A monolithic approach to fluid-structure interaction

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Abstract

This paper compares partitioned and monolithic solution procedures for the numerical simulation of fluid-structure interactions. Their different stability properties are illustrated and the role of structural prediction for a partitioned method is discussed. A grid refinement study has been carried out to assess the temporal accuracy of these methods. Moreover, their computational cost as well as their computational efficiency is compared. Numerical experiments are presented for a one-dimensional model problem of a piston interacting with a fluid.

Key words: fluid-structure interaction, monolithic schemes, partitioned schemes

1 Introduction

Numerical simulation of fluid-structure interactions has often been done using partitioning. In a partitioned method, the fluid and the structure equations are alternatingly integrated in time and the interface conditions are enforced asynchronously; see, e.g., Refs. [1,2,3]. Partitioned methods are typically energy increasing and, hence, numerically unstable; see, e.g., Refs. [2,4].

The deficiencies of partitioned methods have motivated the investigation of monolithic methods, which treat the interaction of the fluid and the structure at the interface synchronously. The discretized equations are then typically solved by multiple fluid-structure iterations; see, e.g., Refs. [5,6,7,8].

In this work we compare partitioned and monolithic solution procedures in terms of stability, accuracy and computational cost. Numerical results are presented for a

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one-dimensional model problem of a piston interacting with a fluid. This prototypical fluid-structure interaction problem suffices for the above mentioned purpose of our investigation. It is anticipated that the main conclusions extend *in abstracto* to more complicated fluid-structure interaction problems.

The contents of this paper are organized as follows : Sec. 2 describes the governing equations of the piston problem. Sec. 3 presents the discretization methods employed for fluid and structure. Sec. 4 addresses partitioned and monolithic methods and prediction techniques. In Sec. 5 we present numerical experiments and results. Sec. 6 contains concluding remarks.

2 Governing equations

2.1 Fluid

The fluid is described by the one-dimensional Euler equations for compressible flow :

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0, \tag{1}$$

where $U = (\rho, \rho u, \rho e)$ is the state vector of conservative variables, and $F = (\rho u, \rho u^2 + p_f, \rho u e + p_f u)$ is the flux vector. ρ denotes the density, u is the velocity, p_f is the fluid pressure and e is the total energy density, respectively. The equation system is closed by the state equation for a perfect gas :

$$p_f = (\gamma - 1)\left(\rho e - \frac{1}{2}\rho u^2\right), \tag{2}$$

where γ denotes the ratio of specific heats (assumed to be 1.4 for the numerical experiments presented here).

2.2 Structure

The structure is represented by a one-degree-of-freedom model (Fig. 1) which is described by the following differential equation :

$$m\ddot{z} + kz = p_s - p_0, \tag{3}$$

where z denotes the structural displacement, m the mass and k the spring-stiffness of the piston model, p_s is the fluid pressure applied to the structure and p_0 is the atmospheric pressure. A superimposed dot denotes differentiation with respect to time. The surface of the piston is assumed to have a unit area.

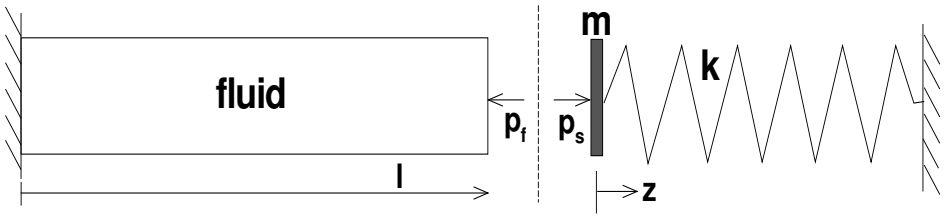

Figure 1. The piston problem (Interface region expanded for clarity).

2.3 Interface conditions

The interface conditions for the fluid-structure system can be expressed as a dynamic condition and two kinematic conditions. Dynamic equilibrium at the interface requires the pressure to be equal at either side of the interface :

$$p_s = p_f. \tag{4}$$

The first kinematic compatibility condition requires that the position of the fluid boundary, l, is equal to its initial position, l_0, plus the structural displacement. The second requires that the fluid velocity at the boundary is equal to the velocity of the moving boundary. This can be expressed as follows :

$$l = l_0 + z, \tag{5a}$$
$$u = \dot{l}. \tag{5b}$$

3 Discrete form of the equations

3.1 Fluid discretization: time-discontinuous Galerkin method

The fluid equations are formulated in space-time, in which moving meshes are accounted for inherently, as part of the formulation. A time-discontinuous Galerkin least-squares discretization with linear isoparametric elements is employed, which is described in detail in Refs. [9,10]. The discretization is done in primitive variables (p_f, u, T), which makes boundary condition implementation and coupling to the structure straightforward.

