1. Consider the following Markov chain for shuffling $n$ cards. In each step, we pick two cards independently and uniformly at random and exchange their positions.

- Prove that this Markov chain will converge to the uniform distribution (of all permutations) from any initial permutation.
- Prove that $\tau(\epsilon) \leq O(n^2 \log(1/\epsilon))$.

2. We repeatedly toss a fair coin and we use a counter to keep track of the length of the last run of heads. The counter has $m$ bits and when it reaches $n = 2^m - 1$ it stays there as long as heads continue appearing. For example, when $m = 2$, for the sequence 011010011110..., the counter is in states 0,0,1,2,0,1,0,0,1,2,3,3,0,... This defines a Markov chain with states \{0,...,n\} and the current state of the chain is the minimum of $n$ and the length of the last run of heads.

   (a) What is the stationary distribution $\pi$ of the Markov chain?
   (b) What is its mixing time? Hint: The mixing time does not depend on $n$.

Assume now that we reverse the direction of time and we observe the counter (after it had reached the stationary distribution). This defines a new Markov chain, with the same set of states and transition probabilities $\hat{p}_{i,j} = p_{j,i}\pi_j/\pi_i$, where $p$ and $\pi$ are the transition probabilities and stationary distribution of the previous chain.

Answer the above two questions for this Markov chain.

3. A gambler with an initial amount of $k$ units plays a sequence of unfair games: at each time step, the gambler bets a unit and wins it back with probability $p < 1/2$. The gambler quits when he either becomes bankrupt or he reaches a fixed amount $n$.

   (a) What is the probability that he becomes bankrupt?
   (b) What is the expected number of steps?

4. A fair coin is tossed repeatedly. What is the expected number of tosses until the pattern 101 appears for the first time? What about the pattern consisting of $n$ 1's? Describe a way to compute the expected number of tosses before any given pattern $p$ appears for the first time.