(Automatically) verifying Morse reductions in Cubical Agda

Proglog meeting
Chalmers University, 13 March 2024

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Background

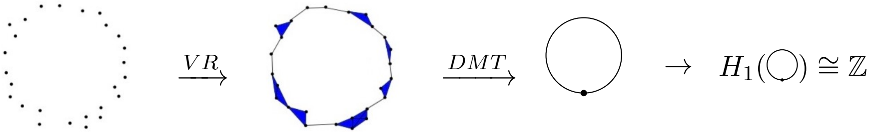
Project: Formalisation in applied topology, *using* and *automating* Cubical Agda's logic for homotopy types.

This talk:

- Overview of results.
- Convey "style" of proofs carried out.
- Outlook on improvements that'd be nice for CTT tools.

Topological data analysis

Study the *shape of data* using tools from algebraic topology:



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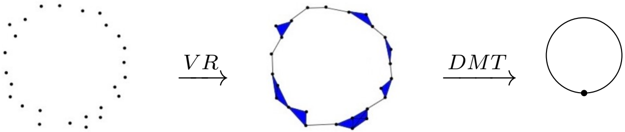
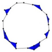


$\rightarrow \quad H_1(\bigcirc) \cong \mathbb{Z}$

Discrete Morse theory (DMT):
Reduce size of complex while retaining its topology.

$\rightarrow$ *establishes that*  $\simeq$ 

Discrete Morse theory for graphs

Input: Graph G and acyclic partial matching (APM) μ
Output: Equivalent Morse complex M of critical cells of G

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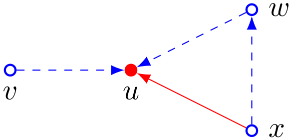

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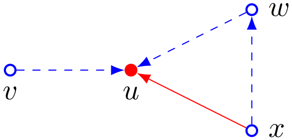

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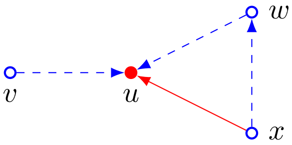

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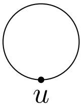

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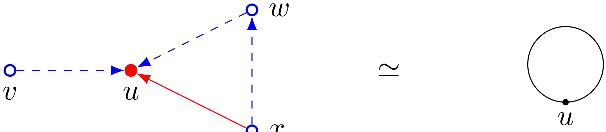

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The Morse theorem in Cubical Agda