3.2 Structural time integration: trapezoidal method

The time integration of the structure employs the trapezoidal method; see, e.g., Ref. [11]. Trapezoidal time integration corresponds to a Newmark method with the parameter choice $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{2}$.

3.3 Discrete interface conditions

The interface conditions, Eqs. (4)–(5), are enforced by identifying displacement, velocity and pressure at the beginning and end of a time interval. Maintaining conservation at the interface is possible under the conditions stated in Refs. [12,13] and has implications for stability and accuracy of the numerical method employed, see Ref. [14]. The trapezoidal method does not generally conserve the energy transferred at the fluid-structure interface, see Ref. [15]. However, as the partitioning error is dominant, a comparison between partitioned and monolithic schemes is still possible.

4 Partitioned vs. monolithic solution methods

4.1 Partitioned solution methods

In a partitioned scheme, the fluid and the structure equations are alternatingly integrated in time and the interface conditions are enforced asynchronously; see, e.g., Refs. [1,2,3]. Typically, partitioned methods are based on the following sequential process :

(1) transfer the motion of the structural boundary to the fluid,
(2) update the position of the moving fluid mesh,
(3) advance the fluid system in time and compute the new pressure,
(4) convert the new fluid pressure into a structural load,
(5) advance the structural system in time under the fluid-induced load.

This sequential process allows for software modularity. Partitioned schemes require only one fluid and structure solution per time step, which can be considered as a single fluid-structure iteration. On the other hand, due to the time lag between the time integration of fluid and structure, the conservation properties of the continuum fluid-structure system are lost. Partitioned schemes are commonly energy-increasing and therefore numerically unstable; see, e.g., Refs. [2,4]. In practice, this introduces a restriction on the admissible time-step size. Moreover, partitioning has

implications for accuracy, see Ref. [14]. The benefits and deficiencies of partitioned methods are discussed in Ref. [1].

4.2 Monolithic solution methods

In a monolithic method, the interaction of the fluid and the structure at the mutual interface is treated synchronously. The discretized equations are then typically solved by multiple fluid-structure iterations; see, e.g., Refs. [5,6,7,8]. For monolithic schemes, maintenance of the conservation properties at the interface is possible, see Refs. [12,13]. This guarantees unconditional stability. The admissible time-step size therefore appears to be limited only by the required accuracy.

4.3 Prediction

The order of the partitioning error can be improved by prediction techniques; see, e.g., Refs. [3,4]. Instead of integrating the fluid equations based on the position of the structural boundary at t_n, a prediction may be used for the position of the structural boundary at t_{n+1} based on an extrapolation of the solution from the current time level. We use the following prediction for the structural displacement d and velocity v :

$$\tilde{d}_{n+1} = d_n + \frac{\Delta t}{2}(3v_n - v_{n-1}), \tag{6}$$

$$\tilde{v}_{n+1} = v_n + \Delta t a_n. \tag{7}$$

Prediction techniques improve the accuracy of the partitioned scheme and lead to better stability; see also Refs. [3,4]. We demonstrate this effect by numerical experiments in Sec. 5.1.

Prediction techniques can also be used to initialize the iterative solution of a monolithic method, which results in a reduced number of iterations, as will be shown in Sec. 5.3.

5 Numerical experiments

The parameters for the piston problem are given in Table 1, where ω_{fs} denotes the frequency of the fluid-structure system. They are similar to the settings used in Ref. [16].

We denote the characteristic time-scales of fluid, structure and fluid-structure sys-

Table 1
Parameters for the piston problem.

ρ	1.3	$\frac{kg}{m^3}$
p	10^5	$\frac{N}{m^2}$
length of the fluid domain L	1.0	m
ω_{fs}	342	$\frac{1}{s}$
m	0.8	kg
k	7911	$\frac{N}{m}$

tem by :

$$T_f = \frac{L}{c}, \qquad T_s = 2\pi\sqrt{\frac{m}{k}}, \qquad T_{fs} = \frac{2\pi}{\omega_{fs}} \tag{8}$$

respectively, where c is the speed of sound. Introducing the following non-dimensional time-scale ratios:

$$\frac{T_f}{T_{fs}} = 0.17, \qquad \frac{T_s}{T_{fs}} = 3.44 \tag{9}$$

and the non-dimensional mass ratio of fluid and structure :

$$\mu = \frac{\rho L}{m} = 1.63 \tag{10}$$

allows to describe the problem in terms of non-dimensional quantities. For the current parameters, the characteristic time-scale for the structure is much greater than the fluid time-scale, which implies that the fluid behaviour can be considered as quasi-steady. Moreover, the mass ratio of fluid and structure is of order one, which implies that fluid and structure contribute equally to the dynamics of the system.

