Effective Interpolation and Preservation in Guarded Logics

Michael Benedikt
University of Oxford
michael.benedikt@cs.ox.ac.uk

Balder ten Cate
LogicBlox and UC Santa Cruz
btencate@ucsc.edu

Michael Vanden Boom
University of Oxford
michael.vandenboom@cs.ox.ac.uk

Abstract
Desirable properties of a logic include decidability, and a model theory that inherits properties of first-order logic, such as interpolation and preservation theorems. It is known that the Guarded Fragment (GF) of first-order logic is decidable and satisfies some preservation properties from first-order model theory; however, it fails to have Craig interpolation. The Guarded Negation Fragment (GNF), a recently-defined extension, is known to be decidable and to have Craig interpolation. Here we give the first results on effective interpolation for extensions of GF. We provide an interpolation procedure for GNF whose complexity matches the doubly exponential upper bound for satisfiability of GNF. We show that the same construction gives not only Craig interpolation, but Lyndon interpolation and Relativized interpolation, which can be used to provide effective proofs of some preservation theorems. We provide upper bounds on the size of GNF interpolants for both GNF and GF input, and complement this with matching lower bounds.

Categories and Subject Descriptors
Theory of computation [Logic]

1. Introduction
Basic modal logic is known to have many attractive properties, both computationally and in its model theory. Given the limited expressiveness of basic modal logic, logicians have searched for more powerful languages that extend it while preserving its positive features.

Andráska et. al. ([2], 4.7) list “meta-properties” of modal logic that one would desire in an extension, including the following.
- The logic should have decidable satisfiability and validity problems.
- The logic should satisfy interpolation theorems, like the Craig Interpolation Property (CIP); if \( \varphi \) and \( \psi \) are in the logic, and \( \varphi \) entails \( \psi \), then there should be a formula \( \theta \) in the logic such that \( \varphi \) entails \( \theta \) and \( \theta \) entails \( \psi \), and \( \theta \) mentions only relations common to both \( \varphi \) and \( \psi \).
- The logic should satisfy preservation theorems such as the Łoś-Tarski theorem: if a sentence in the logic is preserved under superstructures, it should be equivalent to an existential sentence in the logic.
- The logic should have the finite model property: a satisfiable sentence should have a finite model.

Interpolation, in particular, is a desirable property since it implies the well-known Beth Definability Property (BDP), as well as its extension, the Projective Beth Definability Property (PBDP), stating that implicit specifications can be converted to explicit ones. This conversion from implicit to explicit is important in knowledge representation [24] and databases [20]. Interpolation has many applications, both in simplifying definitions and in verification [18].

Andráska et. al. proposed the Guarded Fragment (GF) as a candidate for an attractive logic. They showed that it is decidable, that it has the finite model property, and that it satisfies Łoś-Tarski. Unfortunately Hoogland, Marx, and Otto [14] showed that GF does not satisfy CIP. Later work [7] showed that PBDP fails for GF.

The Guarded Negation Fragment (GNF) is an extension of GF introduced by Bárány, ten Cate and Segoufin in [5]. It also extends the Unary Negation Fragment (UNF) introduced by ten Cate and Segoufin in [23]. Both UNF and GNF were proven decidable and shown to have the finite model property [5, 23]. They were also shown to satisfy many of the properties mentioned earlier: in [23], UNF was shown to have CIP and PBDP; in [7], GNF was shown to have CIP, PBDP, and several preservation properties including the analog of Łoś-Tarski. The combination of interpolation and the finite model property for these logics means that interpolation holds when validity is considered over finite models only. This is attractive for applications of these logics in databases, such as rewriting queries using views [20], since in databases typically only finite models are considered.

In this work we provide further evidence that GNF and UNF are attractive extensions of modal logic. We show that these logics satisfy the Lyndon interpolation property (LIP) and its generalization, the Relativized interpolation property (RIP), and moreover present the first constructive procedures for GNF and UNF interpolation, including CIP, LIP, and RIP. Roughly speaking, the LIP says that if \( \varphi \) entails \( \psi \), we can find \( \theta \) in \( \mathrm{CIP} \), but with a relation symbol occurring positively in \( \theta \) iff it occurs positively in both \( \varphi \) and \( \psi \). The RIP says that the quantification pattern in \( \theta \) must match the quantification pattern in \( \varphi \) and \( \psi \).

We proceed by extending the mosaic method that has been applied in providing decidability for algebras of relations [21], modal logics [19], and temporal logics [17]. We present a variant of the method that gives a sound and complete decision procedure for the validity problem for GNF, but also allows interpolants to be read off from a proof of an entailment.

We show how this algorithm gives interpolants that witness the RIP and hence the LIP and CIP for GNF and UNF. Since the methods of [7, 23] were model-theoretic, this gives the first constructive proof of interpolation for GNF and UNF. We deduce...
bounds on the size of interpolants for GNF, and also on the size of GNF interpolants for GF formulas. We use the RIP to prove that GNF and GF admit effective preservation theorems, which state that formulas satisfying certain semantic properties can be converted to specific forms. To the best of our knowledge, these are the first effective preservation theorems, even for GF.

We complement the upper bounds with lower bounds for the size of GNF interpolants for GF and UNF.

Due to space limitations, most proofs are deferred to the full version of this paper.

2. Preliminaries

Notation and conventions. We use \( x, y, \ldots \) to denote vectors of variables. For a formula \( \phi \), we write \( \phi(x) \) to indicate that the free variables in \( \phi \) are contained in \( x \), and write \( \text{free}(\phi) \) for the actual free variables. The formula \( \phi \) may also use constants \( \text{con}(\phi) \). We use \( \alpha, \beta, \ldots \) to denote atomic formulas, and if we write \( \alpha(x) \) then we assume that \( \text{free}(\alpha) = x \).

Classical query languages. We will make use of some basic notions of database theory – in particular the following “classical” query classes: conjunctive queries (CQs), unions of conjunctive queries (UCQs), first-order logic formulas (FO), existential FO, and positive existential FO. Abitebol, Hull, and Vianu [1] is a good reference for these languages.

The basics of GNF and GF. The Guarded Negation Fragment (GNF) is built up inductively according to the grammar:

\[
\phi ::= R t \mid \exists x. \phi \mid \phi \lor \psi \mid \phi \land \psi \mid \alpha(x) \land \neg \phi(x)
\]

where \( R \) is either a relation symbol or the equality relation \( x = y \), \( \alpha \) is an atomic relation (including equality), and \( t \) is a tuple over variables and constants. Notice that any use of negation must occur conjoined with an atomic relation that contains all the free variables of the negated formula. Such an atomic relation is a guard of the formula.

