Coherence for braided and symmetric pseudomonoids

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Outline

1. Background
   - Result and motivation
   - Full generality from semistrictness
   - Introduction to semistrict bicategories: Gray monoids
   - Quasistrict braiding and symmetry
2. Coherence for pseudomonoids
   - Presentations of pseudomonoids
   - An overview of the proof
   - Proof step 1: Removing unitors
   - Proof step 2: Fixing trees
   - Proof step 3: Showing braid relations

Coherence theorems for braided and symmetric pseudomonoids

- Braided/symmetric *pseudomonoids* are categorifications of commutative monoids (e.g. braided/symmetric monoidal categories in **Cat**, braided/symmetric 2-vector space, etc).
- We prove coherence theorems (biequivalences) for braided and symmetric pseudomonoids.
- Generalises MacLane's theorems [4] — in fact, proves them with string diagrams.
- Categorifies PROs, PROBs, PROPs for monoids and commutative monoids.

Motivation: higher algebra

- This is a first step towards categorification of harder algebraic theories. [1, 7]
- Quantum group theory can be formulated diagramatically [5]. What is e.g. sphericality, modularity, etc. of a Hopf pseudomonoid?
- Coherence results for Frobenius pseudomonoids — TQFT [3], surface foams [2].

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What's new: our results are proved in semistrict braided/symmetric monoidal bicategories

- Quasistrict [9] braided/symmetric monoidal bicategories are simpler than fully weak, but still:

Lemma (Semistrictness of quasistrict braided and symmetric monoidal bicategories)

The free weak braided/symmetric monoidal bicategory $F_W(X)$ on a theory X is (braided/symmetric) monoidally biequivalent to $F_Q(X)$, the free quasistrict braided/monoidal bicategory on that theory.

- The biequivalence identifies certain 1-morphisms and sends corresponding coherence 2-morphisms to the identity.

Semistrictness means our results apply in fully weak monoidal bicategories

- Algebraic theory generated by presentation X.
- An X-algebra in $\mathcal{C}$ is a strict homomorphism $F_W(\Sigma_X) \to \mathcal{C}$.
- Suppose we want to know if two 2-morphisms in $F_W(\Sigma_X)$ are equal.
  - Map along $F_W(\Sigma_X) \xrightarrow{\sim} F_Q(\Sigma_X)$.
  - Biequivalences are faithful on 2-cells, so they are equal iff their images in $F_Q(\Sigma_X)$ are equal.

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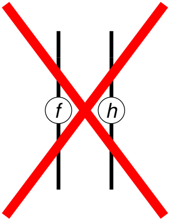
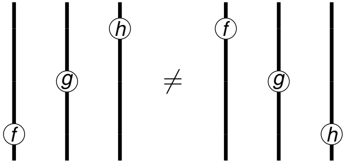
Gray monoids: ordered string diagrams

- To get Gray monoids we weaken planar isotopy of string diagrams in strict 2-categories.
- *Ordered string diagrams* are string diagrams where none of the generating 1-cells occur at the same vertical height. The equivalence relation is *ordered planar isotopy*, where string diagrams must remain ordered throughout the isotopy.



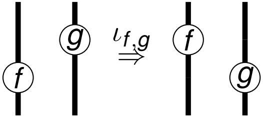


0 and 1-cells in a Gray monoid

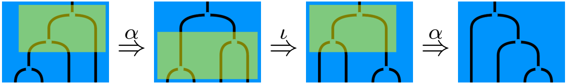
- Generating 0-cells are string labels.
- Generating 1-cells are boxes with specified input and output strings.
- 1-cells in $\mathcal{G}(C_0, C_1, C_2)$ are equivalence classes of ordered string diagrams under ordered planar isotopy.

The interchanger isomorphism

- Instead of equality of 1-cells, we now have an *interchanger* 2-isomorphism to control the isotopy:



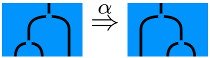
- The interchanger swaps the heights of two vertically adjacent unconnected 1-cells. Example:

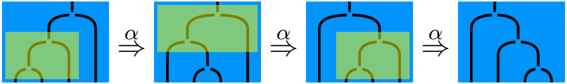


2-cells in a Gray monoid

- Generating 2-cells have ordered string diagrams as source and target.
- 2-cells $D_1 \to D_n$ are sequences $D_1 \overset{\gamma_{1,2}}{\Rightarrow} D_2 \overset{\gamma_{2,3}}{\Rightarrow} \dots \overset{\gamma_{n-1,n}}{\Rightarrow} D_n$, where D_i are 1-cells and $\gamma_{i,i+1}$ are either:
  - A generating 2-cell such that D_i and D_{i+1} differ only by the application of $\gamma_{i,i+1}$ on a rectangular subregion.
  - An *interchanger* 2-cell (next slide).
- If we draw them out, 2-cells are *movies* of ordered string diagrams.
- E.g. the associator:

m : 



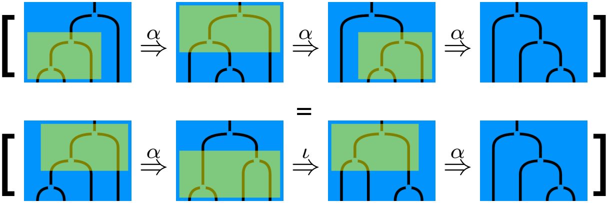


Equalities of 2-cells

- We can also specify certain equalities of 2-cells in our computad, provided they have the same source and target:

