21 *Let* $c : X \to FX$ *and* $d : Y \to FY$ *be F-coalgebras. A* Λ*-bisimulation is nothing but a codensity bisimulation for the family of lifting parameters* $((2, 2, \mathrm{Eq}_2), \tau) = \{((2, 2, \mathrm{Eq}_2), \tau_\heartsuit)_{\heartsuit\in\Lambda}\}$, *that is,* $Z \subseteq X \times Y$ *is a* Λ*-bisimulation if and only if* $(X, Y, Z) \sqsubseteq (c, d)^*(F^2)^{(2,2,\mathrm{Eq}_2),\tau}(X, Y, Z)$ *holds.*

Proof Assume $(X, Y, Z) \sqsubseteq (c, d)^*(F^2)^{(2,2,\mathrm{Eq}_2),\tau}(X, Y, Z)$. Expanding the definitions, the following holds:

If $(x, y) \in Z$, for each $(f, g) : (X, Y, Z) \to (2, 2, \mathrm{Eq}_2)$ and each $\heartsuit \in \Lambda$, $\tau_\heartsuit((Ff)(c(x))) = \tau_\heartsuit((Fg)(d(y)))$ holds.

Let (U, V) be any Z-coherent pair. We define $f : X \to 2$ and $g : Y \to 2$ by $f(x) = \top \iff x \in U$ and $g(y) = \top \iff y \in V$. By Proposition 8.19, $(f, g) : (X, Y, Z) \to (2, 2, \mathrm{Eq}_2)$ is decent. Thus, for each $\heartsuit \in \Lambda$, $\tau_\heartsuit((Ff)(c(x))) = \tau_\heartsuit((Fg)(d(y)))$ holds. By the definition of $\tau_\heartsuit$, this means

$$c(x) \in [\![\heartsuit]\!]_X(U) \iff d(y) \in [\![\heartsuit]\!]_X(V).$$

Since (U, V) is arbitrary, Z is a Λ-bisimulation.

Conversely, assume $Z \subseteq X \times Y$ is a Λ-bisimulation. For every pair $(x, y) \in Z$, modal operator $\heartsuit \in \Lambda$, and Z-coherent pair (U, V):

$$c(x) \in [\![\heartsuit]\!]_X(U) \iff d(y) \in [\![\heartsuit]\!]_X(V)$$

holds. Now, for each decent arrow $(f, g) : (X, Y, Z) \to (2, 2, \mathrm{Eq}_2)$, $(f^{-1}(\top), g^{-1}(\top))$ is Z-coherent by Proposition 8.19. Thus for every pair $(x, y) \in Z$ and modal operator $\heartsuit \in \Lambda$,

$$c(x) \in [\![\heartsuit]\!]_X(f^{-1}(\top)) \iff d(y) \in [\![\heartsuit]\!]_X(g^{-1}(\top))$$

holds. By the definition of $\tau_\heartsuit$, this is equivalent to $\tau_\heartsuit((Ff)(c(x))) = \tau_\heartsuit((Ff)(c(y)))$. Since this holds for any decent $(f, g) : (X, Y, Z) \to (2, 2, \mathrm{Eq}_2)$, $(X, Y, Z) \sqsubseteq (c, d)^*(F^2)^{(2,2,\mathrm{Eq}_2),\tau}(X, Y, Z)$ holds. □

Corollary 8.22 *Let $c : X \to FX$ and $d : Y \to FY$ be F-coalgebras. The codensity bisimilarity $\nu\Phi^{(2,2,\mathrm{Eq}_2),\tau}$ is the largest Λ-bisimulation.* □

In the case where the modal operators are all unary, we can derive a game characterization of Λ-bisimulation from our general framework. Let us first note the following fact:

Proposition 8.23 *The object $(1, 1) \in \mathbf{Set}^2$ is a fibered separator (Definition 5.5) of $\mathbf{BRel} \to \mathbf{Set}^2$.*

Proof Let $(X, Y) \in \mathbf{Set}^2$ and $B_1, B_2 \in \mathbf{BRel}_{(X,Y)}$. Assume $B_1 \neq B_2$. There exists a pair $(x, y) \in X \times Y$ such that exactly one of $(x, y) \in B_1$ and $(x, y) \in B_2$ holds. Consider the arrow $(x, y) : (1, 1) \to (X, Y)$ in $\mathbf{Set}^2$. Then $(x, y)^* B_1 \neq (x, y)^* B_2$ holds. This concludes the proof. □

By Corollary 6.11 and suppressing $\heartsuit$ (which does not affect Duplicator's moves), we obtain the following game characterization.