The purpose of allowing equalities as guards is to ensure that every formula with at most one free variable can always be guarded. Thus GNF includes the Unary Negation Fragment (UNF), which is built up as above, but allowing negation only on formulas with at most one free variable.

A GNF formula is in GN-normal form if in its syntax tree, no disjunction is directly below an existential quantifier or a conjunction; and no existential quantifier is directly below a conjunction sign. Such GN-normal form formulas \( \phi \) can be generated by the following recursive definition:

\[
\phi ::= R t \mid \exists x. \phi \mid \phi \lor \psi \mid \phi \land \psi \mid \alpha(x) \land \neg \phi(x)
\]

where \( R \) is either a relation symbol or the equality relation, \( \alpha \) is an atomic relation (including equality), and \( t \) is a tuple over variables and constants. Thus, GN-normal form formulas can be viewed as formulas built up from atomic formulas using UCQs and guarded negation. Every GNF formula can be brought into GN-normal form, at the cost of an exponential increase in length and linear increase in the number of variables.

GNF should be compared to the Guarded Fragment (GF), typically defined via the grammar:

\[
\phi ::= R t \mid \exists x. \alpha(x) \land \phi(x) \mid \phi \lor \psi \mid \phi \land \psi \mid \neg \phi(x)
\]

where \( R \) is either a relation symbol or the equality relation, \( \alpha \) is an atomic relation (including equality), and \( t \) is a tuple over variables and constants. Here it is the quantification that is guarded, rather than negation. As in GNF, we allow equality guards by default.

It is easy to see that every union of conjunctive queries is expressible in GNF. It is only slightly more difficult to verify that every GF sentence can be expressed in GNF [5].

Turning to fragments of first-order logic that are common in database theory, one can show that common classes of integrity constraints, such as inclusion dependencies (also known as referential constraints) are expressible in GNF, as well as many of the common dependencies used in data integration and exchange. Going further, the relational translations of many of the common description logic languages used in the semantic web (e.g. ALC and ACHTJO [4]) are known to be expressible in GF, and hence in GNF.

Not only is GNF an expressive fragment of FO, but it was shown to be decidable and to have the finite model property.

Theorem 1 ([5]). A GNF sentence is satisfiable over all models iff it is satisfiable over finite models. Satisfiability can be tested effectively (and is \( 2\text{ETIME}\)-complete).

Size of formulas. A formula \( \phi \) given as input (to interpolation, preservation, etc.) is assumed to be in the standard tree representation of a formula, and the size of this representation is the number of symbols in \( \phi \).

For the output, however, we will usually represent the formula and measure its size as a node-labelled DAG (directed acyclic graph). The nodes represent formulas, and the edge relation connects a formula to its subformulas. We will also use DAGs with multiple roots to represent a set of formulas.

We often abuse notation by identifying a formula with a representation: given a representation of a formula \( \phi, [\phi] \) represents the size of the representation: the number of symbols if \( \phi \) is a formula, and the number of edges and labels if the representation is a DAG. Similarly, for a set \( S \) of formulas represented as a DAG, we write \( |S| \) to denote the total number of nodes and edges in the DAG.

The width of \( \phi \), denoted \( \text{width}(\phi) \), is the maximum number of free variables of any subformula of \( \phi \). For GF sentences, the width is bounded by the maximum arity of any relation.

Interpolation. For two formulas \( \varphi_L \) and \( \varphi_R \), we write \( \varphi_L \vdash \varphi_R \) if every model of \( \varphi_L \) is a model of \( \varphi_R \), and in this case say that \( \varphi_L \models \varphi_R \) is a validity. An interpolant for a validity is a formula \( \theta \) such that \( \varphi_L \vdash \theta \vdash \varphi_R \) and \( \theta \) mentions only relations present in both \( \varphi_L \) and \( \varphi_R \). The Craig interpolation theorem [8] states that if \( \varphi_L \) and \( \varphi_R \) are in FO, then there is an FO interpolant \( \theta \) to \( \varphi_L \) and \( \varphi_R \).

We can prove a stronger interpolation property by considering the polarity of relations in the formulas. Given a formula \( \phi \), we define \( \text{occ}(\phi) \) to be the set describing the polarity of occurrences of atomic relations (not including the equality relation). A relation occurs positively (respectively, negatively) in a formula if it is within the scope of an even (respectively, odd) number of negations. For every atomic relation \( S \) in \( \phi \) (where \( S \) is not the equality relation), \( S, + \in \text{occ}(\phi) \) (respectively, \( S, - \in \text{occ}(\phi) \) iff \( S \) occurs positively (respectively, negatively) in \( \phi \). Lyndon’s interpolation theorem [15] states that the interpolant \( \theta \) can be taken so that \( \text{occ}(\theta) \subseteq \text{occ}(\varphi_L \land \text{occ}(\varphi_R)) \).

A more recent relativized interpolation theorem of Otto [22] states that if \( \varphi_L \) and \( \varphi_R \) both use only quantification relativized to a set of unary predicates \( U \), then the same is true of the interpolant. A relativized quantifier for \( \bar{U} \) is \( \exists x. \bigwedge_{U \in \bar{U}} \exists x. \psi(x) \) or \( \forall x. \bigwedge_{U \in \bar{U}} \forall x. \psi(x) \).

Although there are proofs of Craig interpolation for FO that construct interpolants efficiently given proofs that witness the validity (e.g. [10]), these do not give bounds on the size of interpolants relative to the size of \( \varphi_L \) and \( \varphi_R \). In fact, it has been shown that for
FO validities, there is nocomputable bound on the sizes of theinterpolants relative to the sizes of the original validities [11].

We say that a logic has the Craig Interpolation Property (CIP) if given \( \varphi_L \) and \( \varphi_R \) in the logic, there is an interpolant that is again in the logic. Similarly we can talk about the Lyndon Interpolation Property (LIP) and Relativized Interpolation Property (RIP). Hoogland et. al [14] showed that the CIP fails for GF. Marx [16] argued that one of the applications of CIP, the Projective Beth Definability Property (PBDP) holds for GF as well as for an extension called the “Packed Fragment”. He conjectured that the proof could be made effective using the “mosaic method”. The argument in [16] was flawed, and indeed neither the guarded fragment nor the fragment suggested by Marx to show effective CIP, LIP, RIP, and PBDP for GNF (see Theorem 9).