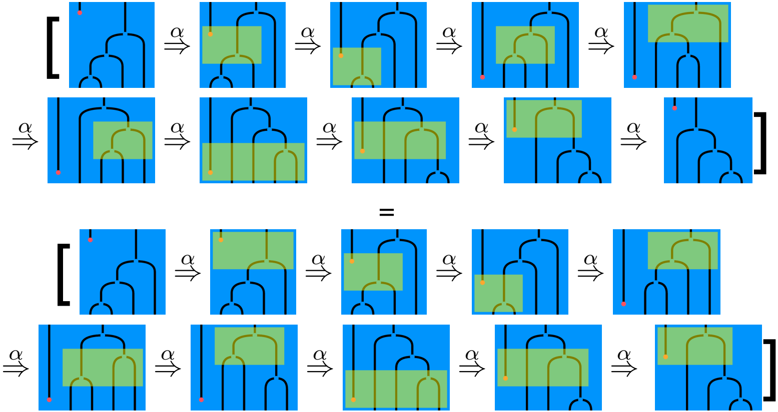
$$[D_s \overset{\gamma_{1,2}}{\Rightarrow} D_2 \overset{\gamma_{2,3}}{\Rightarrow} \ldots \overset{\gamma_{m-1,m}}{\Rightarrow} D_t] = [D_s \overset{\delta_{1,2}}{\Rightarrow} \tilde{D}_2 \overset{\delta_{2,3}}{\Rightarrow} \ldots \overset{\delta_{n-1,n}}{\Rightarrow} D_t]$$

- For example, the 'pentagon':



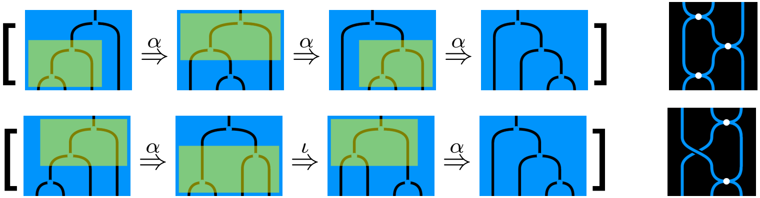
- These generating equalities can be applied to contiguous subsequences on rectangular subregions.

Example: a rewrite of 2-cells in $\mathcal{G}(\mathcal{P})$

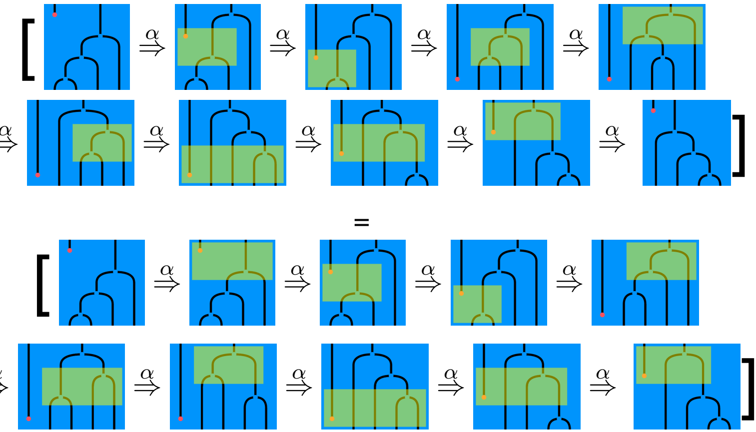


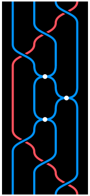
The projection of a 2-morphism

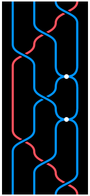
- Viewing a 2-morphism frame-by-frame can be unilluminating.
- At the cost of some information, we can view the whole 2-morphism in one planar diagram.



Revisiting the last example

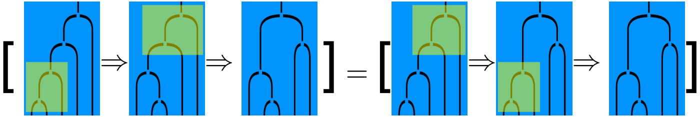




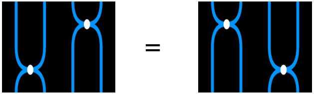


Structural equalities of a Gray monoid: Type I

- If two 2-cells on disjoint rectangular subregions occur one after the other in a movie, we can exchange their order.

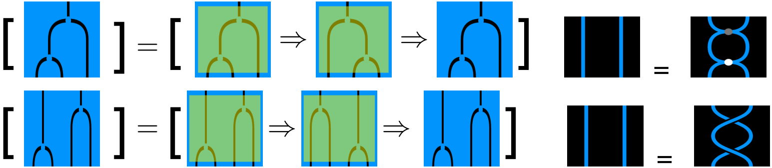


- This corresponds to an interchanger in the projection.



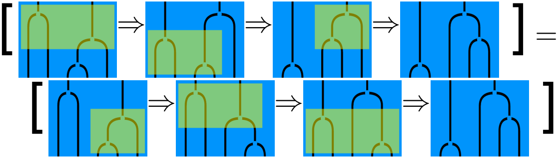
Structural equalities of a Gray monoid: Type II

- We can insert 2-cell and its inverse at any frame containing a rectangular subregion with its source. We call this an *inverse insert*.
- Going the other way is called *cancellation*.

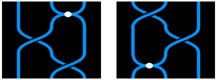


Structural equalities of a Gray monoid: Type III

- Interchanging with all the nodes in a 2-cell, then performing the 2-cell, is equal to performing the 2-cell and then interchanging.