Corollary 8.24 *Let $c : X \to FX$ and $d : Y \to FY$ be F-coalgebras. For a pair of states $(x, y) \in X \times Y$, there exists a Λ-bisimulation containing (x, y) if and only if the position $(x, y) \in X \times Y$ in the game in Table 16 is winning for Duplicator.* □

Table 16 Codensity bisimilarity game for Λ-bisimulation

Position	Player	Possible moves
$(x, y) \in X \times Y$	Spoiler	f and g such that, for some $\heartsuit \in \Lambda$, exactly one of $\tau_{\heartsuit}((Ff)(c(x))) = \top$ and $\tau_{\heartsuit}((Fg)(d(x))) = \top$ holds
$f : X \to 2$ and $g : Y \to 2$	Duplicator	(x', y') such that exactly one of $f(x') = \top$ and $g(y') = \top$ holds

This in turn yields game characterizations of many bisimulation notions, e.g., those listed in [19, Example 13–16].

8.3 Deterministic Automata and the Language Topology

We introduced two versions of *bisimulation topology* for deterministic automata in Example 6.12. They are in close correspondences with accepted languages; therefore, we call them *language topologies*.

For the first topology in Example 6.12 (where $\mathbf{\Omega}$ is the Sierpinski space, modeling the situation where acceptance is only semi-decidable), the corresponding (untrimmed) codensity game is shown in Table 17. It follows from our general results that the game notion is sound and complete.

We have not yet found a good way (e.g., join-dense subsets) of trimming the game arena. This is left as future work.

8.4 Markov Chains and Bisimulation Metric

Recall Examples 3.10, 4.5 and 5.3. Markov chains are $\mathcal{D}_{\le 1}$-coalgebras. We use the $\mathbf{CLat}_{\sqcap}$-fibration $\mathbf{PMet}_1 \to \mathbf{Set}$ (Sect. 2.2.3), taking pseudometrics as a notion of indistinguishability. With the lifting parameter we described in Example 3.10, we get the bisimulation metric as the codensity bisimilarity. We can use the join-dense subset described in Example 5.3 to obtain a trimmed codensity game; the

Table 17 Codensity bisimilarity game for deterministic automata and the bisimulation topology

Position	Player	Possible moves
$\mathcal{O} \in \mathbf{Top}_X$	Spoiler	$a \in \{\varepsilon\} \cup \Sigma$ and $k \in \mathbf{Set}(X, 2)$ such that $\tau_A \circ (A_\Sigma k) \circ c : X \to 2$ is not continuous from $(X, \mathcal{O})$ to $(2, \mathbf{\Omega}_A)$
$a \in \{\varepsilon\} \cup \Sigma$ and $k \in \mathbf{Set}(X, 2)$	Duplicator	$\mathcal{O}' \in \mathbf{Top}_X$ such that $k : X \to 2$ is not continuous from $(X, \mathcal{O}')$ to $(2, \mathbf{\Omega}_A)$

resulting game coincides with the one in Table 3 in the introduction. Therefore, Corollary 5.16 gives an abstract proof for the correctness of the game.

8.5 Continuous State Markov Chains and Bisimulation Metric

To accommodate continuous state Markov chains (for which measurable structures are essential), we consider an example that involves **Meas**. Continuing Sect. 8.4, by the change-of-base along the forgetful functor $U : \mathbf{Meas} \to \mathbf{Set}$, we get another $\mathbf{CLat}_\sqcap$-fibration $U^*(\mathbf{PMet}_1) \rightarrowtail \mathbf{Meas}$. A continuous state Markov chain is a coalgebra $X \to \mathcal{G}_{\le 1}X$ of the so-called *sub-Giry* functor $\mathcal{G}_{\le 1} : \mathbf{Meas} \to \mathbf{Meas}$—see, e.g., [37].

As a parameter of codensity lifting, we take roughly the same thing as used in Example 3.10. The major difference is that we have to equip [0, 1] with some σ-algebra. We use the σ-algebra of *Borel sets* $\mathcal{B}([0,1])$. Let us abuse the notation [0, 1] to mean the object $([0,1], \mathcal{B}([0,1])) \in \mathbf{Meas}$. Then the parameter of codensity lifting we use is

$$(\mathbf{\Omega}, \tau) = \Big(\big([0,1], d_{[0,1]}\big),\ e : \mathcal{G}_{\le 1}[0,1] \to [0,1] \Big),$$

where e is the *expectation function* $e(\mu) = \int r d\mu(r)$, and $d_{[0,1]}$ is the Euclidean metric.

Let us expand the definition of the codensity lifting $\mathcal{G}_{\le 1}^{\mathbf{\Omega},\tau} : U^*(\mathbf{PMet}_1) \to U^*(\mathbf{PMet}_1)$. For $X \in \mathbf{Meas}$ and $(X,d) \in U^*(\mathbf{PMet}_1)$, $\mathcal{G}_{\le 1}^{\mathbf{\Omega},\tau}(X,d) = (\mathcal{G}_{\le 1}X, \mathcal{K}(d))$ holds. Here, $\mathcal{K}(d)$ is a variation of Kantorovich metric. For $\mu, \nu \in \mathcal{G}_{\le 1}X$,

$$\mathcal{K}(d)(\mu,\nu) = \sup_f \big| e((\mathcal{G}_{\le 1}f)(\mu)) - e((\mathcal{G}_{\le 1}f)(\nu)) \big|,$$

where f ranges over all non-expansive and measurable functions from (X, d) to $([0,1], d_{[0,1]})$. Note the similarity with the Eq. (1). The corresponding codensity bisimilarity $\nu\Phi_c^{\mathbf{\Omega},\tau} \in U^*(\mathbf{PMet}_1)$ (Definition 3.9) is a variation of the bisimulation metric from [5] for continuous state Markov chains.