3. The Mosaic Method

In its simplest form, a mosaic is just a finite set of formulas. When it satisfies certain coherence properties, it can be viewed as a collection of formulas describing part of a model. For instance, in the mosaic system for GF described in Section 4.6 of [2], a mosaic can describe the complete subformula type of some guarded set of elements in a model.

For modal and guarded logics, it turns out that the existence of a (finite) set of mosaics satisfying certain coherence properties is enough to ensure the existence of a full model. This allows mosaics to be used to decide satisfiability.

We adapt this method to check whether \( \varphi_L \land \neg \varphi_R \) is unsatisfiable, and if so, extract an interpolant for \( \varphi_L \models \varphi_R \) from the mosaics. The purpose of this section is to describe the general framework that we are using. We utilize this framework in later sections with different versions of mosaics depending, for instance, on whether the input formulas are in GF versus GNF, or whether we are aiming for Lyndon interpo- lation versus Relativized interpolation. As a result, some of the terms in this section are only defined informally; the formal definitions appear in later sections since they depend on the particular mosaic variant that we are using.

We start by fixing some sentences \( \varphi_L \) and \( \varphi_R \) over signatures \( \sigma_L \) and \( \sigma_R \), respectively. The first step is to define a finite set \( \Gamma \) of formulas that are relevant to checking satisfiability of \( \varphi_L \land \neg \varphi_R \). This is usually some extension of the subformula closure of \( \varphi_L \) and \( \varphi_R \). To be suitable for computing an interpolant for \( \varphi_L \models \varphi_R \), we also annotate these formulas with a provenance \( X \in \{L, R\} \) indicating that the formulas are related to \( \varphi_X \). This parallels other constructive interpolation methods, e.g. [10]. Given a provenance \( X \), we write \( X \) for the dual provenance (i.e. \( L = R \) and \( R = L \)). Throughout the paper, we use \( X, Y, Z \) to range over provenances \( L \) and \( R \).

Given some finite set of parameters \( e \), we define \( \Gamma(e) \) as the set of formulas from \( \Gamma \) with parameters from \( e \) substituted for any free variables. Parameters are just a special type of variable. We write \( a, b, \ldots \) to denote vectors of parameters, and given a formula \( \phi \), we write \( \phi(a,b) \) to indicate that the free variables in \( \phi \) are contained in \( a \) and the parameters in \( \phi \) are contained in \( b \). We write free(\( \phi \)) (respectively, par(\( \phi \))) for the actual free variables (respectively, parameters).

For atomic \( \varphi \), if we write \( \varphi(x|z) \) then we assume that free(\( \varphi \)) = \( x \) and par(\( \varphi \)) = \( z \).

A mosaic \( \tau \) over some finite set of parameters \( e \) (of size bounded by the width of the original sentences) is then defined to be a subset of formulas from \( \Gamma(e) \). Each mosaic is annotated with \( L \) or \( R \), and an \( X \)-mosaic has additional restrictions on the \( X \)-formulas appearing in it. This helps prove properties about the interpolants that we later extract from these mosaics.

Mosaics can be linked together. For instance, if a formula \( Y : \exists x. \psi(x|b) \) appears in an \( X \)-mosaic \( \tau \), then this can be linked to some \( Y \)-mosaic \( \tau' \) using parameters \( ab \) and containing the formula \( Y : \psi(ab) \) (for some new parameters \( a \)). In addition, the linked mosaic \( \tau' \) must contain the maximum amount of information from \( \tau \) about formulas using only shared parameters \( b \) (subject to any constraints on formulas in \( \tau' \)) and cannot contain any new information about formulas using only shared parameters. We say an existential requirement \( Y : \exists x. \psi(x|b) \) in \( \tau \) is fulfilled if there is a formula \( Y : \psi(ab) \) in \( \tau \), or there is some \( \tau' \) such that \( \tau' \) and \( \tau \) are linked via \( Y : \exists x. \psi(x|b) \).

In order for a mosaic to correspond to a part of a model it must satisfy certain coherence properties. Within a single mosaic, this means that it must be internally consistent. For instance, we may want to require that if it contains a conjunction, then it contains both conjuncts. The exact definition will depend on the particular version of mosaics that we are considering. A set of mosaics is coherent if every mosaic is internally consistent and it is saturated: every existential requirement is fulfilled by a mosaic in the set.

We then fix some finite set \( M \) of mosaics that suffices to prove the following key mosaic theorem:

\[
\varphi_L \land \neg \varphi_R \text{ is satisfiable if and only if there is a saturated set of internally consistent mosaics from } M \text{ containing a mosaic with } \varphi_L \text{ and } \neg \varphi_R .
\]

In one direction, the idea is to extract a saturated set of internally consistent mosaics from a model. In the other (harder) direction, we must show that a saturated set of internally consistent mosaics from \( M \) can be used to construct a model of \( \varphi_L \land \neg \varphi_R \).

This gives us a procedure for deciding whether or not \( \varphi_L \land \neg \varphi_R \) is satisfiable. Consider the following mosaic elimination procedure over \( M \):

- At stage \( i = 1 \), every internally inconsistent mosaic is eliminated.
- At stage \( i > 1 \), a mosaic \( \tau \) is eliminated if it has an existential requirement that is not fulfilled in \( \tau \), and every linked \( \tau' \) fulfilling it was already eliminated.

This process continues until a fixpoint is reached. The mosaic theorem can then be used to show that \( \varphi_L \land \neg \varphi_R \) is satisfiable if there is a mosaic containing \( \varphi_L \) and \( \neg \varphi_R \) that is never eliminated.

For interpolation, we are interested in the case when \( \varphi_L \land \neg \varphi_R \) is unsatisfiable, since this means \( \varphi_L \models \varphi_R \).

It turns out that we can construct a mosaic interpolant \( \theta_X^\prime \) for each \( X \)-mosaic \( \tau \) eliminated using the procedure above. These mosaic interpolants satisfy the property that \( \tau_X \models \theta_X^\prime \) and \( \theta_X^\prime \models \neg \tau_X \), where \( \tau_X \) denotes the conjunctions of all of the \( Y \)-formulas from \( \Gamma(e) \) in \( \tau \) and the negation of all of the \( Y \)-formulas from \( \Gamma(e) \) that satisfy the constraints on formulas in \( \tau \) but are not in \( \tau \).

We start by constructing these interpolants for mosaics eliminated because of internal inconsistencies. The interpolants for such mosaics are simple: atomic formulas or the negation of atomic formulas. Proceeding by induction on the stage at which the mosaic is eliminated, we then build up more complicated interpolants for the other mosaics. Intuitively, each mosaic interpolant describes the reason the mosaic was eliminated.