- In the projection, this is a pullthrough:



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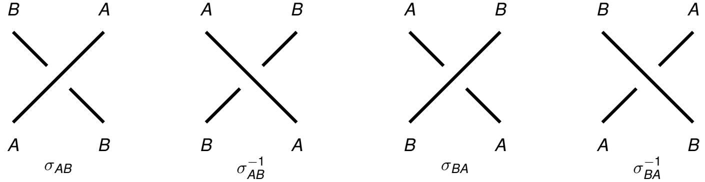
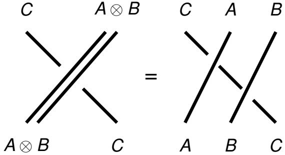
Quasistrict braided structure: I

- A quasistrict braided Gray monoid has an additional four ‘braiding’ 1-cells for every pair A, B of generating 0-cells:

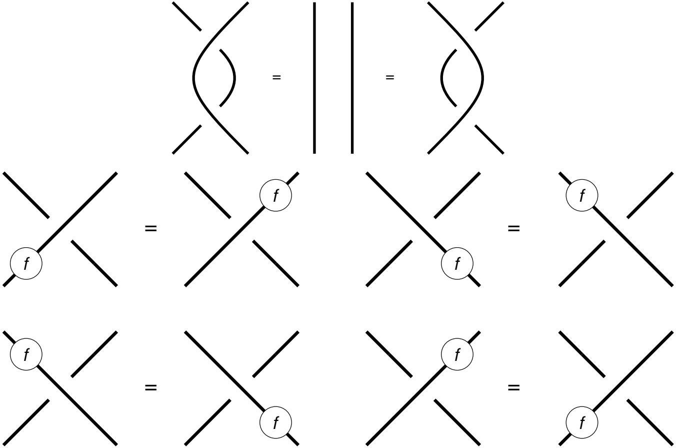


- Braidings on products of generating 0-cells are compositions, e.g $\sigma_{A\otimes B,C} = (\sigma_{A\otimes C} \otimes \mathbb{1}_B) \circ (\mathbb{1}_A \otimes \sigma_{B\otimes C})$:



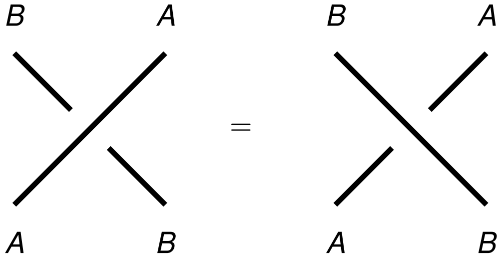
Quasistrict braided structure: II

- Braidings obey the following strict equalities:



Quasistrict symmetric structure

- The symmetric structure [9] is exactly the same as the braided structure, except with the additional equality:



Braided and symmetric monoidal bicategories in Globular

- The braided and symmetric monoidal bicategories in Globular are roughly *Crans* type [3] - weaker than quasistrict.
- In the full paper [10], we use Crans axioms.
- Requires use of simple normalisation routine ('top string normal form') and additional case checking.

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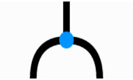
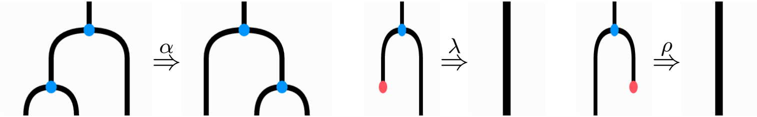
Example: The pseudomonoid computad

- 0-cells: $\{C\}$.
- 1-cells: $m : C \times C \to C$ and $u : I \to C$.

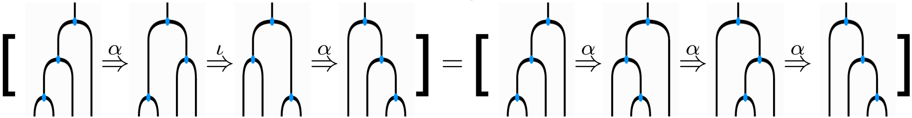
$m =$  $u =$ 

- 2-cells: α (associator), λ (left unitor) and ρ (right unitor), all isomorphisms.

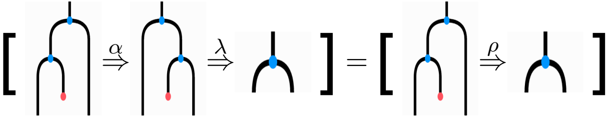


The equalities of the pseudomonoid computad

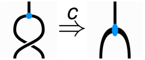
Pentagon



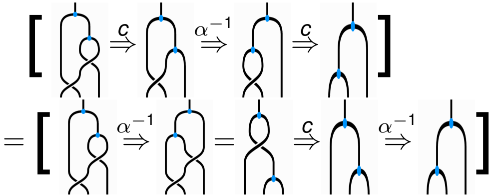
Triangle



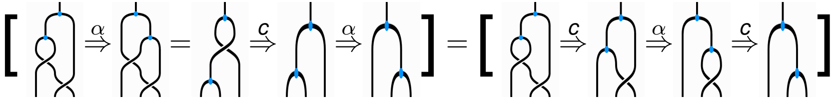
Braided pseudomonoids

- Extra *commutator* 2-isomorphism: 

- Hexagon equality 1:

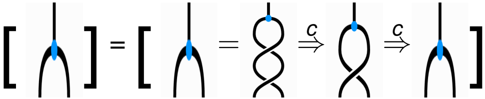


- Hexagon equality 2:



Symmetric pseudomonoids

- Symmetry equality:



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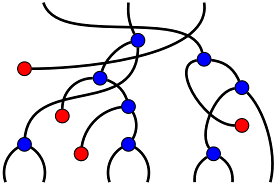
What we will show in this presentation

- In this presentation we consider braided pseudomonoids in symmetric monoidal bicategories.
- In the paper we show a symmetric monoidal biequivalence between the free symmetric Gray monoid on the braided pseudomonoid computad and a certain strict symmetric monoidal bicategory.
- Here we ignore the compositional structure and show two consequences:
  - A ‘normal form’ for 2-morphisms.
  - A solution to the word problem for 2-morphisms.

The PROP for commutative monoids

- In the PROP for commutative monoids, the associator and unitors are strict equalities.
- There is a normal form for 1-cells. Take any 1-cell:



- Pull all the units upwards and eliminate them using the unitor equalities if attached.
- Pull the multiplications up above the braidings...
- Then use commutators and associators to left bracket the trees, with ordered inputs.

