ON DISCRETIZATION OF C*-ALGEBRAS

CHRIS HEUNEN AND MANUEL L. REYES

ABSTRACT. The C*-algebra of bounded operators on the separable Hilbert space cannot be mapped to a W*-algebra in such a way that each unital commutative C*-subalgebra $C(X)$ factors normally through $\ell^\infty(X)$. Consequently, there is no faithful functor discretizing C*-algebras to W*-algebras this way.

1. Introduction

In operator algebra it is common practice to think of a C*-algebra as a noncommutative analogue of a topological space, and to think of a W*-algebra as a noncommutative analogue of a measure space. In particular, just like any topological space embeds into a discrete one, atomic W*-algebras are often viewed as ‘noncommutative sets’ that can carry the ‘noncommutative topology’ of a C*-subalgebra, see e.g. [7, §1]. To make this precise, one needs a way to embed a C*-algebra into a W*-algebra. A standard way is the universal enveloping W*-algebra given by the adjunction

$$
\begin{array}{ccc}
\text{Cstar} & \cong & \text{Wstar} \\
\downarrow & \cong & \downarrow \\
\text{C}(X) & \cong & \ell^\infty(X)
\end{array}
$$

between the category of unital C*-algebras with unital *-homomorphisms and the subcategory of W*-algebras with normal *-homomorphisms, see [6, 3.2]. This construction has the drawback that the resulting W*-algebra is very large. It does not restrict to the commutative case as the embedding $\eta: C(X) \to \ell^\infty(X)$. This leads to the following notion, in keeping with the recent programme of taking commutative subalgebras seriously [18, §4, 19, §3] that has recently been successful [11, 9, 12, 10].

Definition. A discretization of a unital C*-algebra $A$ is a unital *-homomorphism $\phi: A \to M$ to a W*-algebra $M$ whose restriction to each commutative unital C*-subalgebra $C \cong C(X)$ factors normally through $\ell^\infty(X)$, so that the following diagram commutes.

$$
\begin{array}{ccc}
A & \xrightarrow{\phi} & M \\
\downarrow & & \downarrow \text{normal *-homomorphism} \\
C(X) & \xrightarrow{\eta} & \ell^\infty(X)
\end{array}
$$

This short note proves that this construction degenerates in prototypical cases.

Date: February 26, 2015.

2010 Mathematics Subject Classification. 46L30, 46L85, 46M15.

Key words and phrases. noncommutative topology, discrete space, atomic measure.

C. Heunen was supported by EPSRC Fellowship EP/L002388/1.
M. L. Reyes was supported by NSF grant DMS-1407152.
Theorem. If \( \phi : B(H) \to M \) is a discretization for a separable infinite-dimensional Hilbert space \( H \), then \( M = 0 \).

Stated more concretely, this obstruction means that \( A = B(H) \) has no representation on a Hilbert space \( K \neq 0 \) such that every (maximal) commutative \(*\)-subalgebra of \( A \) has a basis of simultaneous eigenvectors in \( K \).

Consequently, discretization cannot be made into a faithful functor.

Corollary. Let \( F : \text{Cstar} \to \text{Wstar} \) be a functor, and \( \eta_A : A \to F(A) \) natural unital \(*\)-homomorphisms. Suppose there are isomorphisms \( F(C(X)) \cong \ell^\infty(X) \) for each compact Hausdorff space \( X \) that turn \( \eta_{C(X)} \) into the inclusion \( C(X) \to \ell^\infty(X) \). If a unital \( C^* \)-algebra \( A \) has a unital \(*\)-homomorphism \( f : B(H) \to A \) for an infinite-dimensional Hilbert space \( H \), then \( F(A) = 0 \).

The proof of the Theorem relies on the existence of normal states in \( \text{W}^* \)-algebras. Intriguingly, this means that it does not rule out faithful functors \( F \) as above from \( \text{Cstar} \) to the category of \( \text{AW}^* \)-algebras (see [12] Section 2 for the appropriate morphisms). A rather different approach to the problem of extending the embeddings \( C(X) \to \ell^\infty(X) \) to noncommutative \( C^* \)-algebras has recently appeared in [16]. We also remark that since the identity functor discretizes all finite-dimensional \( C^* \)-algebras, this truly infinite-dimensional obstruction is independent of the Kochen-Specker theorem, a key ingredient in some previous spectral obstruction results [18, 4].

The rest of this note proves the Theorem and its Corollary.