Since the forgetful functor $\mathbf{Meas} \to \mathbf{Set}$ has a left adjoint, Proposition 5.10 gives us a fibered separator for $U^*(\mathbf{PMet}_1) \to \mathbf{Meas}$: concretely, the two-point set with the powerset σ-algebra $(2, \mathcal{P}2) \in \mathbf{Meas}$ is a fibered separator for $U^*(\mathbf{PMet}_1) \to \mathbf{Meas}$.

By Corollary 5.16, the codensity bisimilarity $\nu\Phi_c^{\mathbf{\Omega},\tau} \in U^*(\mathbf{PMet}_1)$ is characterized by the codensity game (Table 10) specialized in this situation. The game in this case is shown in Table 18.

8.6 Real-Weighted Automata and Bisimulation Seminorm

In Sect. 8.6 we consider *weighted automata*. Here we focus on those with real weights, which are the central subject of works, such as [20, 47]. We identify the *bisimulation seminorm* introduced in [20] as a codensity bisimilarity and derive a

Table 18 Codensity bisimilarity game for (probabilistic) bisimulation metric for a continuous state Markov chain

Position	Player	Possible moves
$(x, y, \varepsilon) \in X^2 \times [0, 1]$	Spoiler	measurable $f : X \to [0, 1]$ such that $\left|e((\mathcal{G}_{\leq 1}f)(c(x))) - e((\mathcal{G}_{\leq 1}f)(c(y)))\right| > \varepsilon$
measurable $f : X \to [0, 1]$	Duplicator	$(x', y', \varepsilon') \in X^2 \times [0, 1]$ such that $\left|f(x') - f(y')\right| > \varepsilon'$

game characterization of it. Note that seminorms are considered a linear-algebraic analogue of pseudometrics here.

We recall the definition of real-weighted automaton. We fix a finite alphabet Σ. The category $\mathbf{Vect}_{\mathbb{R}}$ is the category of real vector spaces (see Definition 2.14).

Definition 8.25 (Real-weighted automaton as a coalgebra in $\mathbf{Vect}_{\mathbb{R}}$) We define a functor $W_\Sigma : \mathbf{Vect}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{R}}$ by $W_\Sigma(V) = \left(\bigoplus_{a\in\Sigma} V\right) \oplus \mathbb{R}$, where $\oplus$ stands for direct sum of vector spaces. For a vector $w \in W_\Sigma(V)$, let $w_\varepsilon \in \mathbb{R}$ be the component corresponding to the right summand $\mathbb{R}$ of $W_\Sigma(V) = \left(\bigoplus_{a\in\Sigma} V\right) \oplus \mathbb{R}$. For $a \in \Sigma$, let $w_A \in V$ be the component corresponding to the a-part of the left summand $\bigoplus_{a\in\Sigma} V$.

A *(real-)weighted automaton* is a pair (c, α) of a W_Σ-coalgebra $c : V \to W_\Sigma V$ and a vector $\alpha \in V$.

In the above definition, α models the initial state and c models both the transitions and the acceptance vector. Note that, since c is an arrow in $\mathbf{Vect}_{\mathbb{R}}$, it is a linear map. The definition coincides with the usual one found in, e.g., [20, Section 2.1]. Since we are interested in the bisimulation metric, we often ignore the initial vector α and focus on W_Σ-coalgebras.

Let us then define a family of parameters to represent the bisimulation metric in [20] as codensity bisimilarity. Now we fix an arbitrary positive real parameter $\gamma > 0$, called the *discount factor*. The index set $\mathbb{A}$ is set to the direct product $\Sigma \times \{1, -1\}$. For each $a \in \Sigma$ and $r \in \{+1, -1\}$, let $\mathbf{\Omega}_{(a,r)} = (\mathbb{R}, |\cdot|)$, where $|\cdot|$ is the absolute value function regarded as a (semi-)norm. The modalities should be arrows $\tau_{(a,r)} : W_\Sigma\mathbb{R} \to \mathbb{R}$ in $\mathbf{Vect}_{\mathbb{R}}$ for each $a \in \Sigma$ and $r = \pm 1$. We define these by

$$\tau_{(a,r)}(w) = w_\varepsilon + r\gamma w_A.$$

Using this family, we have the following result. Recall the fibration $\mathbf{ESemi}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{R}}$ defined in Definition 2.14.