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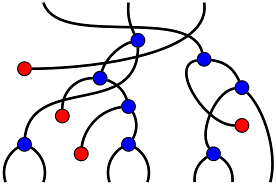

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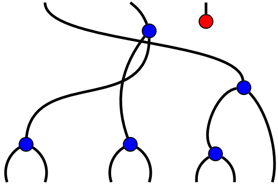

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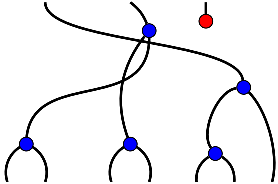

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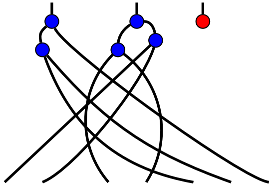

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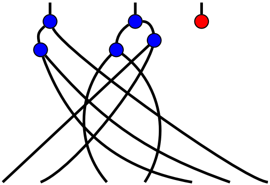

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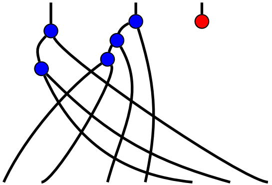

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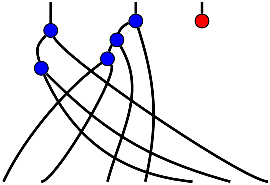
The word problem for the commutative monoid PROP, solved by a normal form

- A 1-cell in this normal form corresponds exactly to a function $\{1, \ldots, m\} \to \{1, \ldots, n\}$, which it is easy to read off [8].



- All the equalities in the commutative monoid PROP preserve this function.
- So this normal form solves the word problem — every 1-cell f is equal to some N_f, and obviously $f = f'$ iff $N_f = N_{f'}$.

Going up to 2 dimensions

- This equality in the 1D case becomes an isomorphism in the 2D case.
- This means that all 1-cells are in the isomorphism class of exactly one 1-cell in this standard form.
- Moreover, two one-cells f, f' are isomorphic iff $N_f = N_{f'}$.
- We want to know when two 2-cells $\alpha, \beta : f \to g$ are equal.

‘Coherence theorem’: normal form for 1-morphisms and ‘word problem’ solution for 2-morphisms

- Everything is an isomorphism.
- So the result for 1-morphisms means that to solve the equality problem for 2-morphisms, we need only solve it for loops on 1-morphisms in normal form:

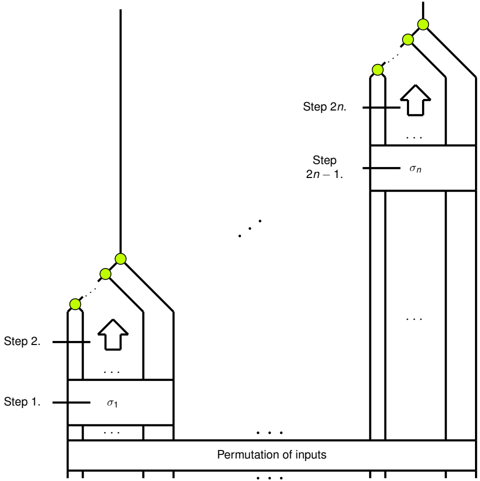
$$
\begin{tikzcd}
N_f \arrow[r, Rightarrow, "\nu"] & f \arrow[r, Rightarrow, bend left=60, "\alpha"] \arrow[r, Rightarrow, bend right=60, "\beta"'] & g
\end{tikzcd}
\qquad \Bigg| \qquad
\begin{tikzcd}
& f \arrow[dr, Rightarrow, "\alpha"] & \\
N_f \arrow[ur, Rightarrow, "\nu"] & & g \arrow[dl, Rightarrow, "\beta^{-1}"] \\
& f \arrow[ul, Rightarrow, "\nu^{-1}"] &
\end{tikzcd}
$$

$$\alpha = \beta \text{ iff } \nu^{-1} \circ \beta^{-1} \circ \alpha \circ \nu = id_{N_f}.$$

A normal form for loops

- To solve the equality problem for 2-loops $\alpha : N_f \to N_f$ on normal form 1-cells N_f, we find a normal form for such loops, such that every $\alpha : N_f \to N_f$ is equal to some unique $N_\alpha : N_f \to N_f$.
- We now specify this normal form.

Normal form for 2-loops on normal form 1-cells

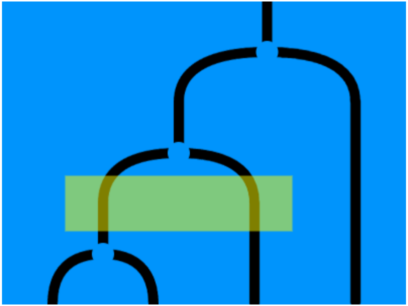


Normal form and equality

- One normal form loop for each choice of product pure braid group element $\sigma_1 \times \cdots \times \sigma_n$ to be absorbed.
- None of the equalities of the braided pseudomonoid (pentagon, triangle, hexagons) change the braid group element absorbed — so none of these normal form loops is equal.

