Probability and Computing
Hilary Term 2019
Exercise Sheet 6

1. Give an FPRAS for the variant of DNF counting in which we are given a DNF formula and an integer k and we want to estimate the number of satisfying assignments in which exactly k of the variables are true.

2. Let $S_1, \ldots, S_m$ be sets of indices in the range $\{1, \ldots, n\}$. Say that a permutation π of $\{1, \ldots, n\}$ is increasing on S_j if $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_{k_j})$ where $i_1 < i_2 < \cdots < i_{k_j}$ are the elements of S_j.

   Given a fully polynomial-time randomised approximation scheme that takes as input n and sets $S_1, \ldots, S_m$, and approximates the number of permutations that are increasing on at least one of the S_j.

3. Consider the following Markov chain for shuffling n cards. In each step, we pick two cards independently and uniformly at random and exchange their positions.

   - Prove that this Markov chain will converge to the uniform distribution (of all permutations) from any initial permutation.
   - Prove that $\tau(\epsilon) \leq O(n^2 \log(1/\epsilon))$.

4. We repeatedly toss a fair coin and we use a counter to keep track of the length of the last run of heads. The counter has m bits and when it reaches $n = 2^m - 1$ it stays there as long as heads continue appearing. For example, when $m = 2$, for the sequence 011010011110..., the counter is in states 0,0,1,2,0,1,0,0,1,2,3,3,0,.... This defines a Markov chain with states $\{0, \ldots, n\}$ and the current state of the chain is the minimum of n and the length of the last run of heads.

   (a) What is the stationary distribution π of the Markov chain?

   (b) What is its mixing time? *Hint*: The mixing time does not depend on n.

   Assume now that we reverse the direction of time and we observe the counter (after it had reached the stationary distribution). This defines a new Markov chain, with the same set of states and transition probabilities $\hat{p}_{i,j} = p_{j,i}\pi_j/\pi_i$, where p and π are the transition probabilities and stationary distribution of the previous chain.

   Answer the above two questions for this Markov chain.