Since the sequence of elimination steps for the mosaics provide a proof that \( \varphi_L \land \neg \varphi_R \) is unsatisfiable, it is not surprising that the desired interpolant for \( \varphi_L \models \varphi_R \) can then be constructed from these mosaic interpolants.

It is helpful to compare this to other constructive interpolation approaches. In [10], an interpolant is constructed from a proof tree for \( \varphi_L \land \neg \varphi_R \) in a tableau proof system for FO. The construction starts at the leaves (where very simple interpolants suffice), and builds up to an interpolant that holds for the original validity at the root of the proof tree. Because of the “one-step” nature of the proof.
rules, there are straightforward interpolation construction rules at each stage.

Using the mosaic method, the definition of the interpolant at the inductive stages is more complicated. Unlike a traditional proof system, however, the mosaics allow precise control over the parameters appearing at each stage. This is advantageous for constructing an interpolant in GNF, since we need to be able to guard certain parameters in order to remain in GNF. Indeed, this precise control over the parameters was one of the motivations for building GNF interpolants from mosaics rather than from a more traditional proof system.

4. GF Mosaics

We start by describing the mosaic method for validity testing and interpolation construction when the input consists of GF sentences $\varphi_L$ and $\varphi_R$ over signatures $\sigma_L$ and $\sigma_R$, respectively. The purpose of this section is to give a more gentle introduction to mosaics, without the complications that come from handling GNF, relativized quantifiers, or equality.

We assume the sentences $\varphi_L$ and $\neg \varphi_R$ are in GF and are in negation normal form (negations are pushed to the inside) and do not use equality. These sentences may use constants, and we let $e := \text{con}(\varphi_L) \cup \text{con}(\varphi_R)$.

Closure. For GF satisfiability, we restrict formulas in mosaics to the subformula closure of $\varphi_L$ and $\neg \varphi_R$, with a slight extension to handle atoms. The closure $\text{cl}(\varphi)$ of a GF formula $\varphi$ is the smallest set $C$ of formulas containing $\varphi$ such that:

- If $\psi_1 \lor \psi_2 \in C$ or $\psi_1 \land \psi_2 \in C$, then $\psi_1, \psi_2 \in C$;
- If $\exists \tau(\varphi(y)) \land \psi(x) \in C$, then $\exists \tau(\varphi(x)) \land \psi(x) \in C$;
- If $\exists \tau(\varphi(x)) \land \psi(x) \in C$, then $\exists \tau(\varphi(y)) \land \psi(y) \in C$;
- If $\exists \tau(\varphi(y)) \land \psi(y) \in C$, then $\exists \tau(\varphi(x)) \land \psi(x) \in C$.

Given parameters $c$, $\text{cl}(e)$ (respectively, $\text{cl}_b(e)$) consists of formulas from $\text{cl}(\varphi_L)$ (respectively, $\text{cl}(\neg \varphi_R)$) with free variables replaced by parameters from $c$ or constants from $e$ and labelled with provenance $L$ (respectively, $R$).

Observe that the formulas in $\text{cl}_L(e) \cup \text{cl}_R(e)$ respect the polarity of relations in the original formulas. This will be important for extracting Lyndon-style interpolants later, where the polarity of relations needs to be respected.

Guards. We say $\alpha$ is an $L$-atom (respectively, $R$-atom) if $(\alpha, +) \in \text{ooc}(\varphi_L)$ (respectively, $(\alpha, -) \in \text{ooc}(\varphi_R)$). Usually, any atom is a possible guard in a GF or GNF formula. For the purposes of interpolation, however, we need atoms in the common signature (catoms for short). We say $\alpha$ is an $L$-catom (respectively, $R$-catom) if $(\alpha, +) \in \text{ooc}(\varphi_L) \cap \text{ooc}(\varphi_R)$ (respectively, $(\alpha, -) \in \text{ooc}(\varphi_L) \cap \text{ooc}(\varphi_R)$).

Given a collection $\tau$ of formulas from $\text{cl}_L(e) \cup \text{cl}_R(e)$, we say $b \in e$ is $X$-guarded in $\tau$ (respectively, $X$-cguards in $\tau$) if $b$ is empty, or if there is some $X \in \exists \tau(\varphi(x)) \in \tau$ where $X$ is an $X$-atom (respectively, $X$-catom). We allow $x$ to be empty, in which case the $X$-guard or $X$-cguard for $b$ is actually of the form $X : \alpha(b)$.

GF mosaics. A GF $X$-mosaic $\tau$ for $\varphi_L \land \neg \varphi_R$ using parameters $e$ is a subset of $\Gamma(e) := \text{cl}_L(e) \cup \text{cl}_R(e)$ such that $X$ is $X$-guarded in $\tau$, and if $\tilde{X} : \psi \in \tau$, then $\text{par}(\psi)$ is $X$-cguarded in $\tau$. We write $\tau(e)$ to emphasize that $e$ uses parameters $e$.

Each $X$-mosaic provides information about an $X$-guarded set of parameters. Unlike the mosaics in [2], we place further restrictions on the formulas in the mosaics; namely, we restrict $X$-formulas in an $X$-mosaic. Roughly speaking, the $X$-formulas are the “dangerous” formulas that could potentially introduce a negation when we try to extract GNF interpolants from these mosaics later, so we require that any such formulas have an $X$-cguard in the mosaic (a guard that could be used in the interpolant).

Let $\tau$ be a GF $X$-mosaic over parameters $e$. Assume $Y : \eta \in \tau$ is an existential requirement i.e. $\eta$ is of the form $\exists x.a(x|b) \land \psi(x)$. Then $\tau$ is linked to $\tau'$ via $Y : \psi$ (written $\tau \rightarrow_{Y, \eta} \tau'$) if:

- $\tau'$ is a GF $Y$-mosaic over parameters $ab$ with $a \cap c = \emptyset$;
- $Y : \alpha(xb)[a|x] \in \tau'$ and $Y : \psi(xb)[a|x] \in \tau'$;
- for all $Y : \psi'(b) \in \tau$ and $Y : \psi'(b) \in \tau'$;
- for all $Y : \psi'(b) \in \tau$ such that $\text{par}(\psi')$ is $Y$-cguarded in $\tau'$,
  $Y : \psi'(b) \in \tau'$;
- for all $Z : \psi'(b) \in \tau'$, $Z : \psi'(b) \in \tau$.

These conditions ensure that the maximum amount of information about the shared parameters $b$ is passed from $\tau$ to $\tau'$ (subject to the restrictions on $Y$ formulas in the $Y$-mosaic $\tau'$), and no new information about $b$ is added in $\tau'$.

