Formalising Topological Data Analysis in Cubical Agda

DANGER 4

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Maximilian Doré
maximilian.dore@cs.ox.ac.uk

Formalising data science

DANGER: data science $\rightarrow$ pure maths

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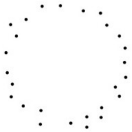
Goal: implement a fully verified tool for topological data analysis

Topological data analysis

Study the *shape of data* using algebraic topology. Typical pipeline:



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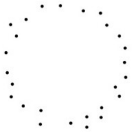
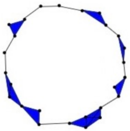


$\xrightarrow{①}$



① associate Vietoris-Rips complex with dataset

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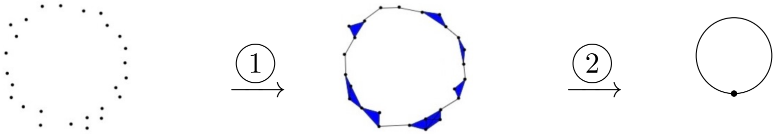

(1) associate Vietoris-Rips complex with dataset

(2) reduce size of complex using discrete Morse theory

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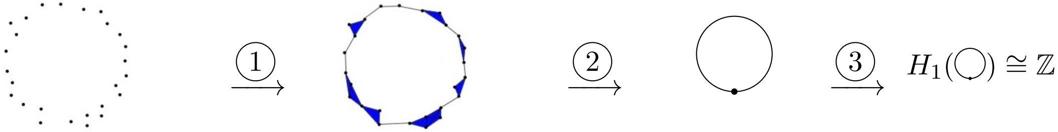

1. associate Vietoris-Rips complex with dataset
2. reduce size of complex using discrete Morse theory
3. compute (persistent) homology

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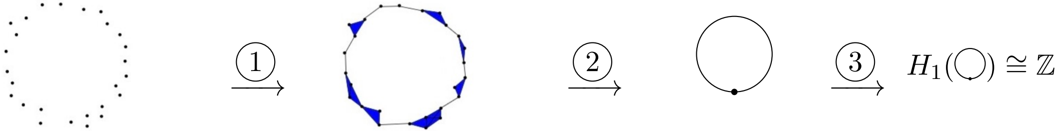

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→ implement this pipeline in a way that is *provably correct*

Writing provably correct programs

Why is $A \to (B \to A)$ a tautology?

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Types can depend on variables, Π and Σ are dependent functions and products.

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Give specification as type T, then program $p : T$ provably meets the specification.

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(1) associate Vietoris-Rips complex with dataset

Write a program of this type:

$\Pi(D : \text{Dataset}).\Sigma(K : \text{Complex}).\Pi(x\ y : \text{points}(D)).d(x, y) < \epsilon \to \text{line}(K, x, y)$

Showing Morse reductions correct

(2) reduce size of complex using discrete Morse theory

Discrete Morse theory removes cells irrelevant for the topology of a complex K, using an *acyclic partial matching* μ that says how to collapse cells.

Morse complex M contains only cells not part of μ. *Morse theorem*: $| K | \simeq | M |$

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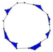
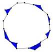
Example: given  and apt matching, Morse theorem gives  $\simeq$ 

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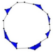
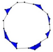
Example: given  and apt matching, Morse theorem gives  $\simeq$ 

Specification for step (2):

$\Pi(K : \text{Complex}).\Pi(\mu : \text{Matching}(K)).\Sigma(M : \text{Complex}').|\, K \,| \simeq |\, M \,|$

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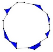
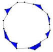
Example: given  and apt matching, Morse theorem gives  $\simeq$ 

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Original work[1] up to homology, modern approach[2] up to homotopy equivalence. Intricate to formalise head-on...

[1] Forman 1998, *Morse Theory for Cell Complexes*
[2] Nanda 2019, *Discrete Morse Theory and Localization*

Homotopy Type Theory and Cubical Agda

Idea for dealing with equality in type theory: there can be different proofs of an equality, and it's meaningful to study equalities between equalities.

Example: For $a, b : A$ have $p, q : a \equiv b$ and $\alpha, \beta : p \equiv q$ and $\phi : \alpha \equiv \beta$ etc...

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Cubical Agda 2019[1] implements these ideas:

- types are really homotopy types, and $\equiv$ behaves like $\simeq$
- *higher inductive types* allow for introducing higher equalities
- properties of cubical sets become principles of logic

[1] based on ideas of Abel, Bezem, Coquand, Cohen, Huber, Mörtberg, Vezzosi, ...

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$M_0 \triangleq \Sigma u \notin \mu, \;\; M_1(c,d) \triangleq \Sigma(uv \notin \mu).(u \rightsquigarrow c) \times (v \rightsquigarrow d)$

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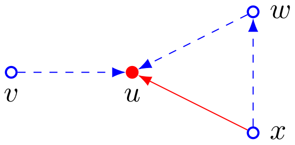

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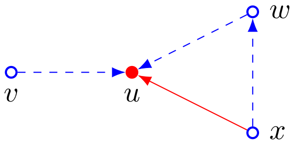

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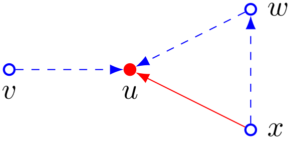

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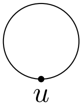

