Introduction to Fractals and Julia Sets

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1 Introduction

1.1 Fractals and a fresh look at the world

Much of classical geometry is concerned with the study smooth and regular forms. Indeed, throughout most our mathematics education, we are taught about 'nice' functions; those which are continuous, those which are differentiable and those which have a simple closed form representation.

This is all well and good; there is much to be said about such functions, but when we look out of the window we do not tend to see the regular and familiar shapes of sines, exponentials and polynomials. Instead, we see intricate detail which seems hard to capture. The boundary of clouds, the profile of mountains, the branches of a tree, the splittng of a river delta and the forking of lightning - all of these phenomena seem to have aspects which lack a classical mathematical description.

Many such shapes, however, have certain properties in common. If you were to look at the edge of a large cloud, for instance, you may be hard pressed to determine its size without additional information for reference; it would look roughly the same whether you were focussing on just part of the profile, or the whole cloud itself. As we shall see, many properties such as types of self-symmetry are present in the examples we will look at. The study of such forms has grown into a branch of mathematics known as *fractal geometry*, which I hope here to provide some motivation for.

In this essay, we will start by looking qualitatively at examples of several objects studied through the centuries which we might now call *fractals*. The properties which these objects all share will motivate several definitions, although we will see that a rigorous definition may be problematic. The remainder of the essay will focus on a particular class of fractals - iterated functions in the complex plane - which we will take the very first steps into understanding. These sets display incredible detail, but are generated from such a simple formulation; they really demonstrate beauty, both aesthetic and mathematical.

2 Fractals Through the Ages

2.1 Weierstrass Function[1]

Possibly the first example of what we now think of as a fractal was described by the German mathematician Karl Weierstrass. He presented a function, to the Prussian Academy of Sciences in 1872, which is continuous everywhere but differentiable nowhere. He originally defined his function

$$f(x) = \sum_{n=1}^{\infty} b^n \cos(a^n x \pi)$$

where a is odd, $b \in [0,1)$ and $ab > 1 + \frac{3}{2}\pi$.

Up until the discovery of this function, it was thought by many mathematicians contemporary to Weierstrass that continuous functions were necessarily differentiable at all but certain points. This idea seems quite intuitive, which is

[1]This subsection uses information from [1, p187]

why its rebuttal was so revolutionary; it kick-started a new branch of analysis relating to such ‘pathological’ functions.

The function is certainly continuous, which can be seen by results from MA244 Analysis III, but we shall not prove non-differentiablity here. Instead, I wish to point out a few qualitative properties. Fig 1 below shows the function on $[-1, 1]$, with a closer look near the origin. We can see that the differentiability is achieved because the function is very ‘spiky’. Looking at the smaller scale section near the origin, the detail appears to remain very fine. Indeed, if we are to believe that it is differentiable nowhere, then the sharp detail would have to remain on arbitrarily small scales. Additionally, we see that the smaller scale blow-up looks broadly similar to the wider plot; we have some form of approximate self-symmetry.

These two properties, a fine structure and some sort of self-symmetry, are seen again in the following examples.

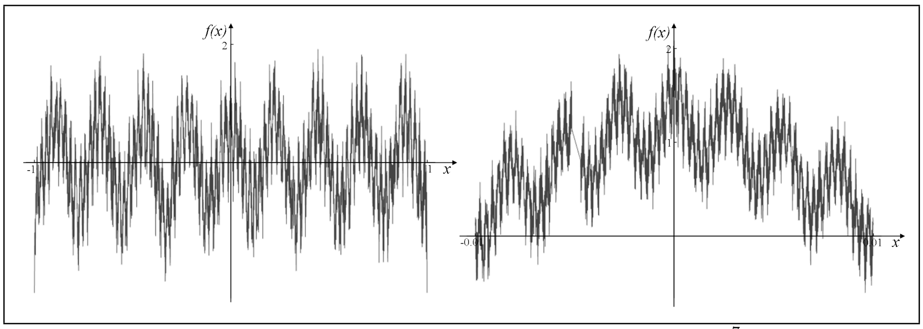

Fig 1: The Weierstrass function with parameters $a = 9$, $b = \frac{7}{10}$

2.2 Cantor Set

Less than a decade after the Weierstrass function first appeared is our next example of a fractal. Published in 1883 by Georg Cantor, another German mathematician, the Cantor Set has a very simple construction but is of interest in many branches of modern mathematics[2, p65].

We construct the *middle third Cantor Set* from the interval $[0, 1] \subset \mathbb{R}$ by means of repeatedly removing certain open intervals. In the first stage, remove the middle third $(\frac{1}{3}, \frac{2}{3})$, leaving $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Call this stage C_1. To proceed to C_2, we remove the middle third from each of the two remaining closed intervals, leaving $[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. In general, we move from step C_k to step C_{k+1} by removing the middle third open interval from each of the 2^k remaining closed intervals.

The Cantor Set C is then those points which remain in C_k for all natural numbers k, or explicitly,

$$C = \bigcap_{i=1}^{\infty} C_i.$$

The length of each closed interval in the k^{th} iteration is 3^{-k}, which tends to zero as k tends to infinity, so a valid question would be whether this removal process leaves anything from $[0, 1]$ at all. Well, some points remain; 0 and 1 for instance are clearly never removed by this process and, in fact, the Cantor Set

C consists entirely of an uncountably infinite number of points[3]. This reveals a few important properties which I wish to highlight.

First, as with the Weierstrass function, we have detail exhibited on arbitrarily small scales.

Secondly, the 'length' of each step C_k is $\left(\frac{2}{3}\right)^k$, so by any usual definition, the limit set C must have length zero. Yet, it is also uncountably infinite, so in one sense is is a big set, and in another, it is very small. We see that the tools of classical analysis let us down when we try and classify the Cantor Set by properties such as 'size'.

Finally, even though C displays an intricately detailed structure, it is actually described in a very straightforward way by means of a simple recursive procedure. It, however, cannot be easily described in a classical sense; it is not the locus of the points that satisfy some simple geometric condition, nor is it the set of solutions of any simple equation[4, p xiv].