2. Proof

We begin with a lemma that characterizes atomic measures. Let \( (X, \Sigma) \) be measurable space with a finite measure \( \mu \). Recall that an atom for \( \mu \) is a measurable set \( V \in \Sigma \) such that \( \mu(V) > 0 \) and for every measurable \( U \subseteq V \), either \( \mu(U) = 0 \) or \( \mu(U) = \mu(V) \). It follows that for every decomposition of \( V \) into a finite (or countably infinite) disjoint union of measurable sets \( V = \bigsqcup V_i \), one of the \( V_i \) has measure \( \mu(V) \) and the rest have measure zero.

The measure \( \mu \) is said to be diffuse if it has no atoms, and atomic if every nonnegligible measurable set contains an atom. Define an interval for a finite measure \( \mu \) on \( (X, \Sigma) \) to be a one-parameter family of measurable sets \( U_t \in \Sigma \) with \( t \in [0, M] \) for a positive real number \( M \) such that \( s \leq t \) implies \( U_s \subseteq U_t \) and \( \mu(U_t) = t \) for all \( s, t \in [0, M] \).

Lemma 1. Let \( (X, \Sigma, \mu) \) be a finite measure space. Then \( (X, \Sigma, \mu) \) has an interval if and only if \( \mu \) is not atomic.

Proof. First suppose that \( \mu \) is not atomic. Any finite measure \( \mu \) decomposes uniquely as \( \mu = \mu_a + \mu_d \) into an atomic measure \( \mu_a \) and a diffuse measure \( \mu_d \) [11] 2.6. Moreover, \( \mu_a \) and \( \mu_d \) are singular [13] 3.3. This means [8] p126 that \( (X, \Sigma, \mu) \) is a disjoint union of an atomic measure space and a diffuse one. The latter is nonempty by assumption and we may restrict to it without loss of generality. But nonempty finite diffuse measure spaces always have an interval, see [2] Lemma 2.5 or [5] Lemma 4.1.

Now suppose that \( \{U_t \mid t \in [0, M]\} \) is an interval in \( (X, \Sigma, \mu) \). Scaling \( \mu \) by \( 1/M \) and restricting to \( S_M \), we may assume \( M = 1 \) and \( U_0 = X \). For any positive integer \( n \), the sets \( K_1 = U_{1/n} \) and \( K_i = (U_{i/n}) \setminus (U_{(i-1)/n}) \) for \( i = 2, \ldots, n \) partition
$X$ into $n$ disjoint subsets of measure $1/n$ each. If $V$ were an atom of $\mu$, because $V = \bigsqcup_n V \cap K_i$ it must be the case that $\mu(V) = \mu(V \cap K_i) \leq \mu(K_i) \leq 1/n$ for some $i$. As $n$ was arbitrary, this means $\mu(V) = 0$. Thus $\mu$ is not atomic. \hfill \Box

Now let $X$ be a compact Hausdorff space, and let $\psi$ be a state on $C(X)$. We say that $\psi$ is atomic if $\psi = \sum \lambda_\rho \rho$ for pure states $\rho$ of $C(X)$ and nonnegative scalars $\lambda_\rho$ with $\sum \lambda_\rho = 1$. The Riesz–Markov theorem shows that $\psi(f) = \int_X f \, d\mu$ for a unique regular Borel probability measure $\mu$ on $X$. Any atoms of such a measure $\mu$ must be singleton sets $\{x\}$ for $x \in X$ \cite[2.IV]{13}. Note that the pure states $\rho$ on $C(X)$ precisely correspond to Dirac measures $\delta_x$ for $x \in X$. Thus the state $\psi$ is atomic if and only if the corresponding probability measure $\mu$ is atomic, in which case it has the form $\mu = \sum_{x \in X} \lambda_x \delta_x$ for scalars $\lambda_x \geq 0$ with $\sum \lambda_x = 1$.

For the separable Hilbert space $H = L^2[0,1]$, write $A = B(H)$ for the algebra of bounded operators on $H$, write $C = L^\infty[0,1]$ for the corresponding continuous maximal abelian subalgebra of $A$, and write $D$ for the discrete maximal abelian subalgebra of $A$ generated as a W*-algebra by the projections $q_n$ onto the Fourier basis vectors $e_n = \exp(2\pi in-)$ for $n \in \mathbb{Z}$.

**Lemma 2.** Let $\psi : A \to \mathbb{C}$ be a state. If its restriction to $D$ is pure, then its restriction to $C$ cannot be atomic.

