Automating Reasoning in Cubical Type Theory

Prospects of Formal Mathematics
Trimester Program at HIM Bonn

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Reasoning with Kan compositions

In cubical type theory, equalities over a type A are captured with paths, i.e., maps from $\mathbf{I}^n$ into A. Fix A in the following.

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$$y \overset{?}{\dashrightarrow} x$$

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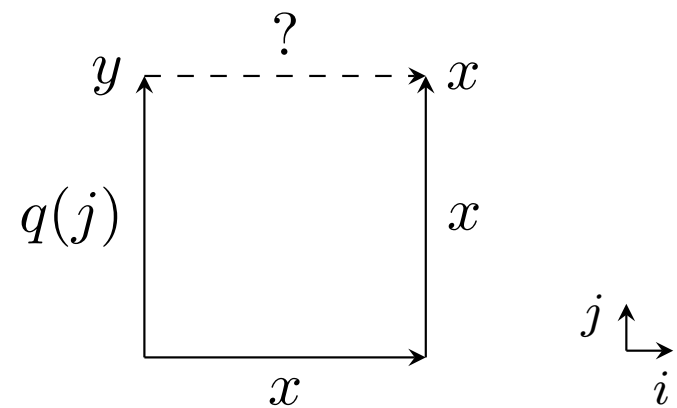

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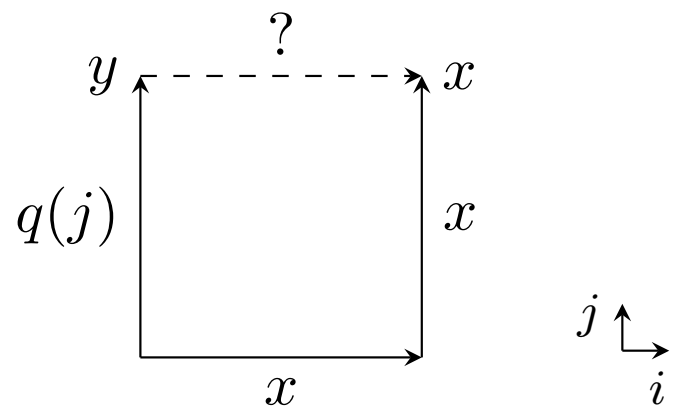

Solution: $? = \mathsf{hcomp}\ (j.[i{=}0 \mapsto q(j) \mid i{=}1 \mapsto x])\ x$

Finding open cubes with constraint solving

$$q(i) : [i{=}0 \mapsto x \mid i{=}1 \mapsto y], r(i) : [i{=}0 \mapsto y \mid i{=}1 \mapsto z] \mid$$
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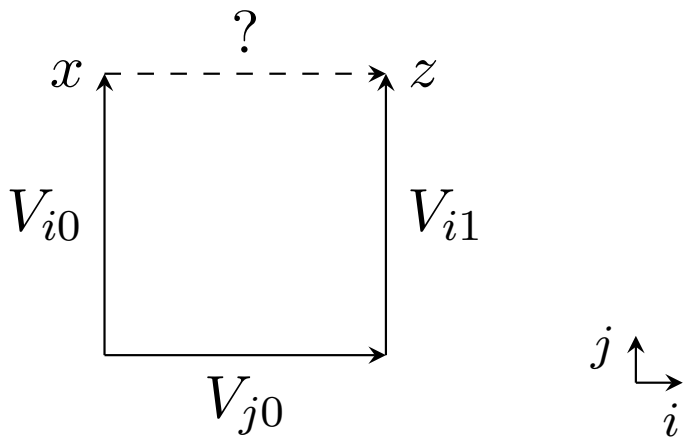

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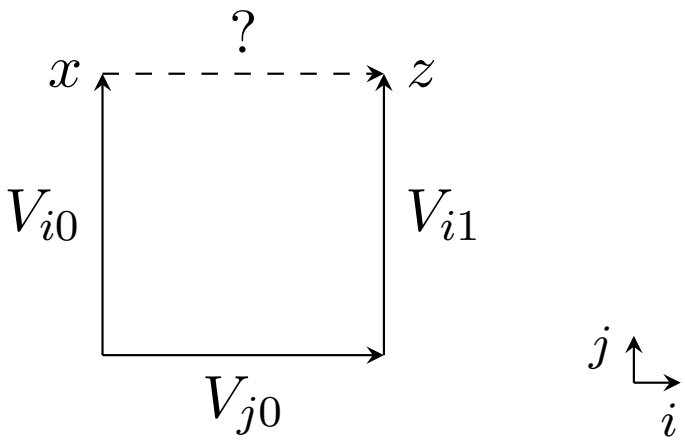