$M_0 \triangleq u$

$M_1(u,u) \triangleq (xu, [(x, xw), (w, wu)], [])$

The Morse theorem in Cubical Agda

Higher inductive type allows us to take the *geometric realisation* of a relation:

```
data |_| {c₀ : Type} (c₁ : c₀ → c₀ → Type) : Type where
  |_|₀ : c₀ → | c₁ |
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Ex. cont'd: $|\ M\ |$ has point $|\ u\ |_0$ with circle $|\ u \Rightarrow u \ni (xu, [(x, xw), (w, wu)], [])\ |_1$

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data |_| {c0 : Type} (c1 : c0 → c0 → Type) : Type where
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Morse theorem for graphs: $|\ G\ |\ \equiv\ |\ M\ |$ formalised in Cubical Agda:

`https://cs.ox.ac.uk/people/maximilian.dore/thesis/html/Morse.Morse.html`

Computing invariants of the space

③ compute (persistent) homology

Computing invariants involves loads of linear algebra. Already partially formalised.[1]

WIP: how to integrate this with my approach.

Algebraic invariants can be captured differently inside Cubical Agda[2], need to see how this can be used for computation.

[1] Heras, Coquand, Mörtberg 2013, *Computing Persistent Homology Within Coq/SSReflect*
[2] work by Brunerie, Cavallo, Lamiaux, Ljungström, Mörtberg on "synthetic" cohomology (rings)

Next steps

- Formalise DMT for 2-dim complexes to compute topology of grayscale images[3]

$H_1($  $) \cong \mathbb{Z}$

- Implement full pipeline in Cubical Agda: turn grayscale image into complex; compute APM; compute cohomology of reduced complex
- Refine pipeline: filtered complexes for persistent homology, cellular sheaves, ...

[3] Robins, Wood, Sheppard 2010, *Theory and Algorithms for Constructing Discrete Morse Complexes from Grayscale Digital Images*,

Conclusions

- Type theory is a natural language for verification of mathematical software: we can write down programs and mathematics in the same language
- Cubical Agda is a powerful (but intricate) logic for homotopy types

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Thank you for your attention!

The Morse theorem in Cubical Agda

Morse theorem: $| \ G \ | \ \equiv \ | \ M \ |$

- establish maps back and forth, for example define $| \ M_1 \ | \to | \ E \ |$:
  $| \ c \Rightarrow d \ni (uv, \gamma, \delta) \ |_1 \mapsto | \ \gamma \ |^{-1} \bullet\bullet | \ u \Rightarrow v \ni uv \ |_1 \bullet\bullet | \ \delta \ |$
- show that these maps are mutually inverse.

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Proving the Morse Theorem

E.g., for any critical edge $\mid u \Rightarrow v \ni uv \mid_1$ show that going along $\mid E \mid \to \mid M_1 \mid \to \mid E \mid$ is coherent:

$$\mid u \Rightarrow v \ni uv \mid_1$$

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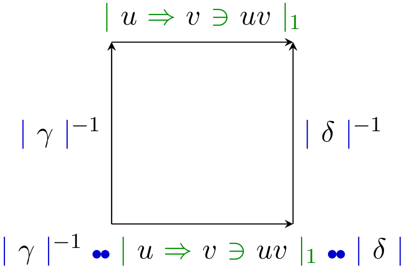

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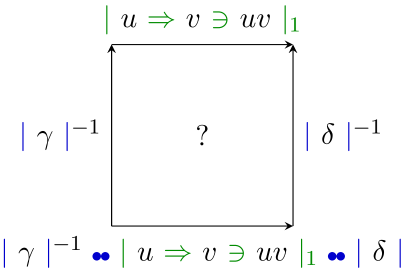

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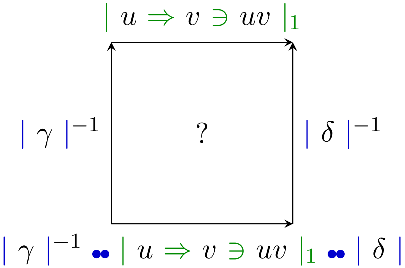
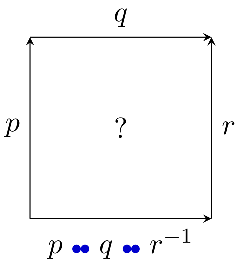
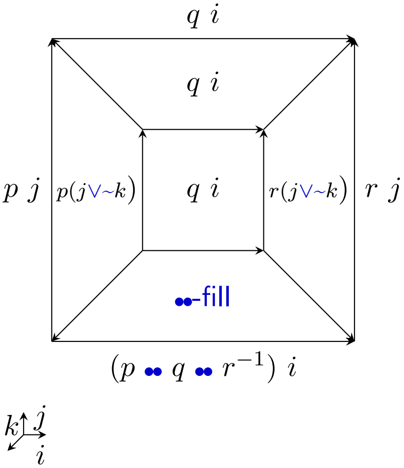




Kan compositions in Cubical Agda




```
lemma : (p : w ≡ x) (q : x ≡ z) (r : y ≡ z)
  → PathP (λ i → (p ∙∙ q ∙∙ sym r) i ≡ q i) p r
lemma p q r i j = hcomp (λ k → λ {
                      (i = i0) → p (j ∨ ∼ k)
                    ; (i = i1) → r (j ∨ ∼ k)
                    ; (j = i1) → q i
                    }) (q i)
```