Proposition 8.26 *Consider the codensity lifting $W_\Sigma^{(\mathbb{R},|\cdot|),\tau}$ defined by the above parameters. Let $(V, s) \in \mathbf{ESemi}_{\mathbb{R}}$, where V is a (plain) vector space and s is a seminorm. Let $(W_\Sigma(V), s') = W_\Sigma^{(\mathbb{R},|\cdot|),\tau}(V, s)$. Then the seminorm s' on $W_\Sigma(V)$ satisfies*

$$s'(w) = |w_\varepsilon| + \gamma \max_{a\in\Sigma} s(w_A)$$

for each $w \in W_\Sigma(V)$.

Proof Since the fiberwise meet in $\mathbf{ESemi}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{R}}$ is pointwise sup, unwinding the definition (Definition 3.1) yields

$$s'(w) = \sup_k \max_{r=\pm 1, a\in\Sigma} |w_\varepsilon + r\gamma k(w_A)|,$$

where $k : V \to \mathbb{R}$ ranges over the linear nonexpansive maps $(V, s) \to (\mathbb{R}, |\cdot|)$. For each such k, case analysis on the sign of $w_\varepsilon \in \mathbb{R}$ implies

$$\max_{r=\pm 1, a\in\Sigma} |w_\varepsilon + r\gamma k(w_A)| = |w_\varepsilon| + \gamma \max_{a\in\Sigma} |k(w_A)|,$$

thus we have

$$s'(w) = |w_\varepsilon| + \gamma \max_{a\in\Sigma} \sup_k |k(w_A)|.$$

Now, it suffices to show $\sup_k |k(w_A)| = s(w_A)$ to conclude the proof. Let $w \in V$ be an arbitrary vector. We show $\sup_k |k(w)| = s(w)$.

We use the Hahn–Banach theorem ([48, Section III.6]):

Let V be a real vector space, $U \subseteq V$ be a subspace, $f : U \to \mathbb{R}$ be a linear function, and $p : V \to \mathbb{R}$ be a non-negative function. Assume that p is *sublinear*, that is, that the following hold:

- For each $r \leq 0$ and $v \in V$, $p(rx) = rp(x)$.
- For each $v, v' \in V$, $p(v + v') \leq p(v) + p(v')$.

Assume also that, for each $u \in U$, $|f(u)| \leq p(u)$ holds.
Then there exists a linear function $k : V \to \mathbb{R}$ satisfying the following:

- For each $u \in U$, $k(u) = f(u)$.
- For each $v \in V$, $|k(v)| \leq p(v)$.

Here, let U be the subspace of V generated by w and let p be the seminorm s. We define $f : U \to \mathbb{R}$ by $f(rw) = r$ for each $r \in \mathbb{R}$. Then all the assumptions are satisfied and there is a linear function $k : V \to \mathbb{R}$ such that, for each $v \in V$, $|k(v)| \leq s(v)$. This k is a linear nonexpansive map $(V, s) \to (\mathbb{R}, |\cdot|)$ and $|k(w)| = s(w)$ holds. Thus $\sup_k |k(w)| = s(w)$ holds. □

In particular, for each W_Σ-coalgebra $c : V \to W_\Sigma V$, the predicate transformer $\Phi_c^{(\mathbb{R},|\cdot|),\tau}$ (Definition 3.7) satisfies

$$\Phi_c^{(\mathbb{R},|\cdot|),\tau}(s)(v) = |(c(v))_\varepsilon| + \gamma \max_{a\in\Sigma} |s((c(v))_A)|$$

for each seminorm s, which coincides with the predicate transformer $F_{A,\gamma}$ defined in [20, Section 3].

Definition 8.27 (γ-bisimulation seminorm) For each W_Σ-coalgebra $c : V \to W_\Sigma V$, the greatest fixed point $\nu\Phi_c^{(\mathbb{R},|\cdot|),\tau}$ of the above predicate transformer is called the γ *-bisimulation seminorm* of c.

Note that, in the above definition, the greatest fixed point is taken w.r.t. the indistinguishability order (Notation 2.10). This means that, numerically, it is the least fixed point. If $\nu\Phi_c^{(\mathbb{R},|\cdot|),\tau}$ assigns a finite seminorm value to each vector, it coincides with the γ-bisimulation seminorm defined in [20, Section 3]. If some vector has the extended seminorm value ∞, then the γ-bisimulation seminorm in the original sense is not defined.

Using this fact, we can derive a game characterization of γ-bisimulation seminorm from our framework. First, let us mention the following:

Proposition 8.28 *The object* $\mathbb{R} \in \mathbf{Vect}_\mathbb{R}$ *is a fibered separator* (*Definition* 5.5) *of* $\mathbf{ESemi}_\mathbb{R} \to \mathbf{Vect}_\mathbb{R}$.