Domains for variables

- $D_{i0} = \{x\}$
- $D_{i1} = \{z, r(j)\}$
- $D_{j0} = \{x, y, z, q(i), r(i)\}$

subject to constraints

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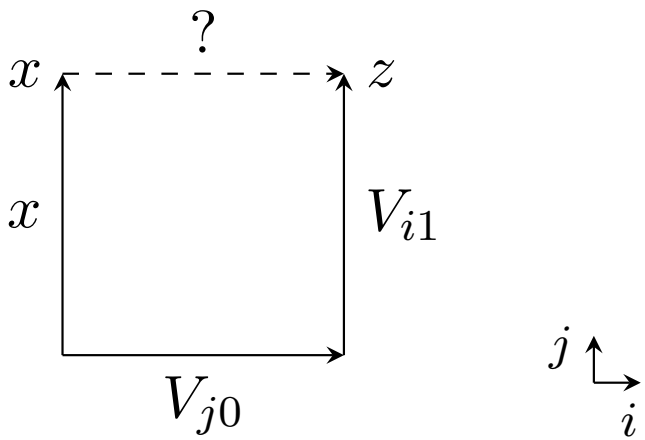

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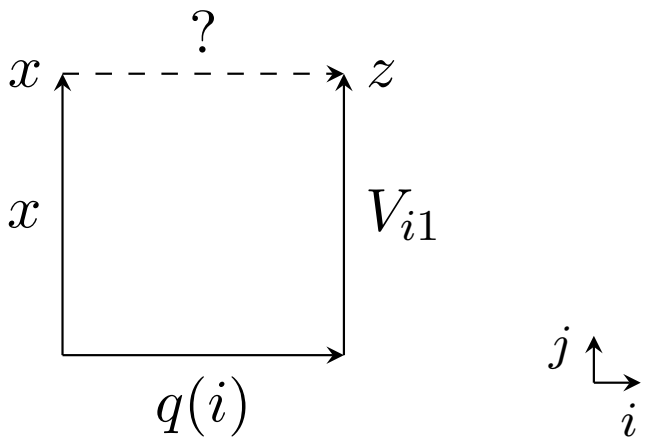

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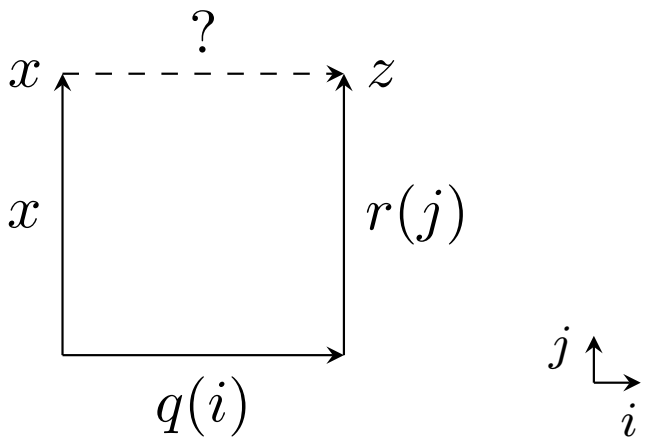

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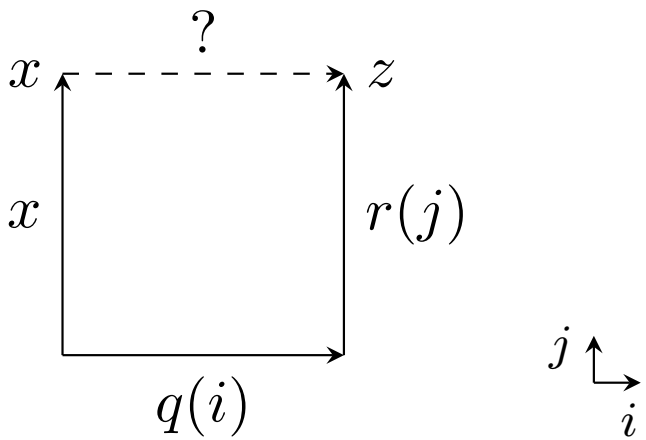