The computation is started from initial conditions which are derived from the solution of the linearized problem with an initial structural displacement of $z_0 = 0.01$. The non-dimensional quantities are scaled such that one oscillation period is equal to a unit time, if not stated otherwise.

5.1 Monolithic vs. partitioned schemes

Figs. 2 – 4 show a comparison between the monolithic and the partitioned scheme with trapezoidal time integration. The structural displacement is plotted versus time for five cycles of oscillation. In Fig. 2 the amplitude computed with the partitioned scheme grows in time, whereas the amplitude computed with the monolithic scheme remains constant. When using a structural predictor for the partitioned scheme, the growth in amplitude is substantially reduced so that partitioned and monolithic results visually coincide (Fig. 3 with the same number of time steps per cycle as in Fig. 2). However, if the number of time steps per cycle is decreased,

e.g. by a factor of ten (Fig. 4), the error in amplitude computed with the partitioned scheme emerges again, in spite of using a structural predictor. The amplification of the solution obtained with the partitioned method can be attributed to artificial energy production at the interface, which can induce numerical instability of the combined fluid-structure system; see, e.g., Refs. [2,4]. In practice, this imposes a restriction on the admissible time-step size. The monolithic scheme, on the other hand, does not become numerically unstable. It appears that for a monolithic scheme, there does not exist a restriction on the admissible time-step size other than the required accuracy.

We note that for a single-mode problem such as the one considered, stability and accuracy are closely related. Only for multiple-mode problems stability and accuracy can be distinguished clearly. For partitioned schemes, the stability of modes with time-scales smaller than one actually wants to resolve can impose a severe restriction on the admissible time-step size. On the other hand, for monolithic schemes there is no such stability restriction, but the time-step size is restricted by accuracy considerations only. This discussion is similar to the one about time-step restriction for explicit vs. implicit time integration.

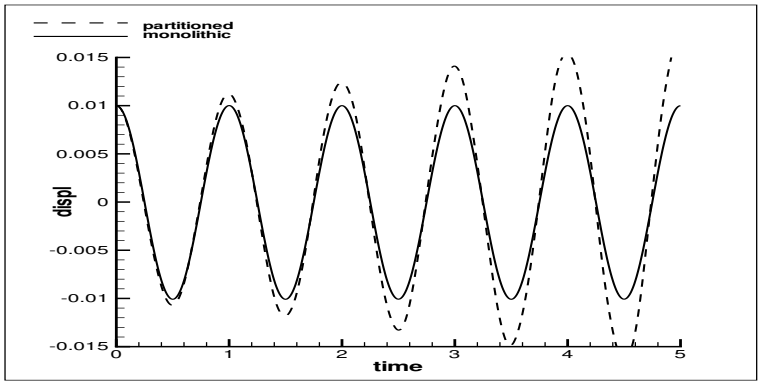

Figure 2. Structural displacement computed with a monolithic and a partitioned scheme without structural prediction, 100 time steps / cycle.

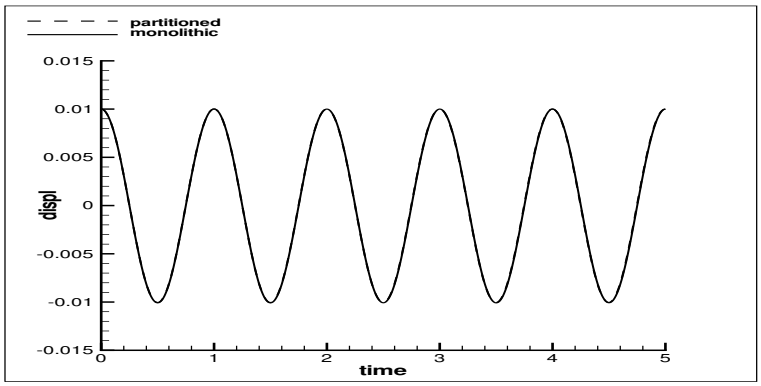

Figure 3. Structural displacement computed with a monolithic and a partitioned scheme with structural prediction, 100 time steps / cycle.