These will all be important notions when we come soon to define what we mean by a fractal.

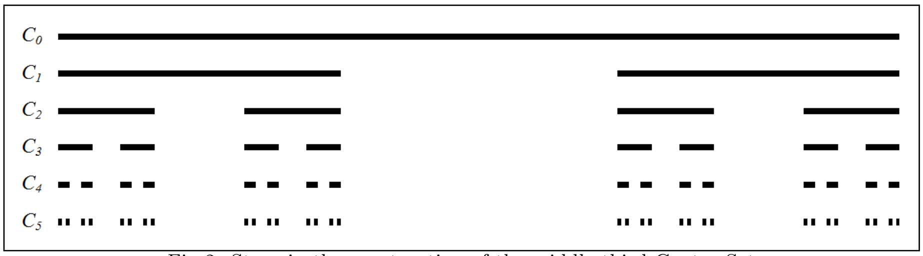

Fig 2: Steps in the construction of the middle third Cantor Set

2.3 The von Koch curve

Our third example of a fractal is the von Koch Curve, the picture of which is probably familiar to readers. In 1904, more than two decades after the Cantor Set first appeared, the Swedish mathematician Helge von Koch published what we now call the von Koch Curve. Its construction is iterative, similar to that of the Cantor Set, and, like the Weierstrass function, is continuous but nowhere differentiable[5, p368].

To construct the curve, we start with a line segment of unit length, V_0, and a generator, V_1. In this case, the generator is the unit line segment with the middle third replaced by all but the base of an equilateral triangle (see Fig 3 below). V_2 is generated by replacing each straight line segment in V_1 with a suitably scaled and rotated copy of the generator.

In general, we move from the iterate V_k to the iterate V_{k+1} by replacing each straight line segment of V_k with a suitably scaled and rotated copy of V_1, the generator. Let us denote by f the function which takes V_k and returns V_{k+1}, so that $f^k(V_0) = V_k$, with f^k representing the k-fold application of f. Then, the von Koch curve, V, is

$$V = \lim_{k\to\infty} \left(f^k(V_0)\right).$$

In the construction, every time a straight line segment is altered by f, it is replaced by four straight line segments, each of length one-third of that removed.

So, V_k, the k^{th} step of the construction has length $\left(\frac{4}{3}\right)^k$. The limit curve V, therefore, has infinite length.

If you take V_0 as the interval $[0,1]$ on the x-axis as a subset of $\mathbb{R}^2$, the area enclosed by V and the x-axis is not infinite; certainly it is bounded above by the area of the rectangle $[0,1] \times [0, \frac{\sqrt{3}}{6}]$.

Thus, we have a curve of infinite length which encloses a finite area. This fact captures, in some sense, that the detail on the curve is very fine indeed. The curve itself does not, however, occupy any area in the plane and so we are once again let down when trying to talk about the 'size' of this curve in a classical sense.

There are also other properties that the von Koch curve has in common with our previous two examples. First, it displays a kind of self symmetry. Taking any part of the curve and blowing it up will give you a picture somewhat like the original. Second, we would struggle to describe it adequately using traditional geometrical language and, finally, we can actually define it in a very simple, albeit recursive, way.

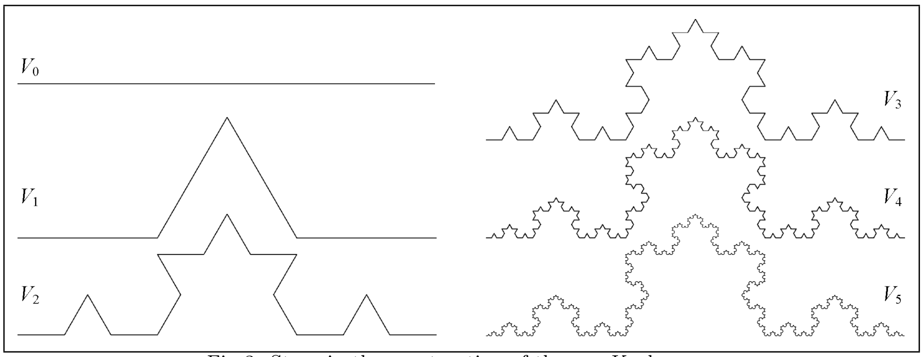

Fig 3: Steps in the construction of the von Koch curve

2.4 Sierpiński Triangle

Our final specific example in this section is a fractal described by the Polish mathematician Wacław Sierpiński in 1915, but is a very recognisable shape with its roots in 13th-century Italian art. It is also called the Sierpiński gasket or sieve and is formed by a simple recursive algorithm[3].

To generate the fractal, we start with a filled equilateral triangle, call it S_0. To move to S_1 we remove the inverted equilateral triangle from S_0 so as to leave the shape with three identical filled equilateral triangles. In general, to move from step S_k to S_{k+1}, we remove an inverted equilateral triangle from each of the 3^k remaining filled triangles, leaving 3^{k+1} such triangles; the first few applications of this process can be seen in Fig 4, below.

To achieve the fractal itself, we take a limiting process with this algorithm. Let f be the function which removes the 3^k inverted triangles at step k such that $f(S_k) = S_{k+1}$ and $f^k(S_0) = S_k$ with, again, f^k being the k-fold application of f. Then the Sierpiński Triangle, S is given by

$$S = \lim_{k \to \infty} \left(f^k(S_0) \right).$$

We can see that at the k^{th} stage, there are 3^k remaining filled triangles. The perimeter of each small triangle is half that of the larger one which generated it in the previous stage. We therefore have that the total fractional perimeter of filled trianges in S_k is $\left(\frac{3}{2}\right)^k$, which tends to infinity with k.

The filled area, on the other hand, decreases at each stage. The inverted triangle removed represents precisely $\frac{1}{4}$ of the larger area, so the fractional filled area of S_k is $\left(\frac{3}{4}\right)^k$, which tends to zero as k goes to infinity.

So, the limiting shape S occupies zero area in the plane, is bounded, but has infinite perimeter. Again, we would find it difficult to categorize the size of this shape using classical geometrical concepts.

