Probability and Computing
Hilary Term 2018
Exercise Sheet 6

1. Consider the following Markov chain for shuffling n cards. In each step, we pick two cards independently and uniformly at random and exchange their positions.

   - Prove that this Markov chain will converge to the uniform distribution (of all permutations) from any initial permutation.
   - Prove that $\tau(\epsilon) \leq O(n^2 \log(1/\epsilon))$.

2. We repeatedly toss a fair coin and we use a counter to keep track of the length of the last run of heads. The counter has m bits and when it reaches $n = 2^m - 1$ it stays there as long as heads continue appearing. For example, when $m = 2$, for the sequence 011010011110..., the counter is in states 0,0,1,2,0,1,0,0,1,2,3,3,0,.... This defines a Markov chain with states $\{0, \ldots, n\}$ and the current state of the chain is the minimum of n and the length of the last run of heads.

   (a) What is the stationary distribution π of the Markov chain?

   (b) What is its mixing time? *Hint*: The mixing time does not depend on n.

   Assume now that we reverse the direction of time and we observe the counter (after it had reached the stationary distribution). This defines a new Markov chain, with the same set of states and transition probabilities $\hat{p}_{i,j} = p_{j,i}\pi_j/\pi_i$, where p and π are the transition probabilities and stationary distribution of the previous chain.

   Answer the above two questions for this Markov chain.

3. A gambler with an initial amount of k units plays a sequence of unfair games: at each time step, the gambler bets a unit and wins it back with probability $p < 1/2$. The gambler quits when he either becomes bankrupt or he reaches a fixed amount n.

   (a) What is the probability that he becomes bankrupt?

   (b) What is the expected number of steps?

4. A fair coin is tossed repeatedly. What is the expected number of tosses until the pattern 101 appears for the first time? What about the pattern consisting of n 1's? Describe a way to compute the expected number of tosses before any given pattern p appears for the first time.