Proof Let $V \in \mathbf{Vect}_\mathbb{R}$ and $(V, s_1), (V, s_2) \in \mathbf{ESemi}_\mathbb{R}$. Assume that $(V, s_1) \neq (V, s_2)$. There exists a vector $v \in V$ such that $s_1(v) \neq s_2(v)$. Consider the linear map $f : \mathbb{R} \to V$ defined by $f(r) = rv$. Let $(\mathbb{R}, t_1) = f^*(V, s_1)$ and $(\mathbb{R}, t_2) = f^*(V, s_2)$. Then $t_1(1) = s_1(v) \neq s_2(v) = t_2(1)$. Therefore, $(\mathbb{R}, t_1) \neq (\mathbb{R}, t_2)$. This concludes the proof. □

Note also that the set of linear maps $\mathbb{R} \to V$ is naturally isomorphic to the underlying set of V, and that a seminorm on $\mathbb{R}$ is uniquely specified by its value on $1 \in \mathbb{R}$, which must be non-negative.

Using Corollary 6.11 and the above two identifications, we obtain the following game characterization.

Theorem 8.29 *Let* $c : V \to W_\Sigma V$ *be a* W_Σ*-coalgebra and let* $s : V \to \mathbb{R} \cup \{\infty\}$ *be the* γ*-bisimulation seminorm* (*Definition* 8.27) *of* c*. Then, for* $v \in V$ *and* $t \geq 0$, $s(v) \leq t$ *if and only if the position* $(v, t) \in V \times [0, \infty]$ *in the game in Table* 19 *is winning for Duplicator.*

Proof Using Corollary 6.11 and Proposition 8.28, we have the game in Table 20.

Table 19 Codensity bisimilarity game for γ-bisimulation seminorm

Position	Player	Possible moves
$v \in V$ and $t \in [0, \infty]$	Spoiler	f such that, for some $a \in \Sigma$ and some $r \in \{+1, -1\}$, $\lvert(c(v))_\varepsilon + r\gamma f((c(v))_A)\rvert > t$ holds
$f : V \to \mathbb{R}$	Duplicator	v' and t' such that $f(v') > t'$

Table 20 Codensity bisimilarity game for γ-bisimulation seminorm, obtained by expanding the definitions

Position	Player	Possible moves
$g : \mathbb{R} \to V$ and a seminorm s on $\mathbb{R}$	Spoiler	f such that, for some $a \in \Sigma$ and some $r \in \{+1, -1\}$, $\lvert (c(g(x)))_\varepsilon + r\gamma f((c(g(x)))_A) \rvert \not\leq s(x)$ holds for some $x \in \mathbb{R}$
$f : V \to \mathbb{R}$	Duplicator	g' and s' such that $\lvert f(g'(x)) \rvert \not\leq s'(x)$ for some $x \in \mathbb{R}$

Here, we use the following fact.

For seminorms s and s' on $\mathbb{R}$, $s(x) \leq s'(x)$ holds for all $x \in \mathbb{R}$ if and only if $s(1) \leq s'(1)$.

Using this, we replace the seminorms in Table 20 and obtain the game in Table 21.

The set of linear maps $\mathbb{R} \to V$ is naturally isomorphic to the underlying set of V by the map

$$\mathbf{Vect}_{\mathbb{R}}(\mathbb{R}, V) \ni g \mapsto g(1) \in V.$$

Using this, we replace $g : \mathbb{R} \to V$ in Table 21 and obtain Table 19. □

9 Conclusions and Future Work

Motivated by some recent works [8, 10, 11, 13], and especially by the similarity of the two games in Tables 2 and 3, we introduced a fibrational framework that uniformly describes the correspondence between various bisimilarity notions and games. The fibrational abstraction allows us to accommodate new games for several known examples (such as Λ-bisimulation in Sect. 8.2 and γ-bisimulation seminorm in Sect. 8.6) and a new example (bisimulation topology in Sect. 8.3). Moreover, the structural theory developed in Sects. 6 and 7 provides new insights to the nature of bisimilarity, identifying the crucial role of observation maps ($k : X \to \Omega$ in Definition 3.1) in bisimulation notions.

Table 21 Codensity bisimilarity game for γ-bisimulation seminorm, after eliminating seminorms on $\mathbb{R}$

Position	Player	Possible moves
$g : \mathbb{R} \to V$ and $t \in [0, \infty]$	Spoiler	f such that, for some $a \in \Sigma$ and some $r \in \{+1, -1\}$, $\lvert (c(g(1)))_\varepsilon + r\gamma f((c(g(1)))_A) \rvert > t$
$f : V \to \mathbb{R}$	Duplicator	g' and t' such that $\lvert f(g'(1)) \rvert > t'$

As future work, we are interested in using games with more complex winning conditions (e.g., parity); they have been used for (bi)simulation notions for Büchi and parity automata [49]. In addition, we will pursue the algorithmic use of the current results.