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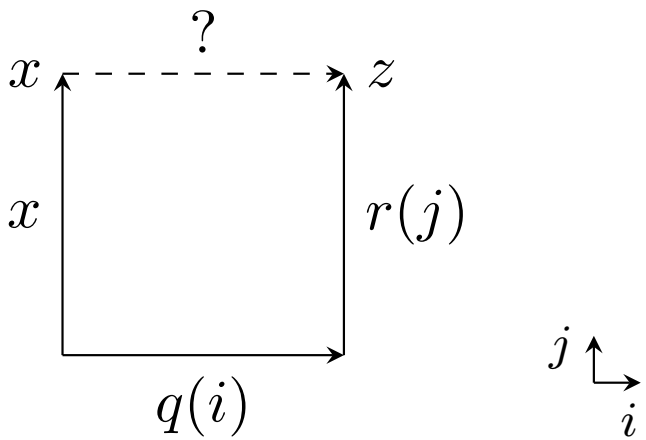

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Find Kan compositions using finite domain constraint solving.

Searching for nested Kan compositions

$p(i) : [i{=}0 \mapsto w \mid i{=}1 \mapsto x], q(i) : [i{=}0 \mapsto x \mid i{=}1 \mapsto y], r(i) : [i{=}0 \mapsto y \mid i{=}1 \mapsto z]$
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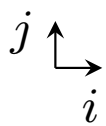

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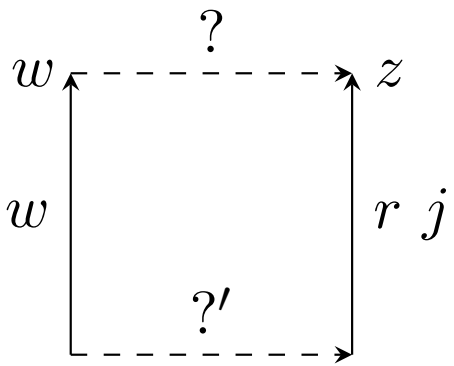

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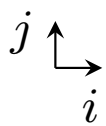

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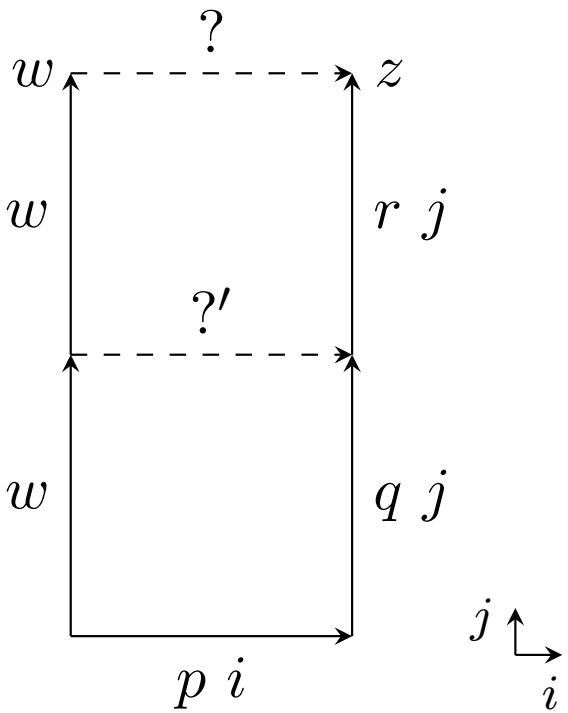

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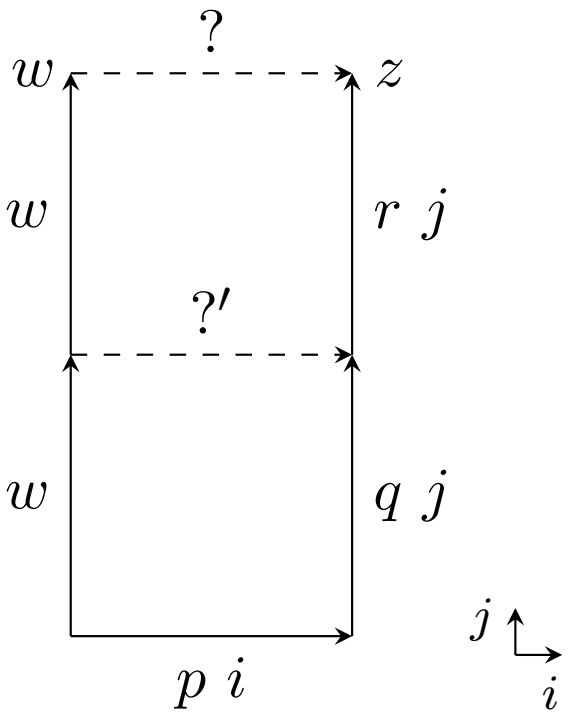

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Use as many simple terms as possible, fill the remaining sides with Kan compositions afterwards.