As with the other fractals we have seen so far, we see a fine structure exhibited as well as (this time very clear) self symmetry and a geometrically difficult-to-describe object with a very simple recursive formulation.

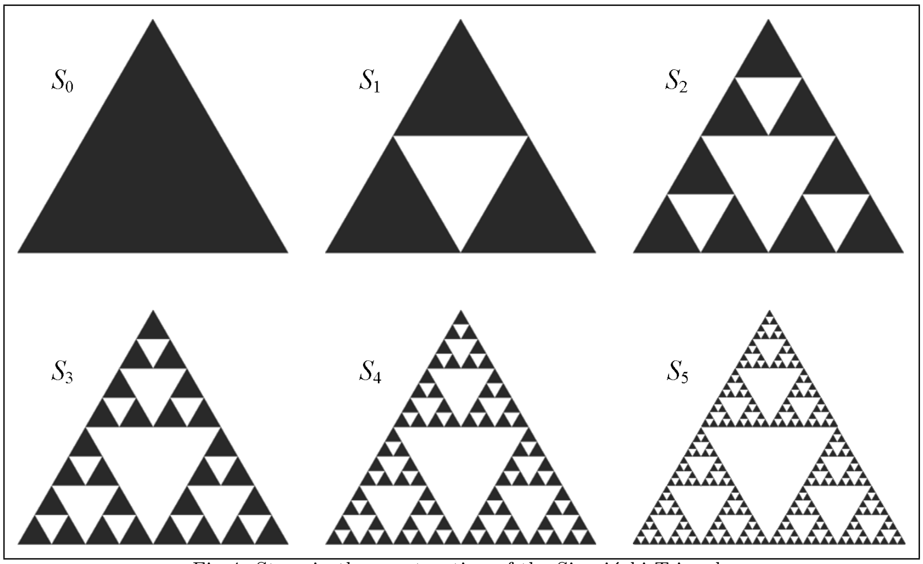

Fig 4: Steps in the construction of the Sierpiński Triangle

2.5 Iterated functions in the complex plane

So far we have now seen four important early examples of what we might now call fractals. A key property that all four have in common is the difficulty in describing them using our classical geometric toolkit. Initially, there was very little investigation into these sorts of irregular sets and functions as they were often regarded as ‘individual curiosities’ and dismissed as objects which did not necessarily have a more general theory associated to them[4, p xiii].

The story changed, however, fairly early in the 20th-Century when the French mathematicians Pierre Fatou and Gaston Julia started investigating iterated functions in the complex plane - in 1917, both Fatou and Julia announced several results regarding ‘the iteration of rational functions of a single complex variable’[6, p1], and it is here, with the work of these two mathematicians, that our investigations into Julia sets begin.

Before then, though, we will try and unpick exactly what we mean by *fractal*.

3 What is a Fractal?

So far we've seen four examples of what we might term fractals, but haven't yet discussed what we mean by the word. For an answer to this question, we look forward several decades from the work of Fatou and Julia to the 1960s, and to the father of Fractal Geometry, Benoît Mandelbrot.

In 1967, Mandelbrot published a famous paper entitled 'How long is the coast of Britain?' He observed that when you measure an intricate geographical shape such as a coastline, the length you get depends on how much detail you take into account when making the measurements - "as even finer features are taken into account, the total measured length increases"[8] - which we see is exactly the problem we met when categorizing the size of the von Koch curve.

Thus, Mandelbrot considered such self-similar shapes to have some constant measure of dimension associated with them which is not necessarily an integer. Such shapes, he said, lie 'between dimensions' and that "in particular, [the dimension] exceeds the value associated with ordinary curves."[8] Mandelbrot himself coined the word 'fractal', from the Latin *fractus* meaning irregular, to describe these shapes with fractional dimension[7].

3.1 Mandelbrot's original definition of a Fractal

Definition 3.1.

> "A fractal is by definition a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension."

This original definition by Mandelbrot[9, p11] requires a few further definitions to make sense of it, however I intend to keep discussion on this brief and informal as it is both fairly technical and bares little interest for the rest of this essay.

3.1.1 Hausdorff dimension[2]

The Hausdorff Dimension, often called the Huasdorff-Besicovitch dimension, is a non-negative real number associated with a set, and is defined in a metric space. The precise definition is not important to us here, so we shall proceed in a rough way.

First, let U be a non-empty subset of a metric space M. We recall that the diameter of U, $|U|$, is given by

$$|U| = \sup\{|x - y| : x, y \in U\}.$$

We consider a subset F of M, and ask how many sets of diameter at most δ we need to form a cover of F. Call this number $N(\delta)$. Then, obviously, as we constrain δ to a smaller value, $N(\delta)$ will increase. We are interested in the value of $N(\delta)$ for small δ.

Taking δ to zero, if $N(\delta)$ increases proportionally with δ^{-d} then we may say that F has dimension d. This is, more or less, the Hausdorff dimension of F, and will certainly suffice for our purposes.

[2]For information and a much more thorough account, see [4, §2].

What this dimension number captures is how intricate a set is; in the case of a bounded smooth line in the plane, one can imagine covering it with balls of radius δ placed boundary to boundary. As δ decreases, the number of balls required in such a cover will simply increase proportionally with the reciprocal of that radius, giving a Hausdorff dimension, according to our rough definition above, of 1.

A intricate shape like the von Koch curve, however, tells a different story. We can imagine that when the radius of covering balls decreases, we will actually need more balls than the simple inverse relationship above, since there is this fine 'wiggly detail' to capture. Thus, intuitively, the Hausdorff dimension is larger than 1.

In fact, the Hausdorff dimension of the von Koch curve[3] is $\frac{\log(4)}{\log(3)} \approx 1.26$. This sits nicely with our intuition; the value lies between 1 (which we would expect for a smooth curve in the plane) and 2 (which we would expect for the subset of the plane in which the curve sits) which captures that the curve is, in a sense, 'big', while still occupying no area in $\mathbb{R}^2$.