The task of rewriting a given loop

- We are presented with some loop on a normal form 1-cell.
- Our task: rewrite it into normal form using structural equalities, pentagon, triangle and hexagons. For example:



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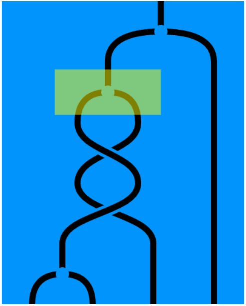

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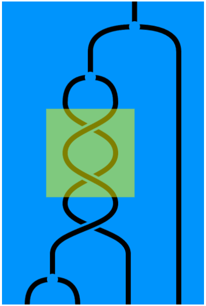

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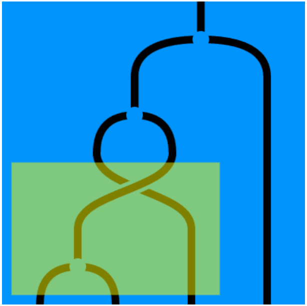

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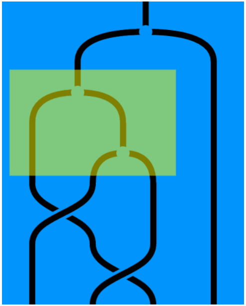

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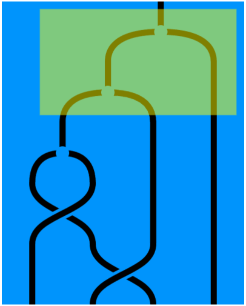

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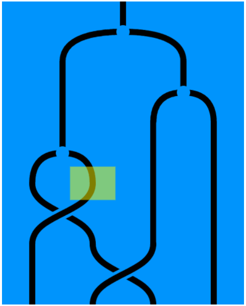

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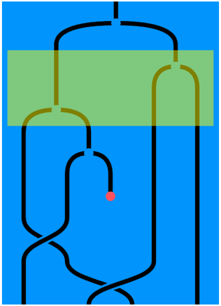

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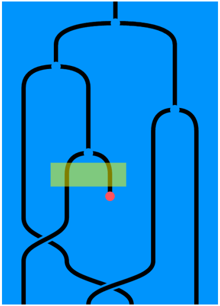

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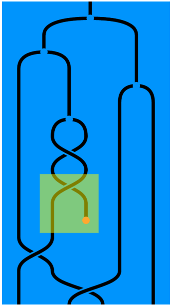

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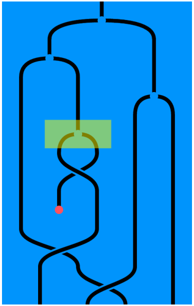

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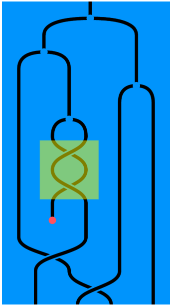

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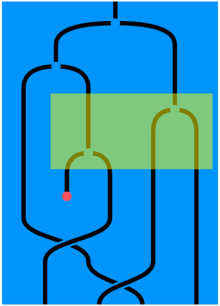

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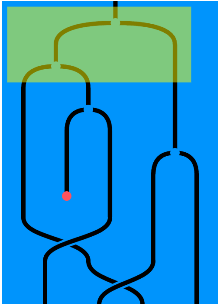

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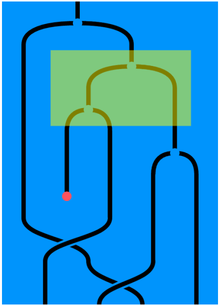

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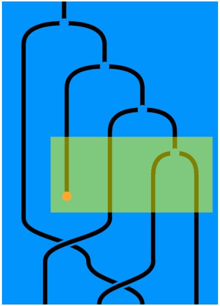

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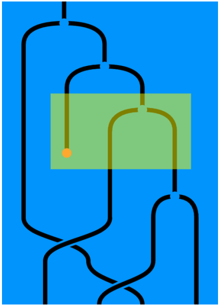

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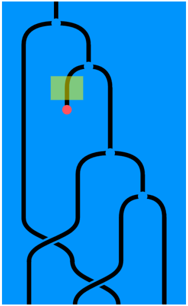

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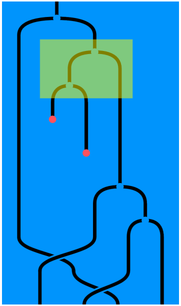

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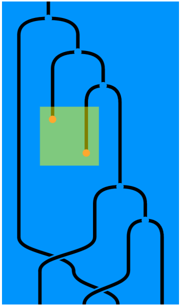

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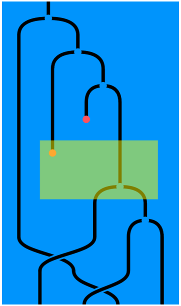

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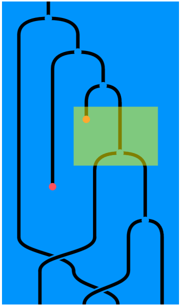

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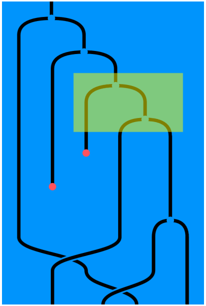

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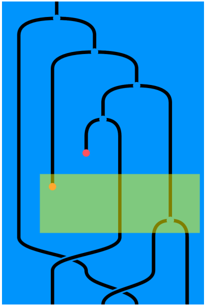

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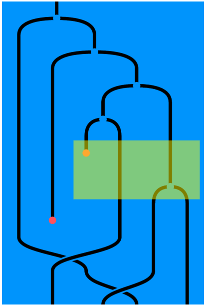

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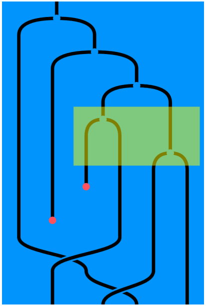

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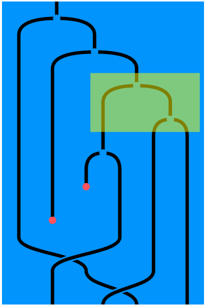

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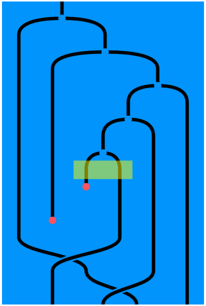