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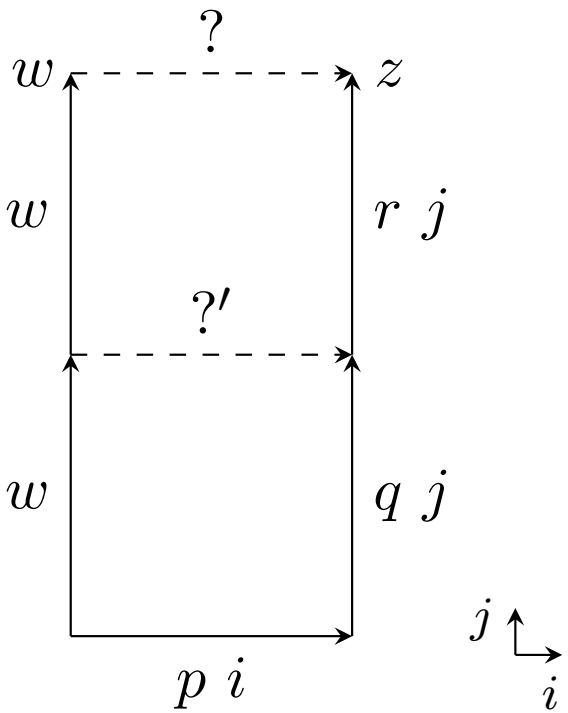

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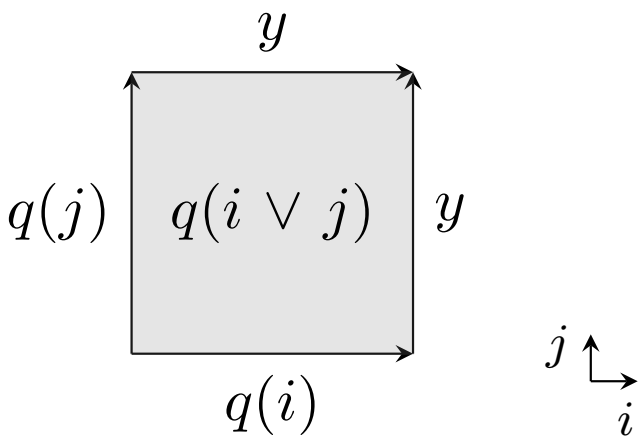
Constructing cubes with contortions

CTTs often come with operations on the interval variables.

$$i, j \vdash q(i \vee j) : [i{=}0 \mapsto q(j) \mid i{=}1 \mapsto y \mid j{=}0 \mapsto q(i) \mid j{=}1 \mapsto y]$$



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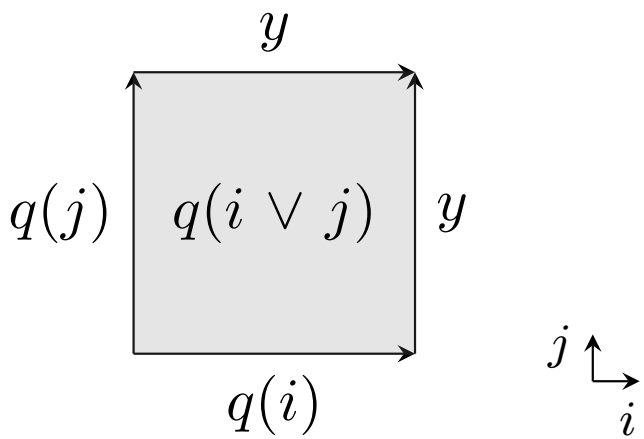

Trade-off: contortions makes large class of Kan compositions superfluous, but makes search for contorted cubes non-trivial—*Dedekind* contortions $\wedge$ and $\vee$ yield $D(n)$-th ways to turn 1-cube into an n-cube (e.g., $D(6) = 7\,828\,354$)

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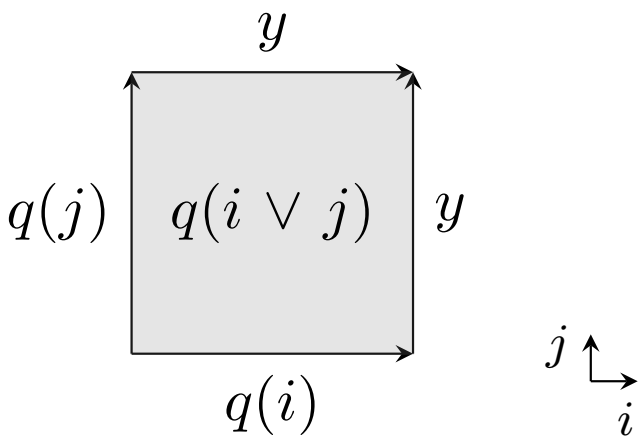

