# Classical simulation of quantum contextuality


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**Abstract**

Recently, several inequalities involving sequences of measurements have been proposed which hold for non-contextual models but which are violated in quantum mechanics. They also have been found to be violated in experiments and the violation is independent of the prepared quantum state. A typical contextual model explaining this violation uses a classical memory. We investigate the required memory size and show that in order to simulate certain effects of quantum contextuality for two qubits, more than two classical bits are required.


## 1 Noncontextuality inequalities

Let us start with explaining the noncontextuality inequalities, which are consequences of the Kochen-Specker theorem [1]. For that, we take the one introduced in Ref. [2], see also Ref. [3] for more discussion. Consider a single system with nine observables, $A, B, C, a, b, c, \alpha, \beta$ and $\gamma$, and then the mean value

$$\langle\chi_{\rm KS}\rangle = \langle ABC\rangle + \langle abc\rangle + \langle\alpha\beta\gamma\rangle + \langle Aa\alpha\rangle + \langle Bb\beta\rangle - \langle Cc\gamma\rangle. \tag{1}$$


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Here, the expressions $\langle ABC \rangle$ etc. denote the mean value of the product of the three measurement values, if $A, B, C$ are measured simultaneously or in a sequence. If the measurements in each expectation value are compatible it does not matter whether one measures them simultaneously or in a sequence. Then, any noncontextual hidden variable model has to assign fixed values to each of the nine occurring measurements, independent of the column or row that is used as a measurement context. In this case one can see that

$$\langle \chi_{\mathrm{KS}} \rangle \leq 4, \tag{2}$$

as one cannot get rid of one minus sign in one term. However, in a two-qubit system, one may choose the observables of the Mermin-Peres square [4],

$$\begin{array}{lll} A = \sigma_z \otimes \mathbb{1}, & B = \mathbb{1} \otimes \sigma_z, & C = \sigma_z \otimes \sigma_z, \\ a = \mathbb{1} \otimes \sigma_x, & b = \sigma_x \otimes \mathbb{1}, & c = \sigma_x \otimes \sigma_x, \\ \alpha = \sigma_z \otimes \sigma_x, & \beta = \sigma_x \otimes \sigma_z, & \gamma = \sigma_y \otimes \sigma_y. \end{array} \tag{3}$$

The observables in any row or column commute and are therefore compatible. Moreover, the product of the observables in any row or column equals $\mathbb{1}$, apart from the last column, where it equals $-\mathbb{1}$. Hence, for *any* quantum state,

$$\langle \chi_{\mathrm{KS}} \rangle = 6 \tag{4}$$

holds. The remarkable fact is that this result shows that any quantum state reveals nonclassical properties if the measurements are chosen appropriately. This inequality has been tested in three recent experiments [5] and also the state independence of the violation has been confirmed.

## 2 Possible contextual models

In the experiments mentioned above, the observables are measured sequentially. This allows for a simple explanation of the violation: One could imagine that the quantum system "remembers" the measurements made before and flips the values of the later measurements accordingly. This is similar to an explanation of a Bell inequality violation by communication between the parties. Of course, such a model with memory is contextual.

The question arises: How many memory bits are needed in order to reproduce the violation of the noncontextuality inequality? More precisely, one could formulate this as a game: The first player Alice asks the second player Bob a sequence $Q_1 Q_2 Q_3 \ldots$ of arbitrary length of questions $Q_i$. The single questions are always out of a set of nine possible questions $Q_i \in \{A, B, C, a, b, c, \alpha, \beta, \gamma\}$. The second player has to give an answer $\pm 1$ directly to each of the questions and the answers have to fulfill two conditions: First, if at any point in the sequence one of the rows or columns of the Mermin-Peres square is measured (in an arbitrary permutation), then the answers have to fulfill the perfect correlations predicted by quantum mechanics (e.g. $\langle Cc\gamma \rangle = -1$) for these six contexts. Second, if a subsequence consists only of mutually compatible observables (e.g. from the set $\{A, a, \alpha\}$) then the values of these observables do not change during this subsequence.