Use HIT to introduce homotopy type of a relation:

```
data |_| {c₀ : Type} (c₁ : c₀ → c₀ → Type) : Type where
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$$|\_\Rightarrow\_\ni\_|_1 : (x\ y : c_0) \to c_1\ x\ y \to |\ x\ |_0 \equiv |\ y\ |_0$$

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Central result: $|\ G\ |\ \equiv\ |\ M\ |$

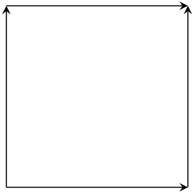
- establish maps back and forth, e.g., define $|\ M_1\ | \to |\ E\ |$: $|\ c \Rightarrow d \ni (uv, \gamma, \delta)\ |_1 \mapsto |\ \gamma\ |^{-1} \bullet\bullet\ |\ u \Rightarrow v \ni uv\ |_1 \bullet\bullet\ |\ \delta\ |$
- show that these maps are mutually inverse.

https://cs.ox.ac.uk/people/maximilian.dore/thesis/html/Morse.Morse.html

Proving the Morse Theorem

E.g., for any critical edge $\mid u \Rightarrow v \ni uv \mid_1$ show that going along $\mid E \mid \to \mid M_1 \mid \to \mid E \mid$ is coherent:

$\mid u \Rightarrow v \ni uv \mid_1$



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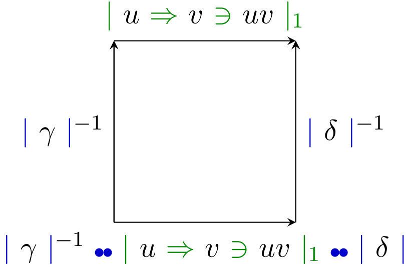

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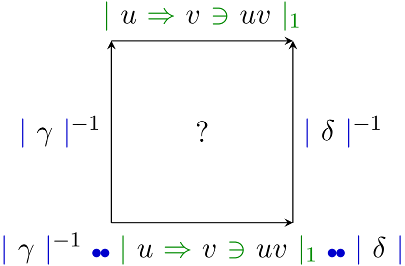

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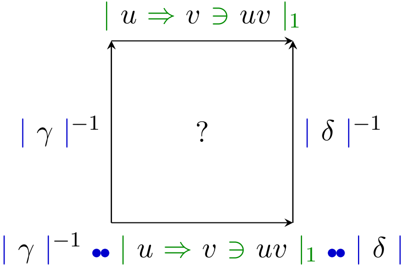

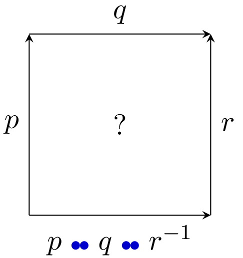

Automating Boundary Filling in Cubical Agda[1]

Search problem: Given cell context Γ and boundary $\Gamma \vdash \phi$, give a cell $\Gamma \vdash t : \phi$ constructed from

- **Contortions**: interval substitutions with $\wedge$, $\vee$, $\sim$.
- **Kan compositions**: gives completion of open cube with ϕ on missing side.

[1] jww with Evan Cavallo & Anders Mörtberg, arXiv:2402.12169

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$$x \overset{?}{\dashrightarrow} z$$

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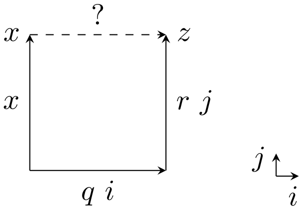

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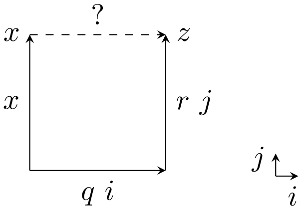

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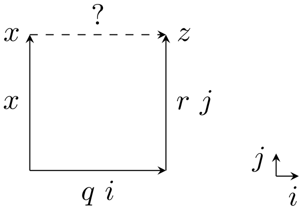

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Find Kan compositions using finite domain constraint solving.