Appendix 1: Direct Proof of Equivalence of the Two Game Notions Characterizing Probabilistic Bisimilarity (Tables 2 and 4)

Table 4 ⇝ Table 2

Assume that Duplicator wins Table 4 from (x, y), and let Spoiler play some Z in Table 2. There are two cases to consider which are essentially identical, but we write them down separately for reference.

- If $\tau(x, Z) > \tau(y, Z)$ then we make Spoiler select $s = x$ and play Z in Table 4. To this Duplicator responds with some $Z' \supseteq Z$ such that $\tau(x, Z) \leq \tau(y, Z')$, which implies that $Z' \neq Z$. Pick any $y' \in Z' \setminus Z$ and play it as Spoiler in Table 4; when Duplicator responds with some $x' \in Z$, play the pair x' and y' as Duplicator in Table 2.
- If $\tau(x, Z) < \tau(y, Z)$ then we make Spoiler select $s = y$ and play Z in Table 4. To this Duplicator responds with some $Z' \supseteq Z$ such that $\tau(y, Z) \leq \tau(x, Z')$, which implies that $Z' \neq Z$. Pick any $y' \in Z' \setminus Z$ and play it as Spoiler in Table 4; when Duplicator responds with some $x' \in Z$, play the pair x' and y' as Duplicator in Table 2.

Table 2⇝ Table 4

This is a less straightforward implication. A winning strategy for Duplicator in Table 4 is built not from a single strategy in Table 2, but rather from an entire collection of winning positions.

Formally, assume that Duplicator wins Table 2 from (x, y), and let Spoiler choose $s \in \{x, y\}$ and play some Z in Table 4. We define

$$\bar{Z} = \{w \in X \mid \exists v \in Z \text{ such that Duplicator wins Table 2 from } (v, w)\}.$$

One basic observation is that $Z \subseteq \bar{Z}$, since Duplicator wins from all positions of the form (w, w). As a result, we have

$$\tau(x, Z) \leq \tau(x, \bar{Z}) \qquad \text{and} \qquad \tau(y, Z) \leq \tau(y, \bar{Z}). \tag{A1}$$

Another observation is that Spoiler wins Table 2 from the position $\bar{Z}$. To see this, consider any Duplicator's response $x' \in \bar{Z}$, $y' \notin \bar{Z}$. Then there is some $v \in Z$ such that Duplicator wins Table 2 from (v, x'). If Duplicator could win Table 2 from (x', y') then she could win from (v, y') as well, which contradicts the assumption that $y' \notin \bar{Z}$.

Since we assume that Duplicator wins Table 2 from (x, y), $\bar{Z}$ cannot be a legal move for Spoiler from (x, y), hence

$$\tau(x, \bar{Z}) = \tau(y, \bar{Z}).$$

Together with (A1) this implies that

$$\tau(x, Z) \leq \tau(y, \bar{Z}) \qquad \text{and} \qquad \tau(y, Z) \leq \tau(x, \bar{Z}),$$

so $Z' = \bar{Z}$ is a legal move for Duplicator in the stage (ii) of Table 4, no matter if Spoiler chose $s = x$ or $s = y$ in the stage (i). To this, in the stage (iii) Spoiler replies with some $y' \in \bar{Z} \setminus Z$. By the definition of $\bar{Z}$, there is some $v \in Z$ such that Duplicator wins Table 2 from (v, y'), so Duplicator can respond with $x' = v$.

Appendix 2: Introduction to $\mathbf{CLat}_{\sqcap}$-Fibration

We present an introduction to ($\mathbf{CLat}_{\sqcap}$-)fibrations, starting from a functor $F_{\mathbb{E}} : \mathbb{C}^{\mathrm{op}} \to \mathbf{CLat}_{\sqcap}$. The relevance of the latter is explained in Sect. 2.2. For details, readers are referred to [33].

The Grothendieck Construction

In general, the equivalence between index categories $\mathbb{C}^{\mathrm{op}} \to \mathbf{Cat}$ and fibrations is well-known. Here we sketch the *Grothendieck construction* from the former to the latter, focusing the special case of $\mathbb{C}^{\mathrm{op}} \to \mathbf{CLat}_{\sqcap}$ and $\mathbf{CLat}_{\sqcap}$-fibrations. Its idea is to "patch up" the family $\left(F_{\mathbb{E}}X\right)_{X \in \mathbb{C}}$ of complete lattices, and form a big category $\mathbb{E}$, as shown in Fig. 2.

On the right-hand side in Fig. 2, we add some arrows (denoted by $\dashrightarrow$) so that we have an arrow $(F_{\mathbb{E}}f)(Q) \to Q$ in $\mathbb{E}$ for each $Q \in F_{\mathbb{E}}Y$. (On the left-hand side, the correspondence $\mapsto$ depicts the action of the map $F_{\mathbb{E}}f$.) The diagram in $\mathbb{E}$ in Fig. 2 should be understood as a Hasse diagram: those arrows which arise from composition are not depicted.