Note that these conditions do not include all quantum mechanical predictions (e.g. the statistical predictions of quantum mechanics are neglected), Bob has only to reproduce some cases, where quantum mechanics predicts perfect correlations.

If Bob can use quantum mechanics, he can simply measure the corresponding observables of the Mermin-Peres square on an arbitrary two-qubit state and will fulfill the rules. However, if Bob has only access to classical resources, he needs some memory. How large does it have to be?

To formalize Bob's possible classical strategies, we consider that his memory consists of $K$ internal states. For each of the memory states he has some table $T_i$ in which the answers to all nine possible questions are given. Moreover, to any memory state he has an update rule $U_i$ which tells him how to update the memory state depending on the question asked. In information theory, such a scheme corresponds to a Mealy machine [6]. The question remains, how many tables $T_i$ and $U_i$ are needed to avoid any contradictions with the rules.

# 3 Results for the Peres-Mermin square

In the simplest case, Bob aims only to reproduce the perfect correlations of the Mermin-Peres square. For that, consider the Mealy machine with the tables

$$T_1 = \begin{bmatrix} - & - & + \\ + & + & + \\ + & - & - \end{bmatrix} \quad T_2 = \begin{bmatrix} + & - & - \\ + & + & + \\ + & + & + \end{bmatrix} \quad T_3 = \begin{bmatrix} - & - & + \\ - & - & + \\ + & + & + \end{bmatrix} \tag{5}$$

and the update tables

$$U_1 = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \quad U_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{6}$$

The tables should be understood as follows: Let us assume that the memory is in state 1. Then, if some observable (say, $a$) is measured one gives the corresponding value (here, $+1$) and *afterwards* the memory us updated according to the update table (here, one jumps to state 2). In the updating tables, a "0" means that the memory state should not change. A sequence of measurements may then also be written as, e.g. $a_1^+ A_2^+ \alpha_2^+ \beta_2^+ b_3^-$ etc. describing the dynamics when the sequence $aA\alpha\beta b$ is measured starting from memory state 1. One can show that the above machine is optimal:

**Theorem 1.** If a classical machine should reproduce all six perfect correlations for compatible observables from the Mermin-Peres Square, then it requires at least three memory states, i.e. $\log_2(3)$ bits of memory. One optimal solution is given by the tables in Eqs. (5, 6).

The proof of optimality goes along the following lines: Let us assume that there are only two memory states and consider the corresponding tables $T_1$ and $T_2$. The table $T_1$ has at least one contradiction to the Mermin-Peres conditions and we can assume without loosing generality that it is in the third column. If we measure the sequence $Cc\gamma$ starting from $T_1$ the overall product must be $-1$. This means that at some point in this sequence we have to jump to $T_2$; for definiteness, we can assume that the automaton jumps after measurement of $c$. Moreover, there must be at least one observable in the set $\{C, c, \gamma\}$ where the assignments of $T_1$ and $T_2$ differ. Let us assume that they differ in $C$ and that

$T_1(C) = +1$ and $T_2(C) = -1$. Then, we arrive at a contradiction: The results of the sequence $C_1^+ c_1^? C_2^-$ are a contradiction to the mutual compatibility of $C$ and $c$.

While the previous automaton reproduces the six Mermin-Peres correlations, it has the following problem: Starting from $T_1$, it is predicting: $C_1^+ A_1^- \gamma_3^+ C_2^-$. This means that $A$ and $\gamma$ change the value of $C$, although both of them are compatible with $C$ (but note that these three observables are not mutually compatible).

One can debate whether an automaton shall also fulfill this additional requirement. On the one hand, $A, C,$ and $\gamma$ cannot be measured simultaneously and the condition is not directly required in the definition of compatibility [3]. On the other hand, quantum mechanics predicts with certainty that the value of $C$ should not change. Moreover, models which violate this are in an obvious way contextual, as they directly violate the condition of non-contextuality [3]. In the following, we will assume that this *generalized condition on compatibility* holds. In short, one may formulate this as the condition that if a question (say, $A$) is asked two times in the measurement sequence and all the questions in between are compatible with that (here, they would be from $\{B, C, a, \alpha\}$), then the value of the original question ($A$) should not change.