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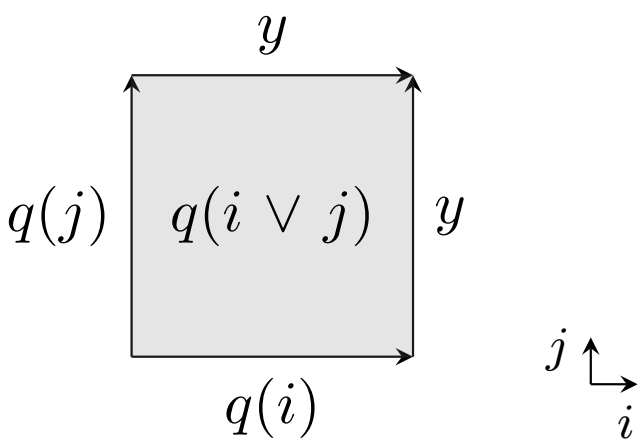

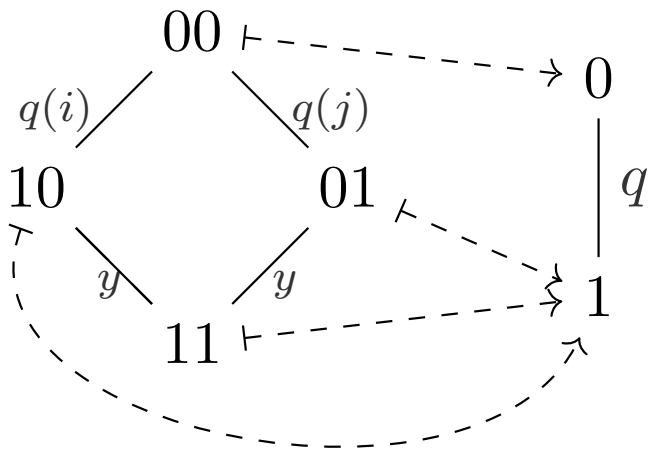

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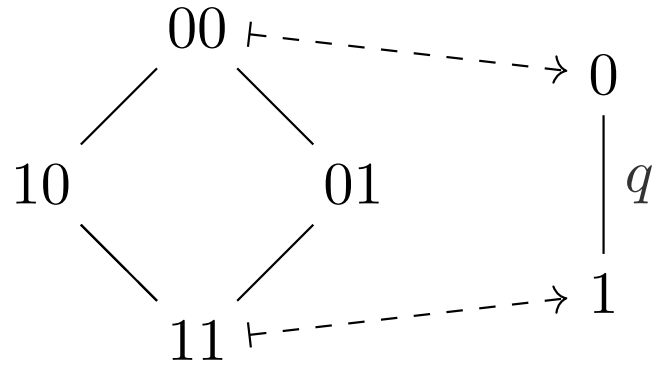
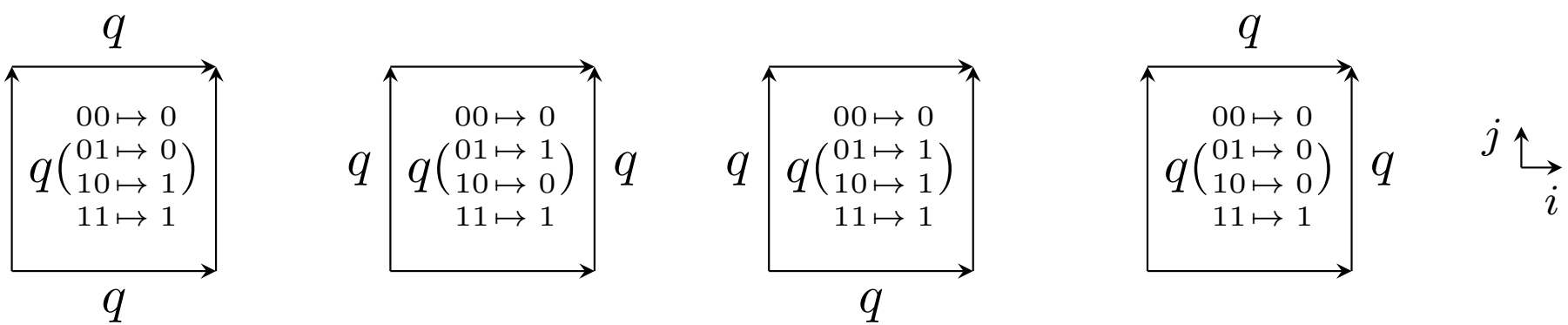
Collections of contortions as potential poset maps