Definition B.1 (The Grothendieck construction) Given a functor $F_{\mathbb{E}} : \mathbb{C}^{\mathrm{op}} \to \mathbf{CLat}_{\sqcap}$, we define the category $\mathbb{E}$ as follows.

- An object is a pair (X, P) of an object $X \in \mathbb{C}$ and an element $P \in F_{\mathbb{E}}X$; and
- An arrow $f : (X, P) \to (Y, Q)$ is an arrow $f : X \to Y$ in $\mathbb{C}$ such that

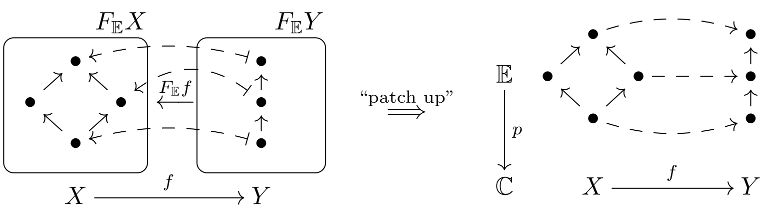


Fig. 2 Grothendieck construction

$$P \sqsubseteq (F_{\mathbb{E}} f)(Q).$$

Here $\sqsubseteq$ refers to the order of the complete lattice $F_{\mathbb{E}}X$.

Thus arises a category $\mathbb{E}$ that incorporates the following.

- the order structure of each of the posets $(F_{\mathbb{E}}X)_{X \in \mathbb{C}}$, and
- the pullback structure by $(F_{\mathbb{E}} f)_{f : \mathbb{C}-\text{arrow}}$.

For fixed $X \in \mathbb{C}$, the objects of the form (X, P) and the arrows id_X between them form a subcategory of $\mathbb{E}$. This is denoted by $\mathbb{E}_X$ and called the *fiber* over X. It is obvious that $\mathbb{E}_X$ is a poset that is isomorphic to $F_{\mathbb{E}}X$.

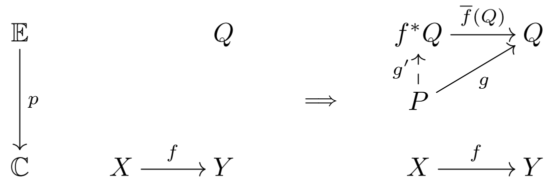
Moreover, there is a canonical projection functor $p : \mathbb{E} \to \mathbb{C}$ that carries (X, P) to X.

Formal Definition of $\mathbf{CLat}_{\sqcap}$-Fibration

We axiomatize those structures which arise in the way described above.

Definition 9.1 ($\mathbf{CLat}_{\sqcap}$-fibration) A $\mathbf{CLat}_{\sqcap}$-*fibration* $\mathbb{E} \xrightarrow{p} \mathbb{C}$ consists of two categories $\mathbb{E}, \mathbb{C}$ and a functor $p : \mathbb{E} \to \mathbb{C}$, that satisfy the following properties.

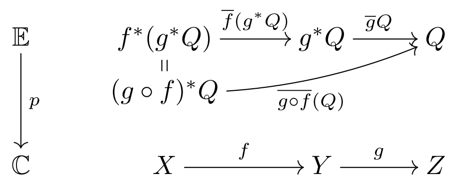
- Each fiber $\mathbb{E}_X$ is a complete lattice. Here the *fiber* $\mathbb{E}_X$ for $X \in \mathbb{C}$ is the subcategory of $\mathbb{E}$ consisting of the following data: objects $P \in \mathbb{E}$ such that $pP = X$; and arrows $f : P \to Q$ such that $pf = \mathrm{id}_X$ (such arrows are said to be *vertical*).
- Given $f : X \to Y$ in $\mathbb{C}$ and $Q \in \mathbb{E}_Y$, there is an object $f^*Q \in \mathbb{E}_X$ and an $\mathbb{E}$-arrow $\overline{f}Q : f^*Q \to Q$ with the following universal property. For any $P \in \mathbb{E}_X$ and $g : P \to Q$ in $\mathbb{E}$, if $pg = f$ then g factors through $\overline{f}(Q)$ uniquely via a vertical arrow. That is, there exists unique g' such that $g = \overline{f}(Q) \circ g'$ and $pg' = \mathrm{id}_X$:



- The correspondences $(\_)^*$ and $\overline{(\_)}$ are functorial:

$$\mathrm{id}_Y^* Q = Q, \qquad (g \circ f)^*(Q) = f^*(g^* Q),$$
$$\overline{\mathrm{id}_Y}(Q) = \mathrm{id}_Q, \qquad \overline{g \circ f}(Q) = \overline{g}Q \circ \overline{f}(g^* Q).$$

The last equality can be depicted as follows:



The category $\mathbb{E}$ is called the *total category* of the fibration; $\mathbb{C}$ is the *base category*. The arrow $\overline{f}Q : f^*Q \to Q$ is called the *Cartesian lifting* of f and Q. An arrow in $\mathbb{E}$ is *Cartesian* if it coincides with $\overline{f}Q$ for some f and Q.

