Probability and Computing
Hilary Term 2019
Exercise Sheet 5

1. A particle takes a random walk on the line starting at position 0. Let X_t be its position at time t and let $Y_n = \max_{t=1,\ldots,n} |X_t|$ be its maximum distance from 0 during the first n steps. We have seen that with high probability $Y_n = O(\sqrt{n \ln n})$.

   Show the opposite. More precisely show that $\mathbb{E}[Y_n] = \Omega(\sqrt{n})$.

   [**Hint:** Use the analysis of the 2SAT algorithm.]

2. Let n equidistant points be marked on a circle. Without loss of generality, we think of the points as being labelled clockwise from 0 to $n-1$. Initially, a wolf begins at 0, and there is one sheep at each of the remaining $n-1$ points. The wolf takes a random walk on the circle. At each step it moves with probability $1/2$ to its clockwise neighbour and with probability $1/2$ to its anticlockwise neighbour. At each visit to a point the wolf eats a sheep if it is still there. Which sheep is most likely to be the last eaten?

3. Consider a random walk on a *directed* graph with vertices $\{0, 1, \ldots, n\}$, and directed edges $\{(i, 0), (i, i+1) : i < n\} \cup \{(n, 0), (n, n)\}$. At vertex i each edge is taken with probability $1/d(i)$ where $d(i)$ is the outdegree of i. Find the stationary distribution of the corresponding Markov chain.

4. A cat and a mouse each independently take a random walk on a connected, undirected, non-bipartite graph $G = (V, E)$ with m edges and n vertices. They start at the same time on different nodes, and each makes one transition in each time step. The cat eats the mouse if they are ever at the same node at some time step.

   (a) Suppose that the cat starts at vertex u and the mouse starts at vertex v. Show that there is an even-length walk from u to v of length at most $2n$.

   (b) Show an upper bound of $O(m^2 n)$ on the expected time before the cat eats the mouse. Carefully justify your answer.

5. Consider a random walk on an undirected tree $T(V, E)$. Show that for each edge $\{u, v\}$, $h_{uv} = 2k - 1$, where k is the number of nodes of the maximum-size subtree that contains u but not v (i.e., the number of nodes of the connected component of u that results after removing edge $\{u, v\}$). Conclude that for every edge $\{u, v\}$, $h_{uv} + h_{vu} = 2|E|$.

   More generally, for a random walk on a connected graph $G(V, E)$, show that

   $$h_{uv} + h_{vu} = 2|E|\,,$$

   when edge $\{u, v\}$ is a bridge. An edge $\{u, v\}$ is called a *bridge* if removing it disconnects the graph.

6. Suppose that we are given n records, $R_1, R_2, \ldots, R_n$. The records are kept in some order. We use the *move-to-front heuristic*: after accessing R_j we move it to the front of list. Assume that the cost of

accessing the j-th record in the order, including move-to-front, is j. For example, if we access R_1 in the list R_2, R_4, R_3, R_1 the total cost is 4 and the resulting list is R_1, R_2, R_4, R_3.

Suppose that at each step record R_i is accessed with probability $p_i > 0$, where $\sum_{i=1}^{n} p_i = 1$. We can then think of the various orders of the records as states of a Markov chain. Give the stationary distribution of this chain. Assuming that the chain is in its stationary distribution, give an expression for the expected cost per access. Your expression for the expectation should be easily computable in time that is polynomial in n, given the p_i.