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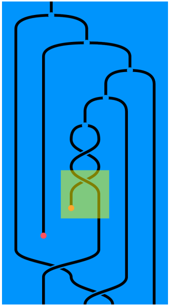

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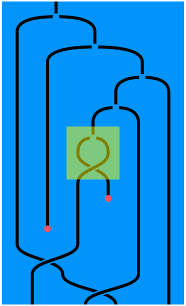

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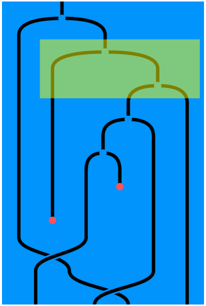

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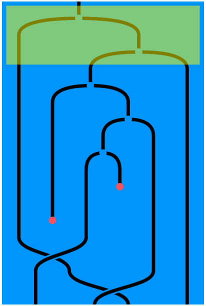

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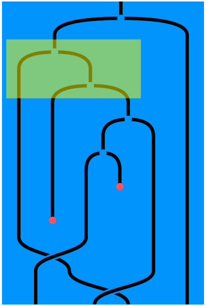

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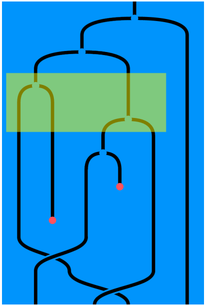

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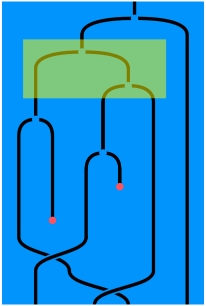

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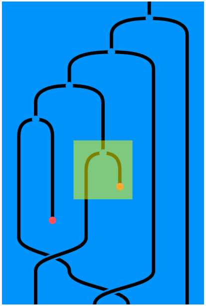

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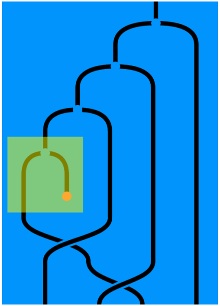

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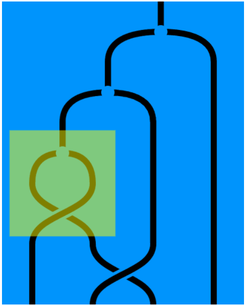

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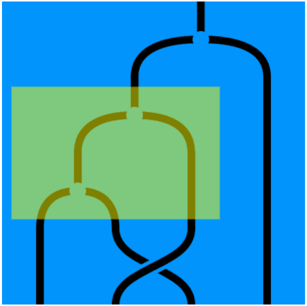

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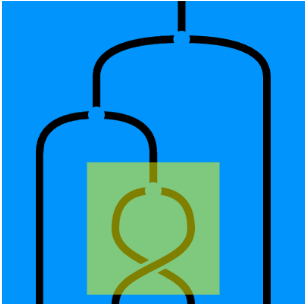

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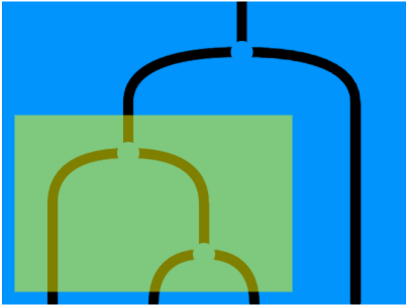

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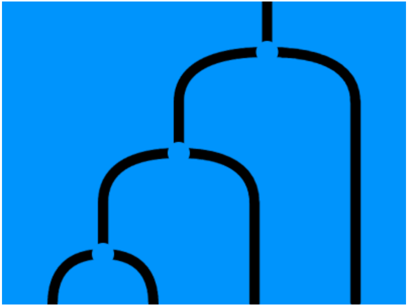

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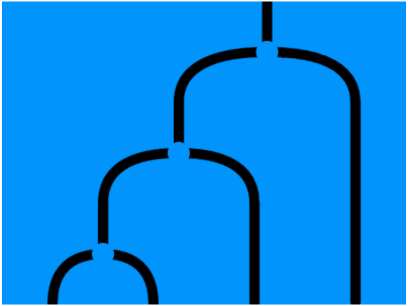


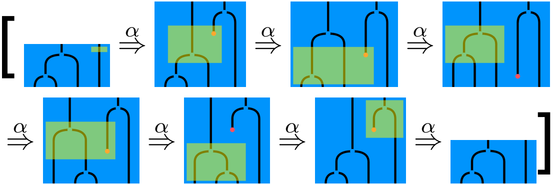
Outline

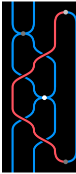
Unitors and inverse unitors cancel

- Notice that there are no attached units on 1-cells in normal form.
- An inverse unitor creates an attached unit.
- The attached unit can't be made unattached using any of the 2-cells.
- Since the movie ends on a 1-cell in normal form, the attached unit must be removed by a unitor.
- So every inverse unitor is paired with a unitor which removes the created unit.
- Idea: move the inverse unitor back to the end of the movie, so that it meets the unitor and so may be cancelled.

Easy case: Part I

- Consider the last inverse unitor in the loop.
- Suppose that no 2-cell affects the created multiplication node throughout the loop, apart from the unitor which removes it.
- We will now show how to cancel the unitor with its inverse.