Save for any element of the domain all possible targets.

$$q\begin{pmatrix} 00 \mapsto \{0\} \\ 01 \mapsto \{0,1\} \\ 10 \mapsto \{0,1\} \\ 11 \mapsto \{1\} \end{pmatrix}$$



Captures four different contortions:



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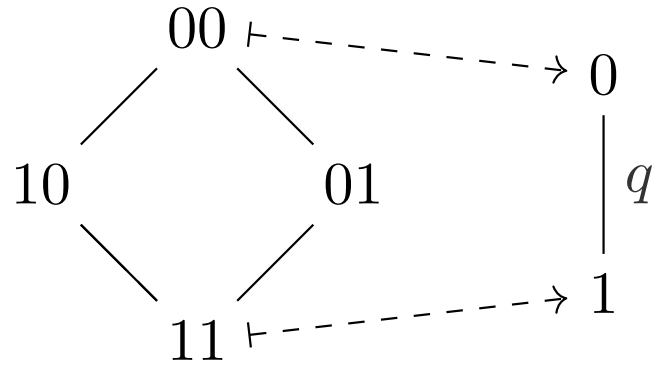

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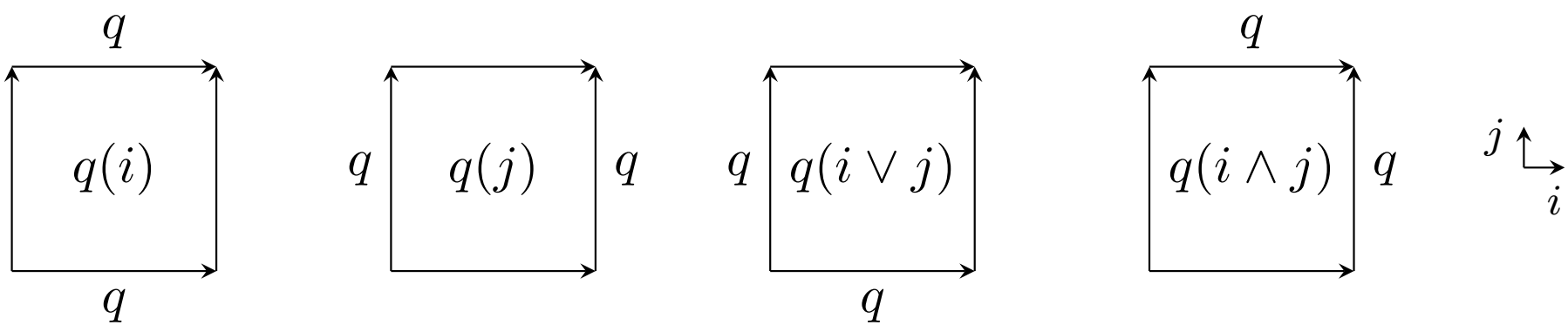

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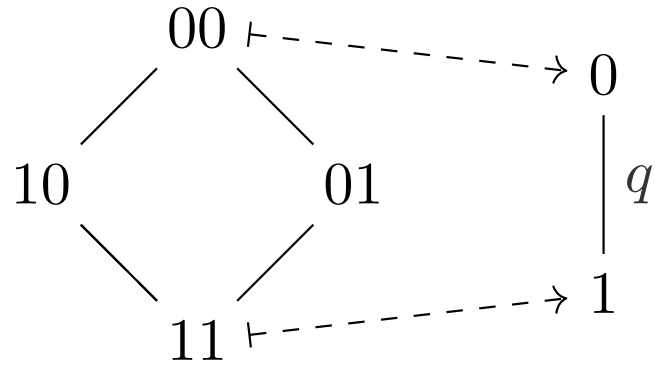