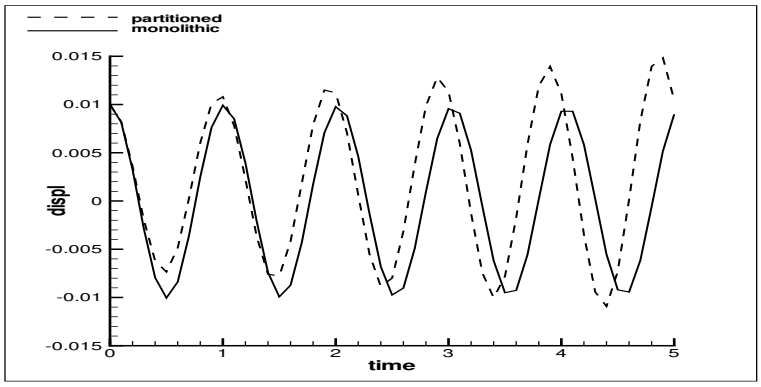

Figure 4. Structural displacement computed with a monolithic and a partitioned scheme with structural prediction, 10 time steps / cycle.

5.2 Mesh refinement study

The order of temporal accuracy of fluid-structure interaction calculations depends on the fluid and structure discretizations separately as well as on their coupling. In the following, the order of temporal accuracy for monolithic and partitioned fluid-structure coupling is investigated. The time-discontinuous Galerkin discretization for the fluid uses linear-in-time shape functions and is therefore second-order time accurate. The trapezoidal method is also second-order time accurate, see Ref. [11]. Therefore the coupled system can be at most second-order time accurate.

We determine the observed order of temporal accuracy by calculating

$$p = \ln\left(\frac{||d_{4\tau} - d_{2\tau}||_2}{||d_{2\tau} - d_{1\tau}||_2}\right) / \ln(2), \tag{11}$$

where p denotes the observed order of temporal accuracy, d the computed structural displacement on meshes of different time-step sizes (denoted by τ, 2τ and 4τ), and the differences are measured in the L_2-norm.

As the fluid equations are formulated and discretized in space-time, refinement of the temporal mesh width requires a corresponding refinement of the spatial mesh width to obtain second-order convergence. If this is not done, the truncation error retains a mixed term, $\Delta t \Delta x$, which will produce only first-order convergence. For the current computations, the non-dimensional spatial and temporal mesh widths in the fluid are taken equal.

The computations were performed on mesh sequences of different time-step sizes and for one period of oscillation equal to $T = 4$. Table 2 shows the observed order of temporal accuracy for monolithic as well as for partitioned fluid-structure coupling with and without structural prediction.

Table 2
Observed order of temporal accuracy for different coupling methods.

mesh sequences with time-step sizes	partitioned without prediction	partitioned with prediction	monolithic
$2^{-4},\ 2^{-5},\ 2^{-6}$	1.0906	1.9941	2.0016
$2^{-5},\ 2^{-6},\ 2^{-7}$	1.0469	1.9972	2.0012
$2^{-6},\ 2^{-7},\ 2^{-8}$	1.0238	1.9986	2.0008
$2^{-7},\ 2^{-8},\ 2^{-9}$	1.0119	1.9993	2.0005
$2^{-8},\ 2^{-9},\ 2^{-10}$	1.0060	1.9997	2.0000

The temporal accuracy shows the expected asymptotic behaviour, i.e., second-order accuracy for the monolithic method and for the partitioned method with prediction, and first-order accuracy for the partitioned method without prediction.

Although the monolithic method and the partitioned method with prediction have formally the same order of accuracy, the error is considerably larger for a partitioned method than for a monolithic method for the same time-step size, as shown in Fig. 5, which plots the error versus the time step for $\tau = 2^{-4} \ldots 2^{-9}$. As a reference solution, we use the solution obtained with the monolithic scheme for a time step $\tau = 2^{-10}$. As the only difference between the curves in Fig. 5 is the number of fluid-structure iterations, it can be concluded that the interface error incurred by partitioning is at least ten times larger than any other sources of error. The figure also indicates that the partitioned method requires smaller time steps than the monolithic method for a specified error tolerance. Conversely, given a certain level of accuracy, a monolithic method can afford larger time steps than a partitioned method.

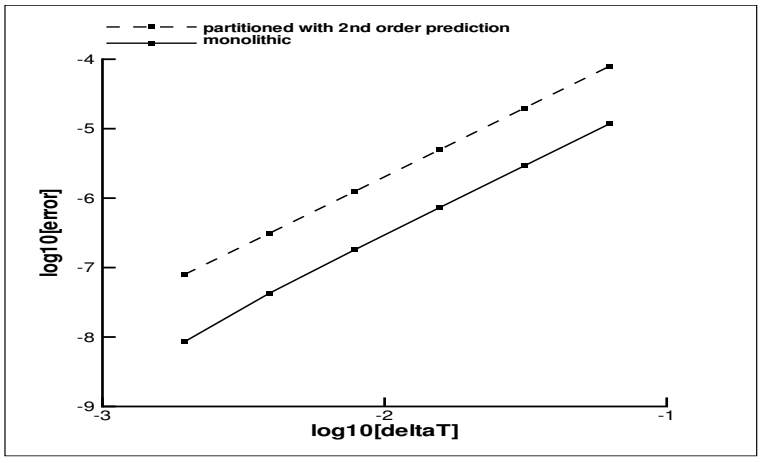

Figure 5. Error for a monolithic and a partitioned method with structural prediction.