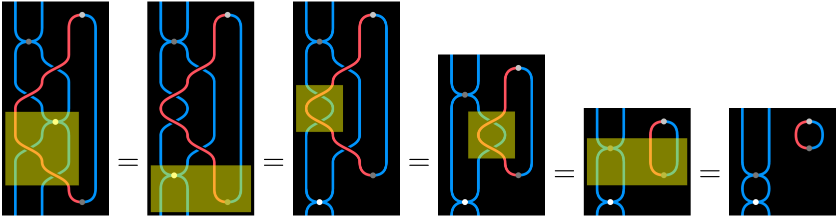


Easy case: Part II

- Because the braiding in the quasistrict category is trivial, the unit and multiplication will form an loop in the projection unentangled with the other morphisms.
- Push the inverse unitor up using Type I rewrites.
- If this becomes impossible, it will be due to a chain of downwards interchangers (it can't be another inverse unitor, since this is the last one).
- Go to the end of the chain of downwards interchangers and push the last one back using Type I rewrites.
- If this is impossible it will be due to an upwards interchanger or a 2-cell acting on the 1-cell with which the unit was interchanged; this may be pulled through.
- Repeating this, we remove the loop.

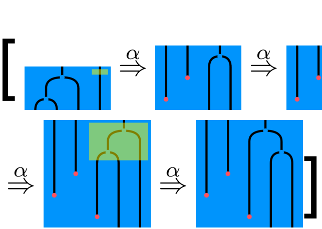
Easy case: Example

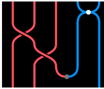
In the projection:



Reducing to the easy case: I

- It may be that there is some 2-cell other than the unitor which acts on the last created multiplication node.



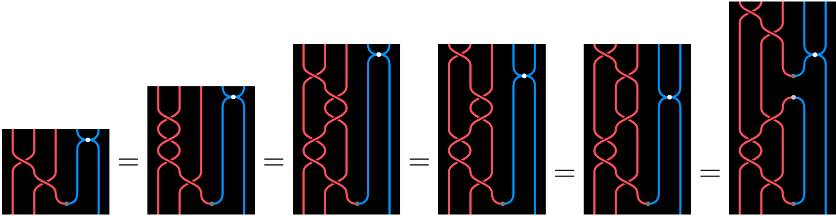


Reducing to the easy case: I

- In this case, use Type II rewrites to insert interchangers and their inverses immediately before the 2-cell so that the unit node goes straight up to the multiplication, returns and then the 2-cell occurs.
- Use Type I rewrites to bring the 2-cell before the pulldowns.
- Insert a unit destruction operator and its inverse immediately before the 2-cell.
- Eliminate the first unit creation operator and the inserted destruction operator (Case I).
- Now use Type I rewrites so that the 2-cell happens immediately after the unit creation.

Reducing to the easy case: I (Example)

- In the projection:

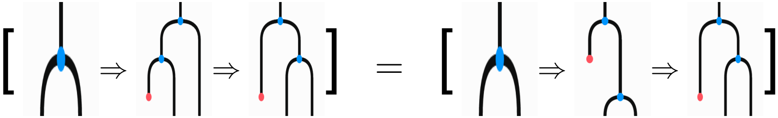


- So now we reduced to the case where the 2-cell happens straight after the inverse unitor.

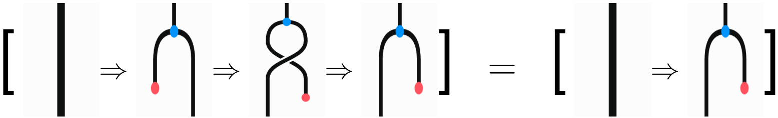
Reducing to the easy case: II

- Now we must consider cases individually.
- We consider each 2-cell that can occur on the multiplication node immediately after the unitor.
- We show that there is always some rewrite that ‘pulls the unitor through’ the 2-cell.
- We show some examples.

Example I: Multiplication node lower partner in associator



Example 2: Commutator acts on multiplication node



Reducing to the easy case: III

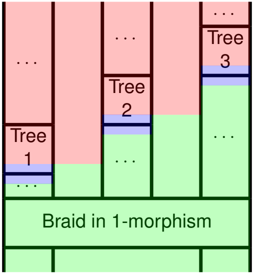
- By using these rewrites for all the 2-cells on the multiplication node, we reduce to the easy case.
- We then cancel the last inverse unitor in the loop.
- Iterating, we remove all the unitors/inverse unitors in the loop.

Outline

1. Background
   - Result and motivation
   - Full generality from semistrictness
   - Introduction to semistrict bicategories: Gray monoids
   - Quasistrict braiding and symmetry
2. Coherence for pseudomonoids
   - Presentations of pseudomonoids
   - An overview of the proof
   - Proof step 1: Removing unitors
   - Proof step 2: Fixing trees
   - Proof step 3: Showing braid relations

Fixing trees: the idea

- The next step is to get the loop into the following form:



- In the red region, only associators occur. In the grey region, only commutators. In the green region, nothing happens (because the braid relations are equalities).
- This uses the pentagon and hexagon equalities.

Outline

Showing the braid relations

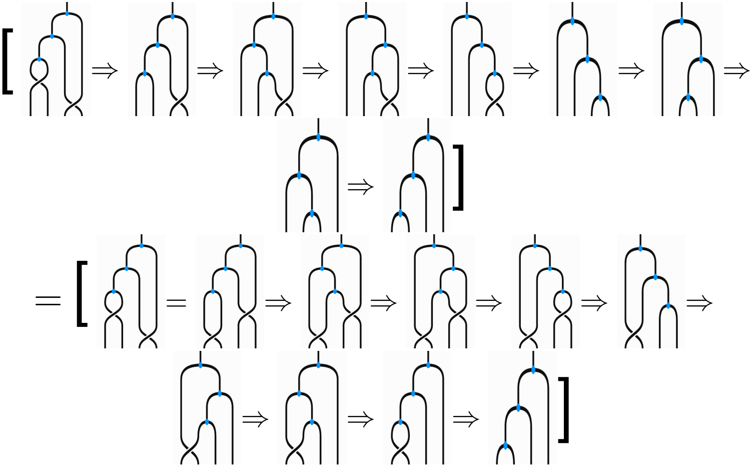
- The movie is in the above form, where some braid is absorbed.
- Recall that our normal form distinguished only between isotopy classes of pure braids absorbed.
- Right now *any* word in the generators of the braid group might be absorbed.
- So, we need to show that the movies can be rewritten using the relations in the braid group.