In the case where $\mathbb{E} \xrightarrow{p} \mathbb{C}$ is induced by an indexed category $F_{\mathbb{E}} : \mathbb{C}^{\mathrm{op}} \to \mathbf{CLat}_{\sqcap}$ via Definition B.1, a Cartesian lifting is given by $f^*(Q) = (F_{\mathbb{E}}f)(Q)$.

In the current paper we focus on $\mathbf{CLat}_{\sqcap}$-fibrations. In a (general) fibration, a fiber $\mathbb{E}_X$ is not just a preorder but a category, and this elicits a lot of technical subtleties. Nevertheless, it should not be hard to generalize the current paper's observations to general, not necessarily $\mathbf{CLat}_{\sqcap}$-, fibrations (especially to the split ones). We shall often denote a vertical arrow in $\mathbb{E}$ (i.e., an arrow inside a fiber) by $\sqsubseteq$.

Appendix 3: Codensity Characterization of Hausdorff pseudometric

Proposition C.1 *Let (X, d) be a pseudometric space. For any $S, T \subseteq X$, we define two functions*

$$d_H(S,T) = \max\left(\sup_{x\in S}\inf_{y\in T} d(x,y), \sup_{y\in T}\inf_{x\in S} d(x,y)\right)$$

and

$$d_c(S,T) = \sup_{k\in\mathbf{PMet}_1((X,d),([0,1],d_{\mathbb{R}}))} d_{\mathbb{R}}\left(\inf_{x\in S} k(x), \inf_{y\in T} k(y)\right).$$

The values of two functions coincide.

Proof First, we show $d_c(S,T) \geq d_H(S,T)$ by contradiction.

Suppose it does not hold. Then, by definition, at least one of

$$\sup_{x\in S}\inf_{y\in T} d(x,y)$$

and

$$\sup_{y\in T}\inf_{x\in S} d(x,y)$$

is greater than $d_c(S,T)$. We can assume the former is greater than $d_c(S,T)$ w.l.o.g.

Therefore, for some $x_0 \in S$,

$$d_c(S,T) < \inf_{y\in T} d(x_0,y)$$

holds.

Now, since $d(x_0, \_)$ is a non-expansive function by the triangle inequality, we have

$$d_c(S, T) \geq d_{\mathbb{R}}\left(\inf_{x \in S} d(x_0, x), \inf_{y \in T} d(x_0, y)\right).$$

However, since $\inf_{x \in S} d(x_0, x) = 0$, we have $d_c(S, T) \geq \inf_{y \in T} d(x_0, y)$, which is a contradiction.

Next, we show $d_c(S, T) \leq d_H(S, T)$ by contradiction.

Suppose $d_c(S, T) > d_H(S, T) + \varepsilon$ for some $\varepsilon > 0$. Then, for some non-expansive $k : X \to [0, 1]$,

$$d_{\mathbb{R}}\left(\inf_{x \in S} k(x), \inf_{y \in T} k(y)\right) > d_H(S, T) + \varepsilon$$

holds.

W.l.o.g. we can assume $\inf_{x \in S} k(x) \leq \inf_{y \in T} k(y)$.

Thus, for some $x_0 \in S$ and $y_0 \in T$ satisfying $k(x_0) \leq \inf_{x \in S} k(x) + \varepsilon/5$ and $k(y_0) \leq \inf_{y \in T} k(y) + \varepsilon/5$,

$$d_{\mathbb{R}}(k(x_0), k(y_0)) > d_H(S, T) + 3\varepsilon/5$$

holds. Since

$$d_H(S, T) \geq \sup_{x \in S} \inf_{y \in T} d(x, y),$$

there exists some $y_1 \in T$ satisfying

$$d_H(S, T) \geq d(x_0, y_1) \geq d_{\mathbb{R}}(k(x_0), k(y_1)).$$

However, we have $k(x_0) \leq k(y_0) + \varepsilon/5 \leq k(y_1) + 2\varepsilon/5$, so

$$d_{\mathbb{R}}(k(x_0), k(y_1) + \varepsilon/5) \geq d_{\mathbb{R}}(k(x_0), k(y_0) + 2\varepsilon/5)$$

and

$$d_{\mathbb{R}}(k(x_0), k(y_1)) + 3\varepsilon/5 \geq d_{\mathbb{R}}(k(x_0), k(y_0))$$

holds.

Then,

$$\begin{aligned} & d_{\mathbb{R}}(k(x_0), k(y_0)) \\ & \leq d_{\mathbb{R}}(k(x_0), k(y_1)) + 3\varepsilon/5 \\ & \leq d_H(S, T) + 3\varepsilon/5 \\ & < d_{\mathbb{R}}(k(x_0), k(y_0)) \end{aligned}$$

holds, which is a contradiction. □

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Declarations

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