Under this condition, we can formulate:

**Theorem 2.** For the Mermin-Peres square with the generalized compatibility condition, quantum contextuality can be simulated with four tables, i.e. two bits of memory. The optimal solution is given by:

$$T_1 = \begin{bmatrix} - & - & + \\ - & - & + \\ + & + & + \end{bmatrix} \quad T_2 = \begin{bmatrix} + & - & + \\ + & - & - \\ + & + & + \end{bmatrix} \quad T_3 = \begin{bmatrix} - & + & + \\ - & - & + \\ + & - & - \end{bmatrix} \quad T_4 = \begin{bmatrix} + & + & + \\ + & - & - \\ + & - & - \end{bmatrix} \tag{7}$$

where the update tables are given by:

$$U_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix} \quad U_2 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U_3 = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}. \tag{8}$$

The fact that these tables fulfill the conditions can directly be checked (preferably by computer). Furthermore, one can show that no three-state solution exists, which proves the optimality. This optimality proof is, however, technical.

## 4 Extended Kochen-Specker inequality

In the previous example, two classical bits were sufficient to simulate quantum contextuality for two qubits. This may not be surprising, as two-qubits cannot be used to store more than two classical bits — a fact known from Holevo's bound.

There is, however, an extended Kochen-Specker inequalities for two qubits which requires more than two bits of memory. This inequality has recently been introduced in Ref. [7]. To motivate it, note that for the observable $A = \sigma_z \otimes \mathbb{1}$ not only $B, C, a, \alpha$ from above are compatible, but also the observables $\mathbb{1} \otimes \sigma_y$ and $\sigma_z \otimes \sigma_y$. Using this for all observables from the Mermin-Peres square, one can consider all possible tensor products of

Pauli matrices and write down an inequality with 15 terms, where the quantum mechanical value is 15, but the classical bound is 9. For this inequality, we have:

**Theorem 3.** In order to simulate classically the extended Kochen-Specker inequality from Ref. [7] with the generalized compatibility condition, one needs at least five memory states, i.e. $\log_2(5)$ classical bits of memory.

In order to prove this, one shows that no four-state solution of the extended inequality exists. This time, the proof is *very* long and technical. The remarkable fact is, however, that the classical simulation of two qubits requires more classical bits of memory than the classical bits that can be stored in the system.

## References

[1] E. Specker, Dialectica **14**, 239 (1960); J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966); S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 (1967).

[2] A. Cabello, Phys. Rev. Lett. **101**, 210401 (2008); see also A. A. Klyachko, M. A. Can, S. Binicioğlu, and A. S. Shumovsky, Phys. Rev. Lett. **101**, 020403 (2008).

[3] O. Gühne, M. Kleinmann, A. Cabello, J.-Å. Larsson, G. Kirchmair, F. Zähringer, R. Gerritsma, and C. Roos, Phys. Rev. A **81**, 022121 (2010).

[4] A. Peres, Phys. Lett. A **151**, 107 (1990); N. D. Mermin, Phys. Rev. Lett. **65**, 3373 (1990).

[5] G. Kirchmair, F. Zähringer, R. Gerritsma, M. Kleinmann, O. Gühne, A. Cabello, R. Blatt, and C. Roos, Nature (London) **460**, 494 (2009); E. Amselem, M. Rådmark, M. Bourennane, and A. Cabello, Phys. Rev. Lett. **103**, 160405 (2009); O. Moussa, C. A. Ryan, D. G. Cory, and R. Laflamme, Phys. Rev. Lett. **104**, 160501 (2010).

[6] G. H. Mealy, Bell Systems Technical J. **34**, 1045 (1955).

[7] A. Cabello, arXiv:1002.3135.