5.3 Computational cost of a monolithic solution method

Fig. 6 shows the required number of fluid-structure iterations per time step for a monolithic scheme with and without prediction during one cycle of oscillation. As the area under the curves corresponds to the computational work, it is clear that prediction techniques can reduce the computational cost of the monolithic solution method.

Still, the computational cost of the monolithic procedure is three to four times the one of a partitioned procedure, which requires only a single fluid-structure iteration per time step. On the other hand, the monolithic scheme is more than ten times as accurate, see Fig. 5. Preference of one method over the other depends on whether the increased accuracy also warrants the additional computational cost. To elaborate this statement let us employ the inverse of the error as a measure of accuracy and the number of fluid-structure iterations as a measure of computational cost. The ratio of accuracy to computational cost can be conceived as computational efficiency. As stated above, for a monolithic scheme the number of fluid-structure iterations required is three to four times the one of a partitioned scheme, but the error is also by a factor of about ten lower. Alternatively, if the time-step size of the partitioned scheme is reduced by a factor of four, then its computational cost increases by a factor of four and becomes comparable to the one of a monolithic scheme. For a second-order accurate scheme, the error reduces then by a factor of 16 and, hence, is of the same order of magnitude as for the monolithic scheme. For the considered test case, monolithic and partitioned schemes therefore have comparable computational efficiency. However, for multiple-mode problems we expect that a monolithic scheme exhibits superior efficiency; cf. Sec. 5.1. Moreover, we remark that there is still potential for reducing the computational cost of the monolithic solution procedure.

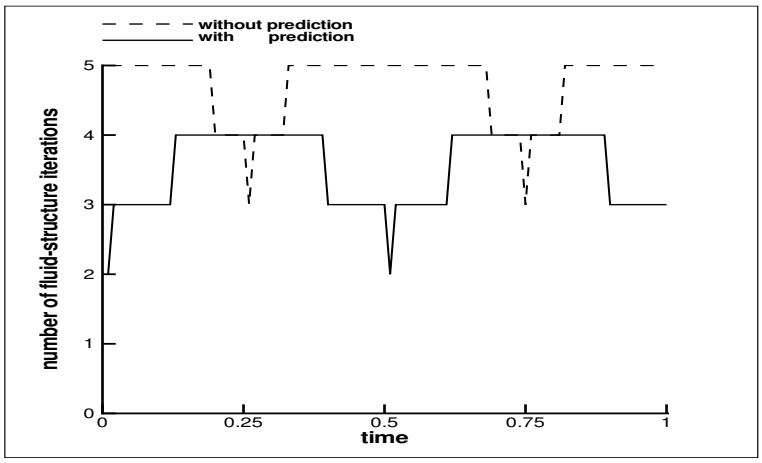

Figure 6. Number of iterations for the monolithic scheme with and without prediction, 100 time steps / cycle.

6 Conclusions

This paper compares partitioned and monolithic solution procedures in terms of stability, accuracy and computational cost. Numerical results were presented for a one-dimensional non-linear fluid-structure interaction problem.

Partitioned schemes require only a single fluid-structure iteration per time step, and therefore their computational cost per time step is lower than for monolithic schemes. On the other hand, the time lag between the integration of the fluid and the structure implies that the interface conditions cannot be satisfied exactly. This induces an algorithmic energy production at the interface, which can cause numerical instability and restricts the admissible time-step size. Although stability and accuracy of partitioned schemes can be improved by means of prediction techniques, their error remains larger than for a monolithic solution method.

In contrast to partitioned schemes, monolithic schemes appear to be unconditionally stable and considerably more accurate. As a consequence, larger time steps can be used than for partitioned schemes for the same level of accuracy. The number of fluid-structure iterations required by a monolithic scheme can be reduced by means of prediction techniques, but monolithic schemes remain computationally more expensive per time step than partitioned schemes. For the considered test case, partitioned and monolithic schemes have comparable computational efficiency. However, for monolithic schemes, there is still potential for reducing their computational cost.

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