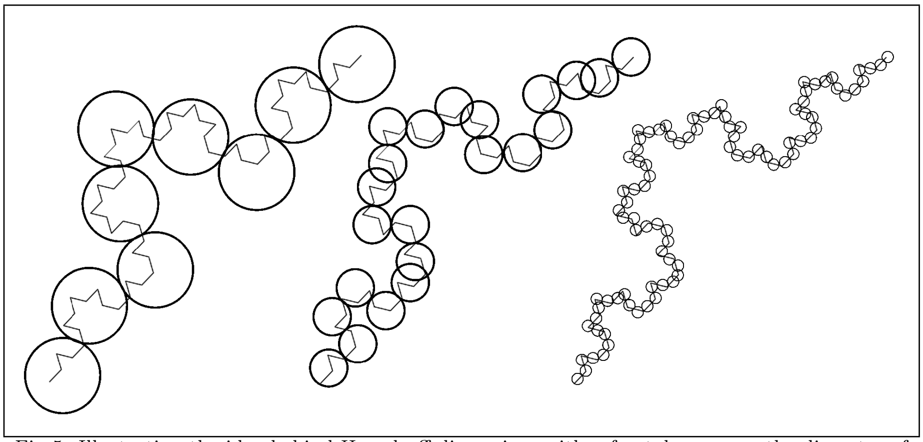

Fig 5: Illustrating the idea behind Hausdorff dimension; with a fractal curve, as the diameter of covering sets tends to 0, we will need more sets than if the curve was smooth

3.1.2 Topological dimension

There are several different notions of Topological dimension which are invariant under homeomorphisms, including the large and small *Inductive dimensions* and the *Lebesgue covering dimension.* All three of these dimensions are, in fact, equivalent in compact metric spaces[10, p7], and we shall not discuss the differences here.

It can be seen directly from the definitions (see, for instance, [10, §1]) that the topological dimension must be an integer, and it can be shown that the topological dimension of $\mathbb{R}^n$ is n. This notion of dimension is, in the case of a Euclidean space, just what we intuitively think of as dimension.

In the case of the von Koch curve, for instance, which is the homeomorphic image of a straight line segment[11], it's topological dimension is 1.

Thus, we see the idea behind Mandelbrot's definition of a fractal; the notion of the Hausdorff dimension captures the intricacy of shapes which is not reflected in the topological dimension.

However, Mandelbrot was eventually not entirely happy with his definition, as he felt that, although correct and precise, it was too restrictive; it excludes, for instance, many fractals useful in physics[9, p11].

3.2 An alternative definition of a Fractal

Throughout the literature I have consulted on fractals, the definition which I feel best to include in this essay is that of Falconer[4, §0] who instead of a rigorous mathematical definition offers a qualitative description in the form of a list of characteristics. He regards a fractal, F as not having any hard and fast definition; rather a set displaying many of the following properties:

- F has a fine structure, i.e. detail on arbitrarily small scales.
- F is too irregular to be described in traditional geometrical language, both locally and globally.
- Often F has some form of self-similarity, perhaps approximate or statistical.
- Usually, the 'fractal dimension' of F (defined in some way) is greater than its topological dimension.
- In most cases of interest, F is defined in a very simple way, perhaps recursively.

With this deliberately woolly definition, which hopefully places the examples of section two nicely into the realm of fractals, we shall depart from our general discussions and continue our journey by delving into the complex plane with Messrs Julia, Fatou and Mandelbrot.

4 Julia Sets

Julia sets provide a vivid example of a simple process giving rise to wonderfully intricate sets. Based in the field of complex analysis, and named after the French mathematician Gaston Julia, Julia sets are born from the iteration of functions in the complex plane. The general theory discussed here follows, in places, [4, §11].

4.1 Definitions

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $n \geq 2$ with complex coefficients,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n, \qquad a_i \in \mathbb{C}, a_n \neq 0.$$

To begin looking at Julia sets, we first require the following:

Definitions 4.1.

(i) Let $f^k(z) = f(f(\ldots(z)\ldots))$ be the k-fold composition of f with itself.

(ii) If $f(w) = w$ for some $w \in \mathbb{C}$, we call w a fixed point of f.

(iii) If $f^p(w) = w$ for some $p \in \mathbb{N}_{\geq 1}, w \in \mathbb{C}$, we call w a periodic point of f.

(iv) If w is a periodic point of f, the least such $p \in \mathbb{N}$ such that $f^p(w) = w$ is called the period of w.

(v) If w is a periodic point with period p, then $\{w, f(w), \ldots, f^{p-1}(w)\}$ is the period p orbit of w.

(vi) If w is a periodic point with period p, and $(f^p)'(w) = \lambda$, we call w a repelling periodic point of f if $|\lambda| > 1$.

(vii) The Julia set $J(f)$ of f may be defined as the closure of the set of repelling periodic points of f.

(viii) The Fatou, or stable, set $F(f) = \mathbb{C} \backslash J(f)$ of f is the complement of the Julia set of f.

Before we start looking at some of these sets, let us recall, from MA222 Metric Spaces, what we mean by the closure of a set. Let T be a general topological space and let H be a subset of T. The closure $\overline{H}$ of a set $H \subset T$ is the set of points $x \in T$ such that every neighbourhood of x meets H. Equivalently, $\overline{H}$ is the smallest closed subset of T which contains H.

Therefore, the Julia set of a function f may be defined as the smallest closed subset of $\mathbb{C}$ which contains the set of all repelling periodic points of f.

4.2 Julia Set for $f(z) = z^2$

In this example, we can calculate explicitly, and directly from the definition, the Julia set for $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$.

Proposition 4.2.
$J(z^2)$ is the unit circle in the complex plane, $\{z \in \mathbb{C} : |z| = 1\}$.

Proof.
First we find all the periodic points of this function:

$$\begin{aligned} f^p(z) = z &\implies z^{2^p} = z \\ &\implies z(z^{2^p-1} - 1) = 0 \end{aligned}$$

So, for periodic points, we have that $z = 0$ or $z^{2^p-1} = 1$. In the latter case, we have roots of unity, so z is of form

$$e^{\frac{2\pi i q}{2^p-1}}, \text{ where } q \in \{0, 1, \dots, 2^p - 2\}.$$

So, all periodic points are either zero, or lie on the unit circle. Which ones, though, are repelling?

