Probability and Computing Hilary 2017

Exercise Sheet 5

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1. Consider a random walk on a *directed* graph with vertices $\{0, 1, \ldots, n\}$, and directed edges $\{(i, 0), (i, i+1) : i < n\} \cup \{(n, 0), (n, n)\}$. At vertex i each edge is taken with probability $1/d(i)$ where $d(i)$ is the outdegree of i. Find the stationary distribution of the corresponding Markov chain.

2. A cat and a mouse each independently take a random walk on a connected, undirected, non-bipartite graph $G = (V, E)$ with m edges and n vertices. They start at the same time on different nodes, and each makes one transition in each time step. The cat eats the mouse if they are ever at the same node at some time step.

   (a) Suppose that the cat starts at vertex u and the mouse starts at vertex v. Show that there is an even-length walk from u to v of length at most $2n$.

   (b) Show an upper bound of $O(m^2 n)$ on the expected time before the cat eats the mouse. Carefully justify your answer.

3. Let n equidistant points be marked on a circle. Without loss of generality, we think of the points as being labelled clockwise from 0 to $n-1$. Initially, a wolf begins at 0, and there is one sheep at each of the remaining $n-1$ points. The wolf takes a random walk on the circle. At each step it moves with probability $1/2$ to its clockwise neighbour and with probability $1/2$ to its anticlockwise neighbour. At each visit to a point the wolf eats a sheep if it is still there. Which sheep is most likely to be the last eaten?

   [**Hint:** Model the situation as a one-dimensional random walk.]

4. In a connected graph $G = (V, E)$, an edge $\{u, v\}$ is called a *bridge* if removing it disconnects the graph. Considering a random walk on G, show that if (u, v) is a bridge then

$$h_{uv} + h_{vu} = 2|E|\,.$$