Fix some set $P$ of parameters of size $2 \cdot \text{width}(\varphi_L \land \neg \varphi_R)$, and let $M$ be the set of GF mosaics over parameters $P$. The following bound on the number of mosaics is easy to show.

Proposition 2. There is a polynomial function $p(\varphi_L, \varphi_R)$ such that if $|\varphi_L| + |\varphi_R| = n$, then there are at most $2^{p(n)}$ mosaics in $M$. Moreover, for a fixed bound on the arity of relations, there are at most $2^{2^{p(n)}}$ mosaics in $M$.

Coherence. We say a GF $X$-mosaic $\tau$ is internally consistent if there is no atomic $\alpha \in \sigma_L \cup \sigma_R$ such that $\forall \alpha, Z : \alpha \in \tau$, and $\tau$ satisfies the following downward closure properties:

- If $Y : \alpha(\varphi(b)) \in \tau$, then $Y : \exists x.\alpha(\varphi(x)) \in \tau$;
- If $Y : \exists x.\alpha(xb) \in \tau$, then $Y : \exists x y.\alpha(xyb) \in \tau$;
- If $Y : \psi_1 \land \psi_2 \in \tau$, then $Y : \psi_1, Y : \psi_2 \in \tau$;
- If $Y : \exists x.\alpha(x) \rightarrow \psi(x) \in \tau$, then $Y : \psi(x) \in \tau$ or $Y : \exists x.\alpha(x) \in \tau$;
- If $Y : \exists x.\alpha(xb) \rightarrow \psi(xb) \in \tau$ and there is a $c \subseteq e \cup e$ such that $Z : \alpha(xb)[a|x] \in \tau$ for some $Z \in \{L, R\}$, then $Y : \psi(xb)[a|x] \in \tau$.

A set of GF mosaics is said to be saturated if for every mosaic $\tau(e)$ in the set and for every existential requirement $Y : \eta \in \tau$ for $\eta$ of the form $\exists x.\alpha(x) \land \psi(x)$, either (i) $Y : \alpha(xb)[a|x]$ and $Y : \psi(xb)[a|x]$ are in $\tau$ for some $\alpha \subseteq c \cup e$ (so $\eta$ is fulfilled in $\tau$) or (ii) $\tau$ is linked to some mosaic $\tau'$ in the set via $Y : \eta$ (so $\eta$ is fulfilled in $\tau'$). Finding a saturated set of internally consistent mosaics containing the original sentences provides a sound and complete method for GF satisfiability testing.

Theorem 3. $\varphi_L \land \neg \varphi_R$ is satisfiable if and only if there is a saturated set of internally consistent mosaics from $M$ that contains a mosaic $\tau$ such that $L : \varphi_L, R : \neg \varphi_R \in \tau$.

We can now use the mosaic elimination procedure described in the previous section, and Theorem 3 ensures that $\varphi_L \land \neg \varphi_R$ is satisfiable if there is some mosaic containing $L : \varphi_L$ and $R : \neg \varphi_R$ that is not eliminated using this process.

Constructive Interpolation. Now assume $\varphi_L \models \varphi_R$, and consider the mosaic elimination procedure for the GF mosaics $M$ for $\varphi_L \land \neg \varphi_R$. We write $\mathcal{N}^\tau$ for the set of mosaics for $\varphi_L \land \neg \varphi_R$ that are eliminated using this procedure, and $\mathcal{N}^{\tau'}$ for the set of mosaics that have been removed by stage $i$. By Theorem 3, we know that every mosaic containing $L : \varphi_L$ and $R : \neg \varphi_R$ must be in $\mathcal{N}^{\tau'}$.

We first show how to construct a mosaic interpolant $\theta_X$ for each $X$-mosaic $\tau(e)$ in $\mathcal{N}^{\tau'}$. Roughly speaking, $\theta_X$ is an interpolant (in the usual sense) for $\tau_X := \eta \rightarrow_{\tau_X}$, where

$$\tau_X := \bigwedge_{Y : \psi \in \tau} \psi \land \bigwedge_{Y : \psi \in \tau} \neg \psi$$

and $\tau$ is the set consisting of formulas $Z : \psi \in \text{cl}_Y(e) \setminus \tau$ such that $Z = X$, or $Z = \tilde{X}$ and $\text{par}(\psi)$ is $X$-cguarded in $\tau$. In other words, $\tau$ contains the formulas from the closure that satisfy the
requirements for being in \( \tau \) (in terms of \( X \)-guardedness), but are not in \( \tau \). Intuitively, \( \theta_X \) describes the reason \( \tau \) was eliminated, i.e. \( \theta_X \) expresses why \( \tau_X \cap \tau_X \) is not satisfied.

We have the following Lyndon-style mosaic interpolant properties.

**Lemma 4.** For each \( GF \) \( X \)-mosaic \( \tau(c) \in N' \) (the set of eliminated mosiacs), we can construct a DAG representation of a formula \( \theta_X \) such that

- \( \tau_X \models \theta_X \) and \( \theta_X \models \neg \tau_X \);
- \( \text{occ}(\theta_X) \subseteq \text{occ}(\varphi_L) \cap \text{occ}(\varphi_R) \) if \( X = L \), \( \text{occ}(\theta_X) \subseteq \text{occ}(\varphi_L) \cap \text{occ}(\varphi_R) \) if \( X = R \); \n- \( \text{con}(\theta_X) \subseteq \text{con}(\varphi_L) \cup \text{con}(\varphi_R) \);
- \( \varphi_X \subseteq \emptyset \);\n- \( \theta_X \) is in GF, even when \( \varphi_X \) variables are viewed as free variables.

Moreover, if \( n = |\varphi_X| + |\varphi_R| \) then there is a DAG representation of \( \Theta_i = \{ \theta_X' : \tau \text{ is a } X \text{-mosaic in } N_i \} \) such that \( |\Theta_i| \leq i \cdot 2^{|\varphi_X|} \) for the fixed arity case, where \( \theta_X' \) is some polynomial function independent of \( \varphi_L \) and \( \varphi_R \).

**Proof.** We first introduce some additional notation. We use \( T \) (respectively, \( \bot \)) as an abbreviation for the GF sentence \( \exists x. x = x \) (respectively, \( \exists x. x = x \land \neg \{ x = x \} \)). During the interpolant construction, we will need to include guards in order to remain in GF, so for any tuple of formulas that are \( X \)-guarded, we define a formula \( \text{gdd}_X(b) \). If \( b = \emptyset \), then set \( \text{gdd}_X(b) := T \). Otherwise, there is some formula \( X \models \exists x. \beta(x) \in \tau \) for some \( X \)-catom \( \beta \), and we set \( \text{gdd}_X(b) := \exists x. \beta(x) \) (note that \( x \) can be empty, in which case this is just an atomic formula). We will also write \( \text{gdd}_X(b) \land \psi \) to indicate some formula \( \exists x. (\beta(x) \land \psi) \).