In the first case, $z_* = 0$, z_* is a fixed point; a periodic point of period $p = 1$, such that $f(z_*) = z_*$.

$$|f'(z_*)| = 2|z_*| = 0$$

Since $0 < 1$, it is not a repelling periodic point.

Our other cases, with z_* of period p, we have

$$|(f^p)'(z_*)| = 2^p \left|z_*^{2^p-1}\right| = 2^p \left|e^{\frac{2\pi i q}{2^p-1}}\right| = 2^p.$$

Since $p \geq 1$, we have $2^p > 1$, so all such periodic points are repelling. Since they are periodic and repelling for all $p \in \mathbb{N}$, for all $0 \leq q \leq 2^p - 2$, they are dense on the unit circle and so their closure is the unit circle.

Thus, $J(z^2) = \{z \in \mathbb{C} : |z| = 1\}$, as required. □

However, in this case, we can arrive at the same set by means of a slightly different approach. Let us consider the fixed points of f; those points such that $f(z) = z$,

$$\begin{aligned} f(z) = z &\implies z^2 = z \\ &\implies z(z - 1) = 0. \end{aligned}$$

So, we have two fixed points, $z_0 = 0$ and $z_1 = 1$. Let us recall from MA133 Differential Equations that a fixed point x_* of a function f is stable, or attractive, if $|f'(z_*)| < 1$, and is unstable if $|f'(z_*)| > 1$.

Here, we have

$$\begin{aligned} f'(z) &= 2z \\ |f'(z_0)| &= 0 \\ |f'(z_1)| &= 2, \end{aligned}$$

so $z_0 = 0$ is a stable fixed point, while $z_1 = 1$ is unstable. We, therefore, expect points near to 0 to move, under f, closer to 0, while points near 1 will move away from 1.

This is actually easy to see for the function $f(z) = z^2$; taking an arbitrary point in the complex plane $w = Re^{i\theta}$, with $R \in \mathbb{R}$ and $\theta \in [0, 2\pi)$, we have that $w^2 = R^2 e^{2i\theta}$.

Thus, f will map any complex number with modulus less than 1 to a number with smaller modulus and greater argument. Hence, under repeated iteration of f, a general point inside the unit circle will spiral anti-clockwise in towards the stable fixed point at 0.

Any point with modulus greater than 1 will be mapped by f to a point with greater modulus and greater argument, so under repeated iteration, f will map points outside the unit circle away from the origin in an anti-clockwise spiral.

Any points on the unit circle itself, however, will remain with a modulus 1 no matter how many times f is applied, so will stay on the unit circle.

We see, therefore, that in this case the Julia set $J(z^2)$ is the same as the boundary of those points which are attracted to the stable fixed point at 0, which is also the boundary of those points which diverge to infinity (in modulus) under repeated iteration of f.

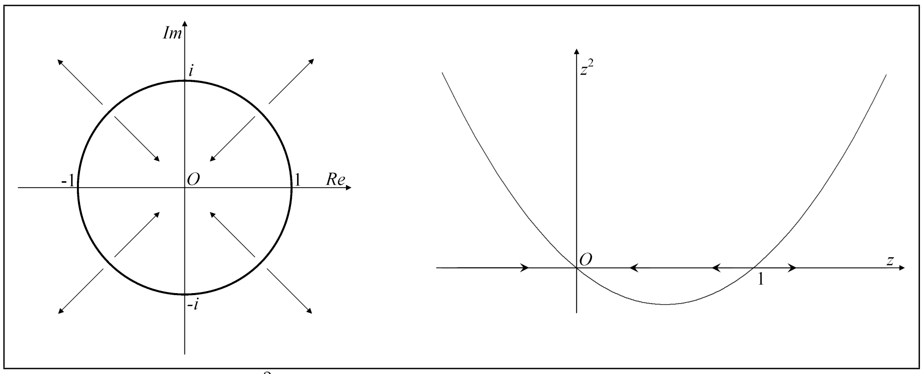

Fig 6: The Julia set $J(z^2)$ (left) and a representation of the stability of fixed points (right)

The question is, does the above generalise beyond this, the most simple of Julia sets?

4.3 An equivalent definition of the Julia set

First, we will generalise slightly the space we are working in. Let $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, or extended complex plane. For the rest of this section, we can assume that all functions f are $f : \mathbb{C}^* \to \mathbb{C}^*$.

When working in $\mathbb{C}^*$, we define $\frac{1}{\infty} = 0$[12, p94], and we say that a fixed point $z_* = \infty$ is stable, or attractive, if $\left|\frac{1}{f'(z_*)}\right| < 1$.

Definition 4.3.
Let w be an attractive fixed point of f. Then

$$A(w) = \{z \in \mathbb{C} : f^k(z) \to w \text{ as } k \to \infty\}$$

is the basin of attraction of w. The basin of attraction of ∞, $A(\infty)$, is defined in the same way.

The following theorem generalises the property which we saw for the Julia set above. We, first, recall from MA222 Metric Spaces that the boundary ∂S of a set $S \subset T$ is the set of points $x \in T$ whose every neighbourhood meets both S and $T \backslash S$.

Theorem 4.4.
Let z_* be an attractive fixed point of f. Then $\partial A(z_*) = J(f)$. The same is true if $z_* = \infty$.

Proof.
The proof of this theorem is beyond the scope of this essay. It is both technical and long, involving work in complex variable theory and relies on Mentel's theorem.

For readers interested in the proof, I recommend similar, but thorough, accounts by Falconer[4, §14] and Helmberg[13, §3]. □

With this new concept, we can continue our investigations with a slightly more interesting Julia set.

4.4 Julia Set for $f(z) = z^2 + c$, $|c|$ small

Let us consider, in general, the Julia set of $f(z) = z^2 + c$ for $c \in \mathbb{C}$ by means of fixed point analysis.

To find the fixed points, we require

$$\begin{aligned} f(z) = z \quad &\Longrightarrow \quad z^2 + c = z \\ &\Longrightarrow \quad z^2 - z + c = 0, \end{aligned}$$

which has solutions

$$z_\pm = \frac{1}{2}\left(1 \pm \sqrt{1-4c}\right).$$

We also have to consider $z_\infty = \infty$, which we notice is also a 'solution' to $f(z) = z$, so we include it as a fixed point.

