1. We say that a function $f(x_1, \ldots, x_n)$ is $c$-Lipschitz if when we change the value of any coordinate causes $f$ to change by at most $c$.

Show that if $f$ is $c$-Lipschitz and $Z_1$ is independent of $Z_2, \ldots, Z_n$

$$|E[f(Z_1, \ldots, Z_n)|Z_1] - E[f(Z_1, \ldots, Z_n)]| \leq c.$$  

Using this, show that if $f$ is $c$-Lipschitz and $Z_i$ is independent of $Z_{i+1}, \ldots, Z_n$ conditioned on $Z_1, \ldots, Z_{i-1}$, then the Doob martingale $X_i = E[f(Z_1, \ldots, Z_n)|Z_1, \ldots, Z_i]$ satisfies $|X_i - X_{i-1}| \leq c$.

2. Let $a = a_1 \ldots a_n$ and $b = b_1 \ldots b_n$ be two binary sequences of length $n$. A longest common subsequence (lcs) of $a$ and $b$ is a subsequence of maximum length common to both; a subsequence is any sequence resulting from keeping only a part (not necessarily contiguous) of a sequence. For example, 001101 is an lcs of 101011010 and 000110111. Suppose now that the symbols of $a$ and $b$ are all chosen independently and uniformly at random from $\{0, 1\}$. Let the random variable $X_n$ denote the length of a lcs of $a$ and $b$.

(a) Use the Azuma-Hoeffding inequality to show that

$$P(|X_n - E[X_n]| \geq \lambda) \leq 2 \exp(-\lambda^2/8n)$$

for any $\lambda > 0$.

(b) How, if at all, does each of the following changes to the problem affect your bound of the first part? Justify your answers rigorously.

i. The symbols of $a$, $b$ are not binary, but are chosen uniformly at random from an alphabet of size $k > 1$.

ii. The symbols of $a$, $b$ are not independent.

iii. There are three strings $a$, $b$, $c$ instead of just two.

Hint: Use the statement of the previous problem.

3. A particle takes a random walk on the line starting at position $i \geq 0$. What is the probability that it reaches 0 before reaching $n$?

4. A particle takes a random walk on the line starting at position 0. Let $X_t$ be its position at time $t$ and let $Y_n = \max_{1 \leq t \leq n} |X_t|$ be its maximum distance from 0 during the first $n$ steps. We have seen that with high probability $Y_n = O(\sqrt{n \ln n})$.

Show the opposite. More precisely show that $\mathbb{E}[Y_n] = \Omega(\sqrt{n})$.

[Hint: Use the analysis of the 2SAT algorithm.]
5. Let $n$ equidistant points be marked on a circle. Without loss of generality, we think of the points as being labelled clockwise from 0 to $n - 1$. Initially, a wolf begins at 0, and there is one sheep at each of the remaining $n - 1$ points. The wolf takes a random walk on the circle. At each step it moves with probability $1/2$ to its clockwise neighbour and with probability $1/2$ to its anticlockwise neighbour. At each visit to a point the wolf eats a sheep if it is still there. Which sheep is most likely to be the last eaten?

[Hint: Model the situation as a one-dimensional random walk and use Problem 3.]