**Proof.** By Kadison–Singer \cite{17}, a pure state on $D$ extends uniquely to a state on $A$ via the canonical conditional expectation $E : A \to D$ that sends an operator $a$ to its diagonal part $\sum q_n a q_n$ with respect to the Fourier basis $e_n$. So $\psi = \psi \circ E$, as we assumed $\psi$ to be pure on $D$. Letting $p_t$ be the projection $\chi_{[0,t]}$ in $C$ for $t \in [0,1]$: \[ \langle p_t e_n, e_n \rangle = \langle \chi_{[0,t]} \cdot \exp(2\pi in-), \exp(2\pi in-) \rangle = \int_0^t \chi_{[0,t]} \cdot e^{2\pi inx} \cdot e^{2\pi inx} \, dx = \int_0^t \chi_{[0,t]}(x) |e^{2\pi inx}|^2 \, dx = \int_0^t 1 \, dx = t. \]

Thus $E(p_t) = \sum q_n p_t q_n = \sum (p_t e_n, e_n) q_n = \sum t q_n = t \cdot 1_A$. It now follows that $\psi(p_t) = \psi(E(p_t)) = \psi(t \cdot 1_A) = t$.

Under an isomorphism $C \cong C(X)$ for a compact Hausdorff space $X$, the projections in the chain $\{p_t\}$ correspond to characteristic functions for clopen subsets $\{U_t\}$ of $X$ and the state $\psi$ corresponds to a state $f \mapsto \int_X f \, d\mu$ for some regular Borel measure $\mu$ on $X$. The condition $\psi(p_t) = t$ means $\mu(U_t) = \int \chi_{U_t} \, d\mu = t$, making $\{U_t \mid t \in [0,1]\}$ an interval of clopen sets in $X$. Lemma \cite{14} implies that $\mu$ is not atomic, so $\psi$ cannot be atomic. \hfill \Box

The first two lemmas suffice to establish the Theorem.

**Proof of Theorem.** Let $M$ be a W*-subalgebra of $B(K)$ for a Hilbert space $K$. Write $C \cong C(X)$ and $D \cong C(Y)$ for compact Hausdorff spaces $X$ and $Y$. The discretization $\phi : A \to M \subseteq B(K)$ is accompanied by the following commutative
diagram.

\[
\begin{array}{ccc}
C = L^\infty[0,1] & \cong & C(X) \\
\downarrow \phi & & \downarrow h \\
B(H) & = & A \\
\downarrow \hat{g} & & \downarrow \subseteq B(K) \\
D = \ell^\infty(Z) & \cong & C(Y)
\end{array}
\]

Given \(y \in Y\), the atomic projection \(\chi(y) \in \ell^\infty(Y)\) has image \(q_y = g(\chi(y)) \in M\). Suppose for a contradiction that \(q_y \neq 0\). Choose a unit vector \(v_y \in K\) in its range. This induces a state \(\psi_y(a) = \langle av_y, v_y \rangle\) on \(A\). For \(d \in D\), considering \(d \in C(Y) \subseteq \ell^\infty(Y)\) we have \(d\chi_y = d(y)\chi_y\), and thus:

\[
\begin{align*}
\psi_y(d) &= \langle dv_y, v_y \rangle \\
&= \langle dq_yv_y, v_y \rangle \\
&= \langle d(y)q_yv_y, v_y \rangle \\
&= \langle d(y)v_y, v_y \rangle \\
&= d(y)\|v_y\|^2 \\
&= d(y).
\end{align*}
\]

That is, \(\psi_y\) restricts to the pure state \(d \mapsto d(y)\) on \(D\). It follows from Lemma 2 that \(\psi_y\) is not atomic on \(C\).

On the other hand, for \(x \in X\) consider the atomic projection \(\chi_{\{x\}} \in \ell^\infty(X)\) and its image \(p_x = h(\chi_{\{x\}}) \in M\). Since \(\sum x_p = 1\), we can decompose \(K = \bigoplus K_x\) along the ranges \(K_x\) of \(p_x\). Write \(v_y = \sum \lambda_x w_x\) for unit vectors \(w_x \in K_x\) and \(\lambda_x \in \mathbb{C}\) satisfying \(\sum |\lambda_x|^2 = 1\). For \(c \in C\), we have \(cp_x = c(x)p_x\) (considering \(c \in C(X) \subseteq \ell^\infty(X)\) as before) and \(cw_x = cp_xw_x = c(x)w_x\), so that:

\[
\begin{align*}
\psi_y(c) &= \langle cw_y, v_y \rangle \\
&= \sum_{x,x'} \lambda_x \lambda_{x'} \langle cw_x, w_{x'} \rangle \\
&= \sum_{x,x'} \lambda_x \lambda_{x'} c(x) \langle w_x, w_{x'} \rangle \\
&= \sum_x |\lambda_x|^2 c(x).
\end{align*}
\]

Thus the restriction of \(\psi_y\) to \(C\) is an atomic state.