We have that $f'(z) = 2z$, so to ascertain the stability of each fixed point, we must evaluate $|2z|$ at $z_\pm$, and $\left|\frac{1}{2z}\right|$ at z_∞.

$$\begin{aligned} |f'(z_+)| &= \left|1 + \sqrt{1-4c}\right| \\ |f'(z_-)| &= \left|1 - \sqrt{1-4c}\right| \\ |(f'(z_\infty))^{-1}| &= 0 \end{aligned}$$

When $|c|$ is sufficiently small, it is guaranteed that $\sqrt{1-4c} > 0$ and so our first fixed point will be unstable for such c and our second one stable. What we mean precisely by 'sufficiently small' is discussed in the next section. Our third fixed point, ∞, is stable for all c.

Now we know that there is an attracting fixed point, for small $|c|$, near the origin and the only other attracting fixed point is at ∞. So, we will see two distinct types of behaviour in the complex plane; some starting points will iterate towards z_- and some will diverge to infinity.

Thus, by Theorem 4.4, the Julia set is the boundary of either of these basins of attraction;

$$\begin{aligned} J(z^2+c) &= \partial A(z_-) \\ &= \partial A(z_\infty). \end{aligned}$$

The Julia sets for this behaviour, then, we expect to look something like circles. There will be some set around the point z_-, the boundary of which is $J(f)$. Such shapes we shall call quasi-circles, and we shall look at these again in the next section.

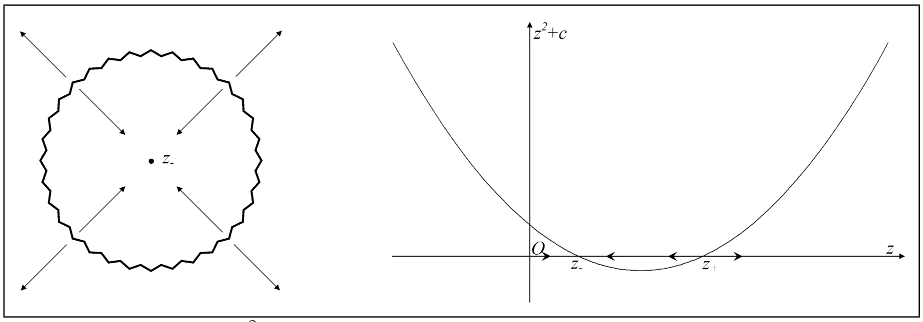

Fig 7: The Julia set $J(z^2+c)$ (left) and a representation of the stability of fixed points (right)

We shall not discuss any general theory of Julia sets here, however much more information is available in both Flaconer and Helmberg. I shall, however, quote several results summarised by Falconer[4, p204]:

> "The Julia set $J(f)$ is the closure of the repelling periodic points of the polynomial f. It is an uncountable compact set containing no isolated points and is invariant under f and f^{-1}."

Instead of proving these properties, we shall move on to look at our final example of a fractal, which is intimately linked to the Julia sets we have just looked at. In general, however, Julia sets are incredibly complex and wonderful shapes. A few examples can be seen in Appendix A, however various excellent and free software is available for rendering these sets; *ChaosPro*, for instance, is an excellent resource for exploring Julia sets visually.

5 The Mandelbrot Set

The Mandelbrot set is based upon specific quadratic functions, iterated in the complex plane. It arises from the study of Julia sets of functions of the form $f_c(z) = z^2 + c$; those which we have briefly studied in the previous section.

5.1 Definition, and a point in M

Let us first recall, from MA222 Metric Spaces, that a set is connected if it cannot be decomposed into two non-trivial open sets with empty intersection. Roughly speaking, a set we are interested in is connected if its figure in the complex plane is not split into more than one part.

Definition 5.1.
We define the Mandelbrot set

$$M = \{c \in \mathbb{C} : J(f_c) \text{ is connected}\}.$$

Certainly, the unit circle $J(z^2)$ is a connected set, and so the point $c = 0$ belongs to M. Therefore, $M \neq \varnothing$.

5.2 Main Cardioid

Looking back to section 4, we decided that certain values of c would have a quasi-circle as their associated Julia set for the function $f(z) = z^2 + c$. Let us now investigate which values of c these are.

The specific behaviour that we are interested in for small $|c|$ is that we have one finite, stable fixed point, near the origin. For which values of c do we get this behaviour?

For a fixed point z_*,

$$f_c(z_*) = z_* \Longrightarrow c = z_* - z_*^2.$$

For z_* to be stable,

$$|f_c'(z_*)| < 1 \Longrightarrow 2|z_*| < 1.$$

On the boundary of this region of stability, the second condition becomes $|z_*| = \frac{1}{2}$. We can then let $z_* = \frac{1}{2}e^{i\theta}$, where θ can be arbitrary, which gives us the boundary

$$c = \frac{1}{2}e^{i\theta}\left(1 - \frac{1}{2}e^{i\theta}\right).$$

Letting θ range from 0 to 2π traces out a cardioid; the interior of which is the main cardioid of the Mandelbrot set. A plot of the Mandelbrot set, along with this cardioid, can be seen in Fig 8.

As already discussed, we would expect Julia sets of $f(z) = z^2 + c$ to be quasi-circles for any c inside the cardioid - however we still do not yet know that such points lie in M at all!

Lemma 5.2.
If c lies within the cardioid $\frac{1}{2}e^{i\theta}\left(1 - \frac{1}{2}e^{i\theta}\right)$, then $J(z^2 + c)$ is connected and, hence, $c \in M$.

Proof.
I shall omit the proof here. For those interested, a proof is given in Falconer[4, pp211-212]. □

It is quite difficult to show that certain Julia Sets are connected, so our definition of M is rather cumbersome. We can, however, derive an equivalent definition which is much more useful for determining whether a parameter c lies in M[4, p205].