We proceed by induction on \( i \), the stage at which \( \tau \) was eliminated. We define the interpolants and then sketch the proof of correctness in the inductive case.

**Base case.** Consider an \( X \)-mosaic \( \tau(c) \in N_i \). Then \( \tau \) has an internal inconsistency, and we consider the following cases.

- Assume one of these conditions holds for \( Y \in \{L, R\} \):
  - \( Y : \alpha \) (i.e. \( Y = L \))
  - \( Y : \neg \alpha \) (i.e. \( Y = R \))
  - \( Y : \exists x. \alpha(x, \theta) \land \neg \alpha \).
- \( Y : \alpha \) (i.e. \( Y = \emptyset \))
  - \( Y : \neg \alpha \) (i.e. \( Y = \emptyset \))
  - \( Y : \exists x. \alpha(x, \theta) \land \neg \alpha \).

**Inductive case.** Now consider an \( X \)-mosaic \( \tau(c) \in N_i \setminus N_{i-1} \). This means there is some \( Y : \eta = Y : \exists x. \alpha(x, \theta) \land \psi(\theta) \in \tau \) that is not fulfilled in \( \tau \), and if \( \tau \models \psi(\theta) \) then \( \tau \in N_{i-1} \).

Assume \( \text{par}(\psi) = b \subseteq \emptyset \), and fix some \( a \in \mathcal{P}' \).

We introduce some additional notation. For \( Z \subseteq \{L, R\} \), we define \( \tau_Z \) to be the \( Z \)-mosaic that results from taking the \( Z \)-formulas in \( \tau \) that only use parameters from \( b \), and then unioning with the \( Z \)-formulas in \( \tau \) whose parameters are contained in \( b \) and are \( X \)-guarded in \( \tau \). For mosiacs \( v, v' \) and \( Z \subseteq \{L, R\} \), we write \( v' \models_Z v \) if \( v' \) uses only parameters from \( ab \) and is obtained from \( v \) by adding only \( Z \)-formulas (so the \( Z \)-formulas in \( v \) and \( v' \) are identical).

If \( Y = X \), then define \( \theta_X \) to be

\[
\bigvee_{\tau'(b) \models \tau', b \models v'} \exists x. \left( \bigwedge_{\tau(b) \models \tau'} \theta_X(x/\alpha) \right).
\]

Informally, this interpolant expresses that there is a way to fulfil the existential requirement by adding some \( X \)-formulas to \( X \), but no matter what \( X \)-formulas we add, the resulting linked mosaic \( \tau' \) has already been eliminated.

If \( Y = X \), then define \( \theta_X \) to be

\[
\text{gdd}_X(b) \land \neg \bigvee_{\tau'(b) \models \tau', b \models v'} \exists x. \left( \bigwedge_{\tau(b) \models \tau'} \theta_X(x/\alpha) \right).
\]

In both cases, notice that the parameters \( ab \) in the inductively defined interpolants \( \theta_X \) are guarded by the atom \( \alpha(ab) \) appearing in the existential requirement \( Y : \exists x. \alpha(x, \theta) \land \psi(\theta) \). We cannot necessarily guard the quantification in the interpolant using \( \alpha(ab) \) since \( \alpha \) may not be in the common signature. This is why the resulting interpolant might not be in GF; even when the input sentences are in GF. However, in case \( Y = X \), we can always guard \( b \) in the common signature using \( \text{gdd}_X(b) \). This guard is guaranteed to exist because the existential requirement is a \( X \)-formula using parameters \( b \) in an \( X \)-mosaic. This ensures that the interpolant is in GF. The occurrence of constants and the polarity of relations follows from the inductive hypothesis and the fact that the polarity of occurrences of relations are preserved when moving to mosiacs.

We now sketch part of the proof of correctness, showing that \( \tau_X \models \theta_X \) and \( \theta_X \models \neg \tau_X \) when \( Y = X \).

Assume there is a model \( M \) for \( \tau_X \). Since \( X : \eta \in \tau \), this means that \( M \models \exists x. \alpha(x, \theta) \land \psi(\theta) \) and there are elements \( a' \in M \) and an expansion \( M' \) of \( M \) with the interpretation \( a'' = a' \) such that \( M' \models (\alpha(a') \land \psi(a')) \). Take \( ab \) to be the union of \( a'' \) and the set of formulas \( X : \psi(\theta) \) in \( M' \). It can be checked that \( \tau \models \psi(\theta) \) and \( \psi(\theta) \models \theta_X \). By construction, \( \theta_X \models \psi(\theta) \). Moreover, for any choice of \( \tau'(b) \models \theta_X \), \( \tau'(b) \models \tau' \) (since the \( X \)-formulas are identical in \( \psi(\theta) \) and \( \psi(\theta) \)).

By the inductive hypothesis, this means that \( M' \models \theta_X \). Hence \( M' \models \text{gdd}_X(b) \land \neg \bigvee_{\tau'(b) \models \tau'} \theta_X \).

Now assume that there is a model \( M \) of \( \theta_X \). Then there is some \( \psi(\theta) \models \tau'(b) \) with \( \tau \models X : \gamma \), \( \psi(\theta) \) elements \( a' \in M \), and an expansion \( M' \) of \( M \) with the interpretation \( a'' = a' \) such that \( M' \models \text{gdd}_X(b) \land \neg \bigvee_{\tau'(b) \models \tau'} \theta_X \).

Note that for all \( \tau'(b) \models \tau' \) such that \( \tau \models X : \gamma \), \( M' \models \theta_X \) and consequently, the inductive hypothesis implies \( M' \models \tau_X \). In particular, consider \( \tau' : \gamma' \land S' \) where \( S' \) is the set of formulas \( X : \psi(\theta) \) in \( M' \) such that \( M' \models \psi(\theta) \), \( \text{par}(\psi) \cap \alpha = \emptyset \), and \( \psi(\theta) \) is \( X \)-guarded in \( \psi(\theta) \) as required in an \( X \)-mosaic. It is clear that \( \tau'(b) \models \tau' \) and \( \tau' \models \tau_X \), so \( M' \models \tau_X \) as described above.