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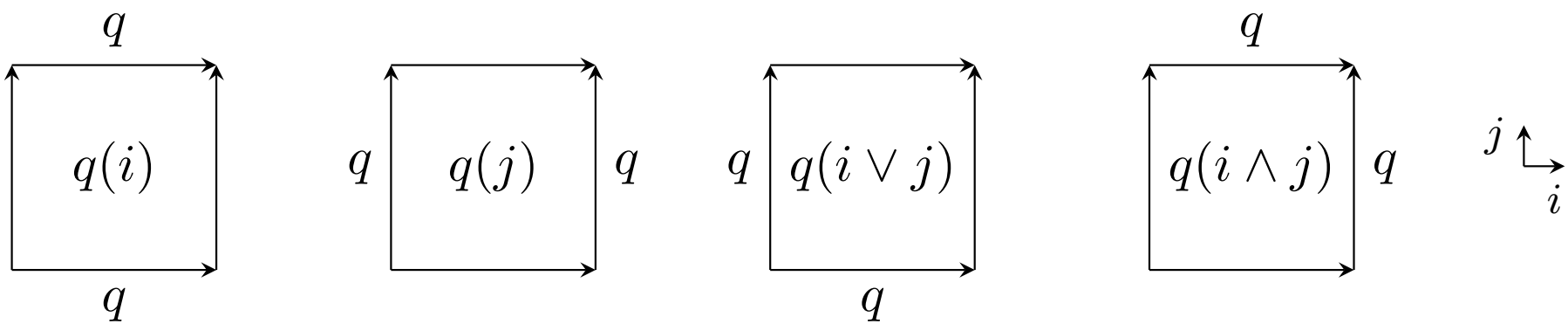
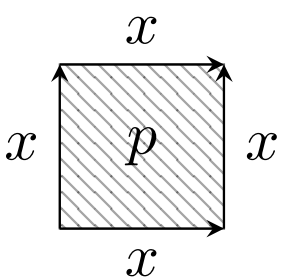


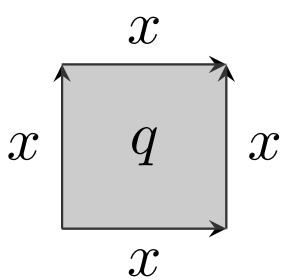
Construct contortions gradually by restricting poset map.

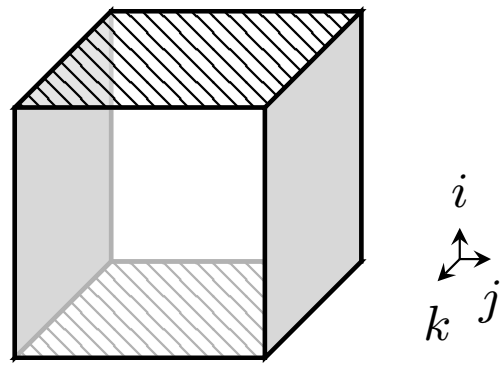
Eckmann-Hilton in a cubical setting

Classical result: concatenation of 2-loops is commutative.

$$x : [], p(i,j), q(i,j) : [i{=}0 \mapsto x \mid i{=}1 \mapsto x \mid j{=}0 \mapsto x \mid j{=}1 \mapsto x] \quad \mid$$
$$i,j,k \vdash ? : [\begin{array}{lll} i{=}0 \mapsto p(j,k) \mid & j{=}0 \mapsto q(i,k) \mid & k{=}0 \mapsto x \\ i{=}1 \mapsto p(j,k) \mid & j{=}1 \mapsto q(i,k) \mid & k{=}1 \mapsto x \end{array}]$$



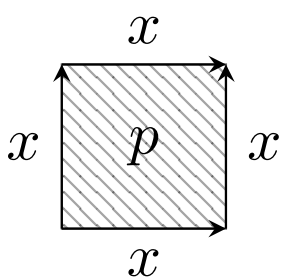


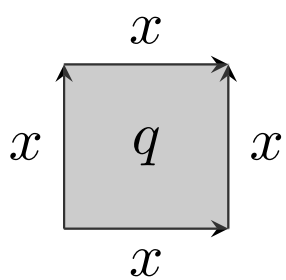


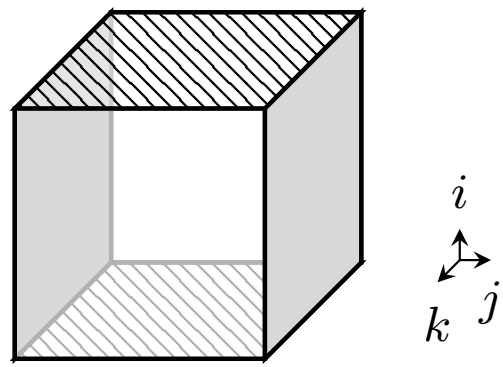
Eckmann-Hilton in a cubical setting

Classical result: concatenation of 2-loops is commutative.