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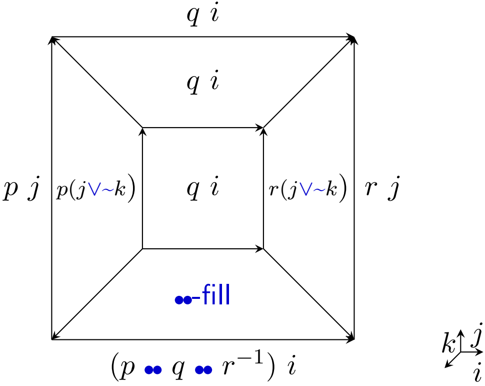
Higher-dimensional Kan composition



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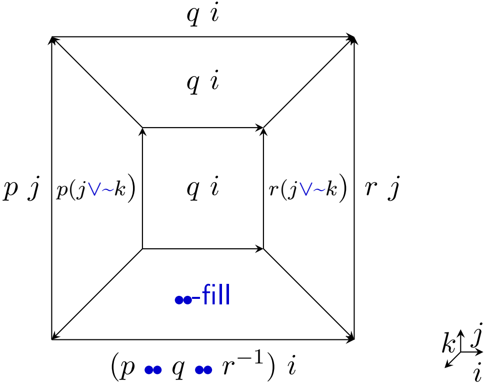

Two issues:

- With growing dimension, there are *a lot* of contortions.
- Sides of the cube can also be the result of Kan composition.

Dedekind contortions

Contortions built only with $\wedge$,$\vee$ can also be seen as poset maps:

$$\text{Dedekind contortions built from } \Psi$$
$$\leftrightsquigarrow$$
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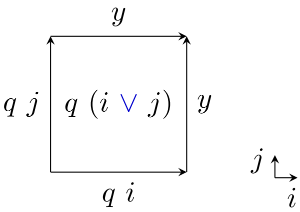

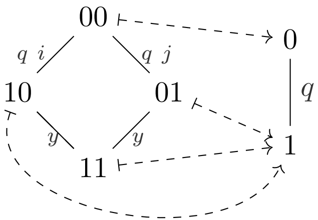

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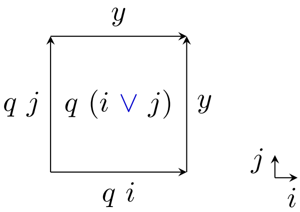
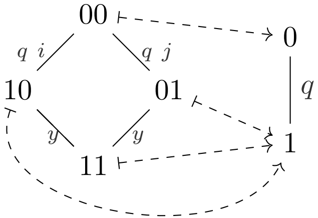
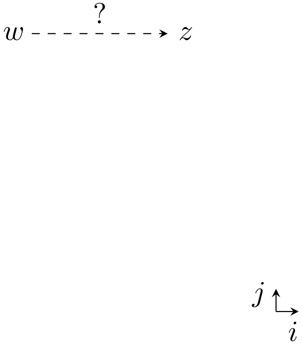




Construct contortions gradually by restricting poset map.

Searching for nested Kan compositions

$$p : w \equiv x,\ q : x \equiv y,\ r : y \equiv z \ \vdash\ ? : w \equiv z$$



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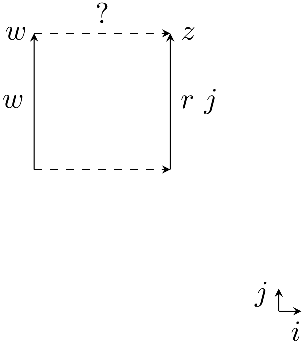

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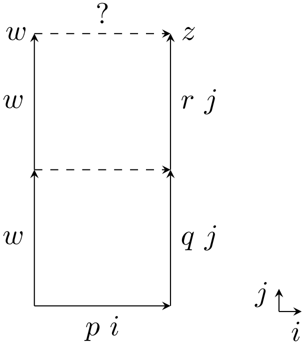

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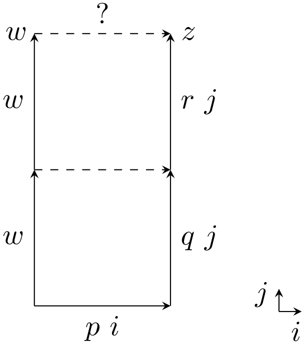


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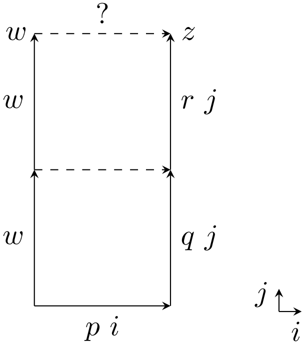


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Use as many contortions as possible, fill the remaining sides with Kan compositions afterwards.

Demo

`https://github.com/maxdore/dedekind`

Summary

- Applied topology is a natural application for Cubical Agda.
- Being able to directly reason about homotopy types is neat, difficult combinatorial steps can be carried out by solver.

Outlook

Developing a tool for TDA fully in Cubical Agda:

- Formalise DMT for 2-dimensional complexes
  $\to$ compute topology of grayscale images[2]
- Implement full pipeline: turn grayscale image into complex; compute APM; compute cohomology of reduced complex.

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Thank you for your attention!

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