Since \( M' \models \tau_X \), there is some conjunct \( \gamma \in \tau_X \) such that \( M' \models \gamma \). Consider some conjunct \( \alpha(ab) \) in \( \tau_X \) that actually uses some parameters from \( a \). Then by choice of \( S' \), \( M' \models \gamma \).
so this formula cannot witness the fact that $M' \models \neg \tau' X$. This means there must be some conjunct $\chi(b)$ in $\tau' X$ that only uses parameters from $b$ such that $M' \models \neg \psi(b)$. If $X : \chi(b)$ is in $\tau'$, then $X \models \chi(b) \in \tau$ by the definition of $\tau \rightarrow X \eta \eta$. Likewise, if $\chi(b) \notin \tau'$, then it must be the case that $\varphi(\chi) = X$-guarded in $\tau'$, so $X \models \chi(b) \notin \tau$ (otherwise it would contradict the definition of $\tau \rightarrow X \eta \eta$). In either case, this means $\chi(b)$ must be a conjunct in $\tau' X$. Hence, $M \models \neg \tau' X$. □

We can prove the following constructive Lyndon interpolation result using the mosaic interpolants from Lemma 4.

**Theorem 5.** Let $\varphi_L$ and $\varphi_R$ be GF sentences without equality over signatures $\sigma_L$ and $\sigma_R$, respectively. If $M \models \varphi_R$ and $|\varphi_L| + |\varphi_R| = n$, then we can construct a DAG representation of a GNF interpolant $\theta$ such that

- $\varphi_L \models \theta$ and $\theta \models \varphi_R$;
- $\operatorname{occ}(\theta) \subseteq \operatorname{occ}(\varphi_L) \cup \operatorname{occ}(\varphi_R)$;
- $\operatorname{con}(\theta) \subseteq \operatorname{con}(\varphi_L) \cup \operatorname{con}(\varphi_R)$;
- the DAG representation of $\theta$ is of size at most $2^{o(n)}$ for some polynomial function $o$ independent of $\varphi_L$ and $\varphi_R$ (and this can be improved to a size of at most $2^{o(n)}$ when the bound on the arity of relations is fixed).

For brevity, in this theorem and throughout the paper, we give only bounds on the output size, not the running time of the algorithms. However the proofs will show that the worst-case running time is bounded by a polynomial in the output size. In the particular, running the time of the interpolation algorithm above is doubly-exponential in the input (but only singly-exponential for bounded arity GF sentences without equality).

**Proof of Theorem 5.** Let $\theta := \bigvee_{\sigma \subset L \cup L \cup \tau' \tau > \sigma \subseteq N^r} \bigwedge_{s \subseteq \omega \gamma} \psi_L'$ where $N^r$ is the set of eliminated mosaics and we write $\psi'() \models \exists x \psi$ if $\psi'$ is over parameters $\emptyset$ and is obtained from $\psi$ by adding only $Z$-formulas.

We prove that $\psi_L \models \theta \models \psi_R$.

Assume there is a structure $\mathcal{M}$ such that $\mathcal{M} \models \varphi_L$. Let $\tau' = L \cup L \cup \tau' \tau > \sigma \subseteq N^r$ be the set of sentences from $\mathcal{M}(\psi_L)$ that are true in $\mathcal{M}$. For any $\tau' \models \tau' \models \tau > \sigma \subseteq N^r$, $\mathcal{M} \models \tau' \models \theta$. By Lemma 4, this means that $M \models \psi_L'$, so $\theta \models \theta$.

Now assume there is a structure $\mathcal{M}$ such that $\mathcal{M} \models \theta$. Then there is some $\tau(\theta) \models \mathcal{M}(\psi_L)$ such that $\mathcal{M} \models \psi_L' \models \tau' \models \tau > \sigma \subseteq N^r$ such that $\tau' \models \tau > \sigma \subseteq N^r$. Let $S' \subseteq \mathcal{M}(\psi_L)$ be the set of formulas $\psi$ in $\mathcal{M}(\psi_L)$ such that $\mathcal{M} \models \psi$. Consider $\tau := \tau \cup S' \cup \{R : \varphi_R \} \models \tau > \sigma \subseteq N^r$. By completeness of the mosaic system (Theorem 3), every mosaic containing $L : \varphi_L$ and $R : \varphi_R$ must be in $N^r$. Hence, $\mathcal{M} \models \psi_L' \models \tau > \sigma \subseteq N^r$. By Lemma 4 this implies that $\mathcal{M} \models \tau > \sigma \subseteq N^r$. But every formula $\psi$ in $S'$ was chosen such that $\mathcal{M} \models \psi$. The only way that $\mathcal{M} \models \tau > \sigma \subseteq N^r$ is if $\mathcal{M} \models \varphi_R$.

The other properties in Theorem 5 follow from Lemma 4. □

Although GF does not have CIP, it was shown in [14] that GF does admit the following weaker form of interpolation.

**Theorem 6 ([14]).** For GF sentences $\varphi_L \models \varphi_R$ there is a GF sentence $\theta$ such that $\varphi_L \models \theta$ and $\theta \models \varphi_R$ and $\theta$ only uses relations in the common signature or relations occurring as a guard for some quantification in $\varphi_L$ or $\varphi_R$.

We remark that our mosaic interpolation method can be modified to produce GF interpolants like this, with the same DAG-size bounds as in Theorem 5, by modifying the inductive case of Lemma 4 to include the guard that is present already in the existential requirement.

**5. GNF Mosaics**

We now aim to describe mosaics that can handle $\varphi_L$ and $\varphi_R$ in GNFS. For now, we assume $\varphi_L$ and $\varphi_R$ are GF sentences, over signatures $\sigma_L$ and $\sigma_R$, that are in GNFS form, use ordinary quantifiers rather than relativized quantifiers, and do not use equality. Let $\psi := \psi_L \cup \psi_R$.

**Closure.** It turns out that just taking the subformula closure is not sufficient to determine satisfiability of $\varphi_L \land \neg \varphi_R$ from a set of coherent mosaics. Instead, the mosaics must contain additional information about CQ-shaped subformulas (subformulas of the form $\exists \psi' \psi(x,y)$).

Our solution is to add specializations to the closure. Informally, each specialization describes a way in which the original CQ-shaped formula could be satisfied in some tree-like structure.