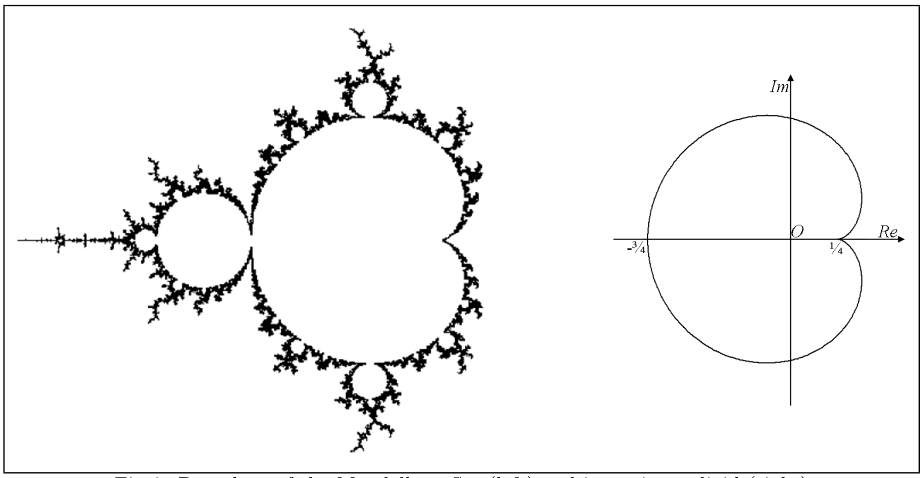

Fig 8: Boundary of the Mandelbrot Set (left) and its main cardioid (right)

5.3 An equivalent definition of the Mandelbrot set[3]

Before we begin, we require a couple of definitions and a lemma, in order to investigate the effect of the transformation f_c on smooth curves.

Definitions 5.3.

(i) A *loop* is a differentiable, closed, simple (non-self-intersecting) curve in the complex plane.

(ii) A *figure of eight* is a smooth closed curve in the complex plane with a single point of self-intersection.

Lemma 5.4.
Let L be a loop in the complex plane.

[3]This section follows the account by Falconer[4, §14.2].

(i) If c is inside L, then $f_c^{-1}(L)$ is a loop, with the inverse image of the interior of L as the interior of $f_c^{-1}(L)$.

(ii) If c lies on L then $f_c^{-1}(L)$ is a figure of eight, such that the inverse image of the interior of L is the interior of the two loops.

Proof.
We shall, again, skip the proof of this lemma, and I direct interested readers to Falconer[4, p206]. □

This lemma, however, gives us the tools required to formulate an alternative definition of the Mandelbrot set.

Theorem 5.5.

$$\begin{aligned} M &= \{c \in \mathbb{C} : \{f_c^k(0)\}_{k\geq 1} \text{ is bounded}\} \\ &= \{c \in \mathbb{C} : f_c^k(0) \nrightarrow \infty \text{ as } k \to \infty\}. \end{aligned}$$

Proof.
To begin, is clear that $f_c^k(0) \nrightarrow \infty$ if and only if $\{f_c^k(0)\}$ is bounded $\forall k \in \mathbb{N}$, so the two sets in the theorem are indeed equal.

First, we shall show that if $\{f_c^k(0)\}$ is bounded, $J(f_c)$ is connected.

Let L be a large circle in $\mathbb{C}$. Because $\{f_c^k(0)\}$ is bounded, we can choose L big enough so that all points $\{f_c^k(0)\}$ lie inside L, and that points outside L iterate to ∞ under f_c^k.

Since $c = f_c(0)$ is inside L, Lemma 5.3(i) tells us that $f_c^{-1}(L)$ is a loop inside the interior of L. Similarly, $f_c(c) = f_c^2(0)$ is in the interior of L, and f_c^{-1} maps everything outside L to everything outside $f_c^{-1}(L)$ and hence, c is inside $f_c^{-1}(L)$. So, by the lemma, $f_c^{-2}(L)$ is a loop contained in the interior of $f_c^{-1}(L)$.

We can proceed in this way, each time adding another loop. $\{f_c^{-k}(L)\}$ is then a sequence of loops, each containing the next in its interior. Let K be the closed set of all points which are on or in the loops $\{f_c^{-k}(L)\}$ for all k.

If $z \in \mathbb{C}\backslash K$ then for some k, $f_c^k(z)$ lies outside L and so $f_c^k(z) \to \infty$. Hence,

$$A(\infty) = \{z \in \mathbb{C} : f_c^k(z) \to \infty \text{ as } k \to \infty\} = \mathbb{C}\backslash K.$$

Then, by Theorem 4.4, $J(f_c)$ is the boundary of $\mathbb{C}\backslash K$, which is just the boundary of K. But, K is the intersection of a sequence of closed, connected sets. By a short topological argument which we shall not prove (but is certainly intuitive in this case), K has connected boundary.

Thus, $J(f_c)$ is connected.

We show the other inclusion, that if $J(f_c)$ is connected then $\{f_c^k(0)\}$ is bounded, by contrapositive. We, therefore, aim to show that $\{f_c^k(0)\}$ being unbounded implies that $J(f_c)$ is disconnected.

We start in the same way as the previous part. Let L be a large circle in $\mathbb{C}$ such that $f_c^{-1}(L)$ is inside L and such that all points outside L iterate to ∞.

This time, however, $\{f_c^k(0)\}$ is unbounded, so let p be such that $f_c^{p-1}(c) = f_c^p(0) \in L$, and with $f_c^k(0)$ either inside or outside L depending on whether k is greater or less than p.

We now construct a series of loops $\{f_c^{-k}(L)\}$ each containing the next. However, when we get to the loop $f_c^{-(p-1)}(L)$, we have $c \in f_c^{-(p-1)}(L)$ and so Lemma 5.3(i) does not apply.

However, let E be the figure of eight $f_c^{-p}(L)$ obtained from Lemma 5.3(ii). E lies inside $f_c^{-(p-1)}(L)$, with f_c mapping the interior of both halves of E onto the interior of $f_c^{-(p-1)}(L)$.

The Julia set $J(f_c)$ must lie in the interior of the loops of E, as other points will diverge to ∞. Since $J(f_c)$ is invariant under the action of f_c^{-1}, we must find parts of it in each loop of E. Therefore, E disconnects $J(f_c)$.