This is a contradiction, so every atomic projection \(\chi_{\{y\}} \in \ell^\infty(Y)\) must have image \(g(\chi_{\{y\}}) = q_y = 0\) in \(M\). Hence the normal \(*\)-homomorphism \(g: \ell^\infty(Y) \to M\) is the zero map. But then \(1_M = \phi(1_A) = g(\eta(1_A)) = g(1_Y) = 0\), so \(M = 0\). \(\square\)

We thank an anonymous referee for informing us that the Theorem can be proved without the full force of Kadison–Singer, as follows. Identifying the algebra \(C(\mathbb{T})\) of continuous functions on the unit circle \(\mathbb{T}\) with the subalgebra of \(C[0,1]\) satisfying \(f(0) = f(1)\), it is known that \(C(\mathbb{T})\) supports unique extensions of pure states of the discrete masa \(D \subseteq B(H)\). Indeed, the algebra of Fourier polynomials—or more generally, the Wiener algebra \(A(\mathbb{T})\)—is a dense subalgebra of \(C(\mathbb{T})\) and lies in the algebra \(M_0 \subseteq B(H)\) of operators that are \(l_1\)-bounded in the sense of Tanbay [20] with respect to the Fourier basis \(\{e_n \mid n \in \mathbb{Z}\}\). Thus \(C(\mathbb{T})\) lies in the norm closure
M of M₀, and the results of [20] imply that pure states on D extend uniquely to M. A computation as in Lemma 2 shows that this extended state corresponds to the Lebesgue measure on T, hence is not atomic on C(T). The Theorem may now be proved in essentially the same manner, replacing the algebra C with C(T).

The proof of the Corollary uses the following ‘stability’ of discretizations.

**Lemma 3.** Discretizations are stable under precomposition with ∗-homomorphisms and postcomposition with normal ∗-homomorphisms: if φ: B → M discretizes B, f: A → B is a morphism in Cstar, and g: M → N is a morphism in Wstar, then g ∘ φ ∘ f discretizes A.

**Proof.** If C(X) ∼= C ⊆ A is a commutative C*-subalgebra, then so is C(Y) ∼= f[C] ⊆ B, making the top squares of the following diagram commute (where f': Y → X is a continuous function between compact Hausdorff spaces derived from f: C → f[C] via Gelfand duality).

\[
\begin{array}{cccccc}
A & \xrightarrow{f} & B & \xrightarrow{\phi} & M & \xrightarrow{g} & N \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C(X) & \xrightarrow{C(f')} & C(Y) & \xrightarrow{\eta(C(Y))} & \ell^\infty(Y) & & \\
& \xleftarrow{\eta(C(X))} & & \xleftarrow{\ell^\infty(f')} & & \\
& & & & & \\
\end{array}
\]

The bottom triangle commutes by naturality of η. As all dashed arrows are normal, so is their composite.

**Proof of Corollary.** We first prove that φ = η ∘ f: B(H) → F(A) is a discretization. If C(X) is a commutative C*-subalgebra of B(H), its image under f is a commutative C*-subalgebra of A and hence of the form C(Y). Consider the following diagram.

\[
\begin{array}{cccccc}
& & & & \phi & \\
& B(H) & \xrightarrow{f} & A & \xrightarrow{\eta_A} & F(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C(X) & \xrightarrow{C(f)} & C(Y) & \xrightarrow{\eta(C(Y))} & F(C(Y)) & \\
\downarrow \quad \ell^\infty(X) & \xrightarrow{\ell^\infty(f)} & & & & \\
& & \quad & & \ell^\infty(Y) & \\
\end{array}
\]

The top-left square commutes by definition, and the top-right square commutes by naturality of η. The bottom-left square commutes by naturality of the inclusion C(X) → ℓ^∞(X), and the bottom-right triangle commutes by assumption. Finally, the dashed arrows are normal: the horizontal one because it is in the image of the functor ℓ^∞, the vertical one because it is in the image of the functor F, and the diagonal one because it is an isomorphism. Thus φ is a discretization.

Since H is infinite-dimensional, it is unitarily isomorphic to L²[0, 1] ⊗ H. This gives rise to a unital ∗-homomorphism i: B(L²[0, 1]) → B(L²[0, 1]) ⊗ B(H) ∼= B(H)
given by $i(a) = a \otimes 1$. Precomposing $\phi$ with this map induces a discretization $\phi \circ i: B(L^2[0,1]) \to F(A)$ according to Lemma 3, so the Theorem guarantees that $F(A) = 0$. □

We leave open whether there exists any state on $B(H)$ that restricts to an atomic state on each (maximal) abelian $*$-subalgebra.

References


Department of Computer Science, University of Oxford, Wolfson Building, Parks Road, OX1 3QD, Oxford, UK
E-mail address: heunen@cs.ox.ac.uk

Department of Mathematics, Bowdoin College, Brunswick, ME 04011–8486, USA
E-mail address: reyes@boudoin.edu