$$x : [], p(i,j), q(i,j) : [i{=}0 \mapsto x \mid i{=}1 \mapsto x \mid j{=}0 \mapsto x \mid j{=}1 \mapsto x] \quad \mid$$
$$i,j,k \vdash ? : [\begin{array}{lll} i{=}0 \mapsto p(j,k) \mid & j{=}0 \mapsto q(i,k) \mid & k{=}0 \mapsto x \\ i{=}1 \mapsto p(j,k) \mid & j{=}1 \mapsto q(i,k) \mid & k{=}1 \mapsto x \end{array}]$$



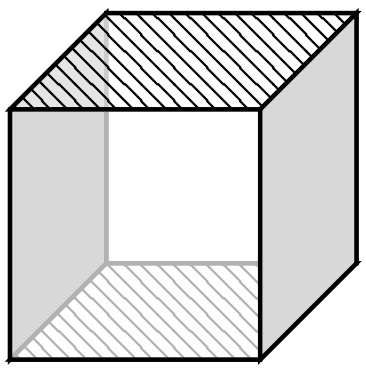


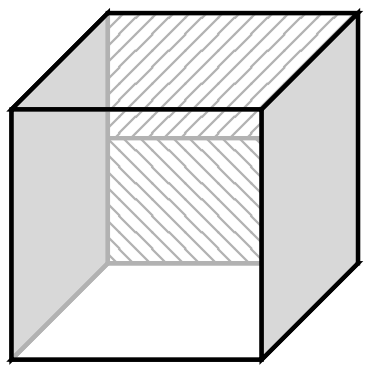


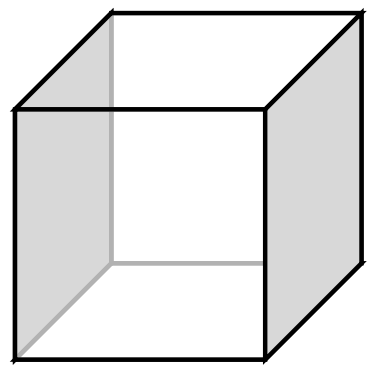
→ Construct as Kan composition of open 4-cube.

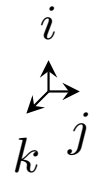
The cubical Eckmann-Hilton argument

$$
\begin{array}{lll}
 & i{=}0 \mapsto p(j, k \vee l) & \mid\ i{=}1 \mapsto p(j, k \vee l) \\
i, j, k \vdash \mathsf{hcomp}\ (l.[\ & j{=}0 \mapsto q(i, k) & \mid\ j{=}1 \mapsto q(i, k) \qquad])\ q(i, k) \\
 & k{=}0 \mapsto x & \mid\ k{=}1 \mapsto p(j, l)
\end{array}
$$









Summary

We studied the problem of constructing certain cubes using

- *Contortions*: operations on the interval variables giving basic shapes (decidable but huge search space),
- *Kan compositions*: gives missing side of open cube (undecidable).

Summary

We studied the problem of constructing certain cubes using

- *Contortions*: operations on the interval variables giving basic shapes (decidable but huge search space),
- *Kan compositions*: gives missing side of open cube (undecidable).

Our algorithm is based on

1. finite domain constraint solving,
2. exploring nested cubes in a "breadth-first" way,
3. representing collections of contortions as poset maps.

Future work

- Incorporate solver into Cubical Agda.
- Refine heuristics.
- Automate other cubical reasoning principles:
  - paths between types (heterogeneous equality),
  - transporting along paths (transp),
  - instances of univalence (Glue).
- Combine with synthesis of dependent type theory.

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Implementation: `https://github.com/maxdore/dedekind`

Description: `https://arxiv.org/abs/2402.12169`
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Thank you for your attention!

Eckmann-Hilton as its commonly known

```
Sq→Comp : PathP (λ j → q j ≡ q j) p p → p • q ≡ q • p
Sq→Comp  i j = hcomp (λ k → λ {
  (i = i0) → hcomp (λ l → λ {
    (j = i0) → x ; (k = i0) → q (j ∧ l) ; (j = i1) →  l k ;
    (k = i1) → hfill (λ m → λ { (j = i0) → x ; (j = i1) → q m })
    (p (j ∧ k)) ;
  (i = i1) → hfill (λ l → λ { (j = i0) → x ; (j = i1) → p l }) (inS
  (j = i0) → x ; (j = i1) → p k })
  (q j)

EckmannHilton' : p • q ≡ q • p
EckmannHilton' = Sq→Comp p q EckmannHilton
```