Two easy properties

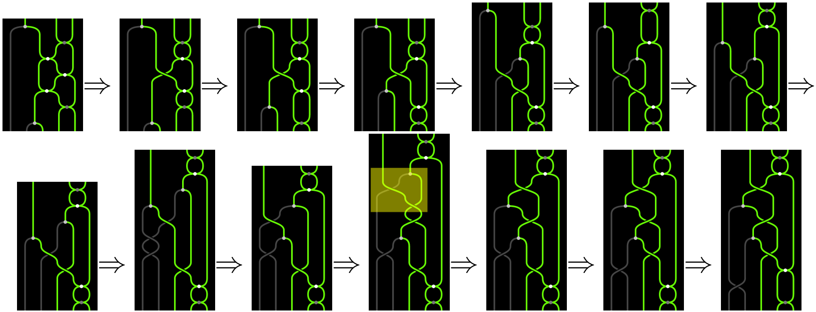
- Associativity of composition is easy, because in the Gray monoid associativity holds strictly.
- For inverses, just perform a cancellation.

Braid relation 1

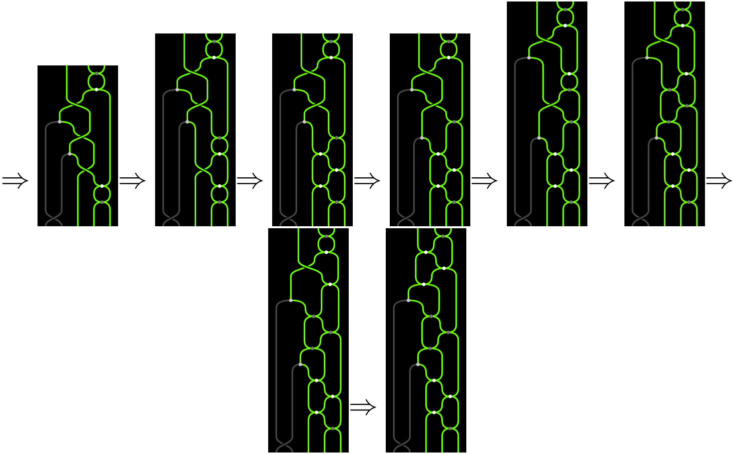
- We need to show that $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$.



Braid relation 1: Proof

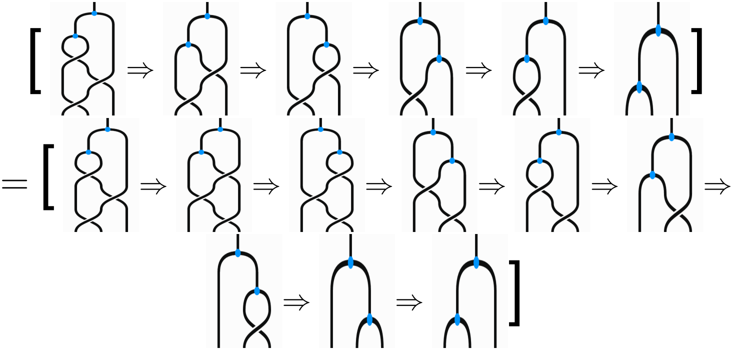


Braid relation 1: Proof

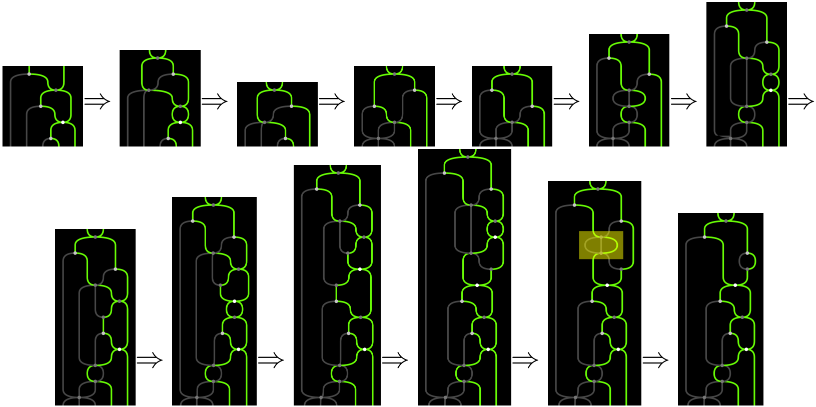


Braid relation 2

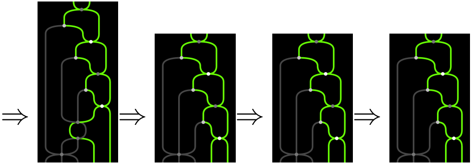
- We need to show that $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.



Braid relation 2: Proof



Braid relation 2: Proof



Conclusion of proof

- So, we rewrote each loop to a normal form loop.
- All the normal form loops are non-equal.
- So this solves the word problem.
- In the paper we use this to build a symmetric monoidal biequivalence with a symmetric monoidal 2-category constructed from the data of normal form loops.

In the paper

- We prove this using weaker Crans axioms.
- We prove biequivalences.
- We prove similar results for all possible combinations:
  - Pseudomonoids in monoidal, braided monoidal and symmetric monoidal bicategories.
  - Braided pseudomonoids in braided monoidal and symmetric monoidal bicategories.
  - Symmetric pseudomonoids in symmetric monoidal bicategories.

Summary

- We can do higher algebra with string diagrams in the weak case.
- Quasistrict bicategories and *Globular* make things a lot easier.
- Outlook
  - Next stop: pseudobialgebras, pseudo-Hopf algebras
  - Can this proof be directed so as to use rewriting techniques [6]?
- Thanks for listening!

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