□

This equivalent definition of the Mandelbrot set is very important as it is an easy basis upon which to investigate M with computers[4, p208]. A computer can easily be used to plot the set itself, as well as corresponding Julia sets using our alternative definition in §4.3. Appendix A shows a plot of the boundary of the Mandelbrot Set, along with a few associated Julia sets.

The boundary of M is an incredibly complicated set, and is certainly a fractal by our definition in section §3.2. It has very fine structure indeed, we have no hope of describing it in traditional geometrical language, and even displays remarkable approximate self-symmetry in various locations. Yet, despite all of this magnificent detail, it can be described simply with the iterative quadratic $f_c(z) = z^2 + c$ - truly an extraordinary and beautiful set.

6 Appendix A

We end our investigation into fractals with a few images which try, but inevitably fail, to capture the beauty of these iterated functions.

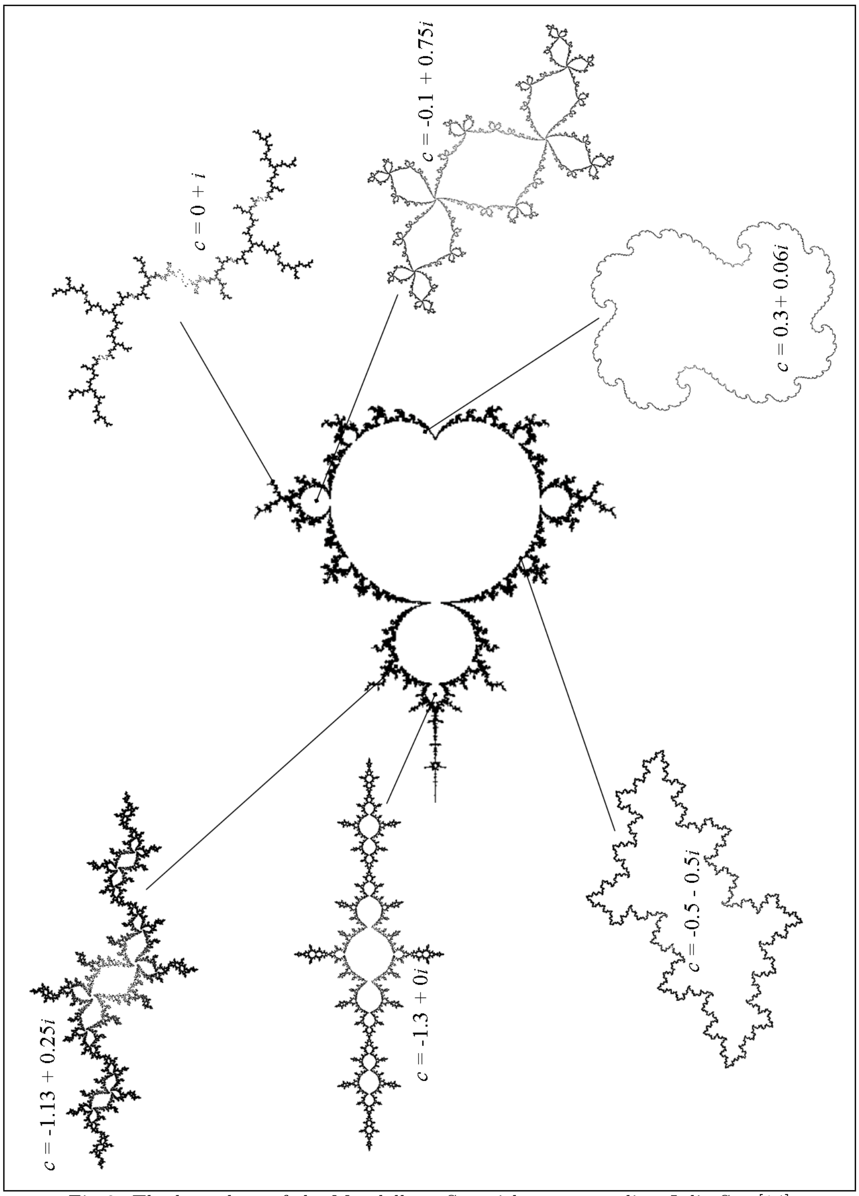

Fig 9: The boundary of the Mandelbrot Set with corresponding Julia Sets[14]

References

[1] *A History of Analysis*. Hans Niels Jahnke. *American Mathematical Society, 2003.*

[2] *Chaos and Fractals: New Frontiers of Science*. H.-O. Pietgen, H. Jürgens & D. Saupe. *Springer-Verlag, 2004*. Second edition.

[3] *Wolfram Mathworld*. http://mathworld.wolfram.com/

[4] *Fractal Geometry: Mathematical Foundations and Applications*. Kenneth Falconer. *John Wiley & Sons, 1990.*

[5] *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences: Vol 1*. Ivor Grattan-Guinness. *The Johns Hopkins University Press, 2003*. New edition.

[6] *A history of complex dynamics: from Schröder to Fatou and Julia*. D S Alexander. *Braunschweig, 1994*.

[7] *Fractals-geometry between dimensions*. Michael Batty. *New Scientist, 4 April 1985*. No 1450.

[8] *How long is the coast of Britain?* Benoît Mandelbrot. *Science, 1985*. 156.

[9] *Fractals*. Jens Feder. *Plenum, 1988.*

[10] *Dimensions, Embeddings, and Attractors*. James C. Robinson. *Cambridge University Press, 2010.*

[11] *The Koch Curve: A Geometric Proof*. James C. Robinson. *Cambridge University Press, 2010*. Šime Ungar. *The American Mathematical Monthly, Jan 2007*. Volume 114.

[12] *A First Course in Complex Analysis with Applications*. Dennis G. Zill. *Jones and Bartlett Publishers Inc, 2009*. 2nd edition.

[13] *Getting acquainted with fractals*. Gilbert Helmberg. *Walter de Gruyter & Co, 2007*. Illustrated edition.

[14] *Julia set images from http://www.markboyd.me.uk/julia/*. Mark Boyd.