Consider a GNFS-formula of $\varphi_R$ is of the form $\exists \psi \exists \psi'(x'y)$, an specialization of $\psi$ is a formula $\phi'$ obtained from $\psi$ by the following operations:
- select a subset $z$ of $x$ (call variables from $y$ the inside variables and variables from $x \setminus z$ the outside variables);
- select a partition $x_1, \ldots, x_k$ of the outside variables, with the property that for every $\psi_j$, either $\psi_j$ has no outside variables or all of its outside variables are contained in some partition element $x_i$;
- let $\chi_j$ be the conjunction of the $\psi_j$ using only inside variables, and let $\chi_j$ be the conjunction of the $\psi_j$ using outside variables and satisfying $\psi_j' \subseteq x_i x_j y z$;
- set $\phi'(y z)$ to be $\chi_0(y z) \land \bigwedge_{j \in \{1, \ldots, k\}} \exists x_i \chi_j(x_i y z)$.

We are now ready to define the closure. Each formula is labelled with a provenance $X \in \{L, R\}$ and a polarity $p \in \{+,-\}$. The closure $\text{cl}(X^p : \psi)$ is the smallest set $C$ of formulas containing $X^p : \psi$ and such that:

- if $X^p : \alpha(\psi) \land \neg \psi(y) \in C$, then $X^p : \alpha(\psi) \land \psi(y) \in C$;
- if $X^p : \alpha(\psi) \land \psi(y) \in C$, then $X^p : \alpha(\psi) \land \psi(y) \in C$;
- if $X^p : \bigwedge \psi_i \in C$ or $X^p : \bigvee \psi_i \in C$, then $X^p : \psi_i \in C$ for all $i$;
- if $X^p : \exists \psi(x,y) \in C$, then $X^p : \phi' \in C$ for all specializations $\phi'$ of $\exists \psi(x,y)$;

Given parameters $e$, we let $\text{cl}_{0}(e)$ (respectively, $\text{cl}_{1}(e)$) consist of formulas from $\text{cl}(L^p : \varphi_L)$ (respectively, $\text{cl}(R^p : \varphi_R)$) with free variables replaced by parameters $e$ or constants from $e$.

Roughly speaking, if $X^p : \psi \in \text{cl}_{0}(e) \cup \text{cl}_{1}(e)$, then the provenance $X \in \{L, R\}$ indicates whether this formula came from a subformula of $\varphi_L$ or $\varphi_R$, and the polarity $p \in \{+,-\}$ indicates whether this subformula occurred positively or negatively (i.e., whether it occurred within the scope of an even or odd number of negations). We write $\psi$ to mean $\psi$ (respectively, $\neg \psi$) if $p$ is $+$ (respectively, $-$).

As in GF mosaics, we remark that the formulas in $\text{cl}_{0}(e) \cup \text{cl}_{1}(e)$ respect the polarity of relations in the original formulas.

**Guards.** Let $\tau$ be a collection of formulas from $\text{cl}_{0}(e) \cup \text{cl}_{1}(e)$. We say $b \subseteq e$ is $X$-guarded in $\tau$ (respectively, $X$-guarded in $\tau$) if $b$ is empty, or if there is some $X^p : \exists a \varphi(\psi) \in \tau$ where $\varphi$ is an $X$-atom (respectively, $X$-catom). The definition of $X$-atom and $X$-catom is the same as in Section 4.)
GNF mosaics. Let $e$ be a tuple of parameters such that $|e| \leq \text{width}(\varphi_1 \land \neg \varphi_2)$. A GNF X-mosaic $\tau$ for $\varphi_1 \land \neg \varphi_2$ using parameters $e$ is a subset of $\Gamma(e) := c_{\varphi_1}(e) \cup c_{\varphi_2}(e)$ such that if $X^p : \psi \in \tau$, then par$(\psi)$ is X-cograduated in $\tau$.

Like the GF mosaics, the $X$-formulas are restricted in X-mosaics. Unlike a GF mosaic, this restriction only applies to positive $X$-formulas.

Let $\tau$ be a GNF X-mosaic over parameters $e$. Assume $Y^+ : \eta \in \tau$ for $\eta$ an existential requirement of the form $\exists x. \psi(x,b)$. Then $\tau$ is linked to $\tau'$ via $Y^+ : \eta$ (written $\tau \rightarrow Y^+ : \eta \tau'$) if

- $\tau'$ is a GF Y-mosaic over $a \cup b$ with $a \cap c = \emptyset$;
- $Y^p : \psi'(b) \in \tau'$;
- for all $Y^p : \psi'(b) \in \tau$, $Y^p : \psi'(b) \in \tau'$;
- for all $Y^p : \psi'(b) \in \tau$ such that either par$(\psi')$ is Y-cograduated in $\tau'$ or $p = \neg Y^p : \psi'(b) \in \tau'$;
- for all $Z^p : \psi'(b) \in \tau'$, $Z^p : \psi'(b) \in \tau$.

As in the GF mosaics, the maximum amount of information about shared parameters is passed through a link, and no new information about shared parameters is added in $\tau'$.

Fix some set $P$ of parameters of size $2^{|\text{width}(\varphi_1 \land \neg \varphi_2)|}$, and let $M$ be the set of GNF mosaics over parameters $\hat{P}$. The following result is proven by a routine calculation.

Proposition 7. There is a polynomial function $p$ (independent of $\varphi_1$ and $\varphi_2$) such that if $|\varphi_1| + |\varphi_2| = n$, then there are at most $2^p(n)$ mosaics in $M$. This bound holds even if $\varphi_1$ and $\varphi_2$ are GNF sentences but are not in GN-normal form.

Coherence. A GNF X-mosaic $\tau(e)$ is downward closed if the following properties are satisfied:

- If $Y^+ : \alpha(b) \land \neg \varphi(b) \in \tau$, then $Y^+ : \alpha(b) \in \tau$ and $Y^+ : \varphi(b) \in \tau$;
- If $Y^+ : \alpha(b) \land \neg \varphi(b) \in \tau$ and $Z^+ : \alpha(b) \in \tau$, then $Y^+ : \varphi(b) \in \tau$;
- If $Y^+ : \bigwedge_{i} \varphi_i \in \tau$ (respectively, $Y^+ : \bigvee_{i} \varphi_i \in \tau$), then $Y^+ : \bigwedge_{i} \varphi_i \in \tau$ (respectively, $Y^+ : \bigvee_{i} \varphi_i \in \tau$) for all $i$;
- If $Y^+ : \bigwedge_{i} \varphi_i \in \tau$ (respectively, $Y^+ : \bigvee_{i} \varphi_i \in \tau$), then $Y^+ : \bigwedge_{i} \varphi_i \in \tau$ (respectively, $Y^+ : \bigvee_{i} \varphi_i \in \tau$) for some $i$;
- If $Y^+ : \exists x. \psi(x,b) \in \tau$, then for all specializations $\psi'(y,z)$ of $\exists x. \psi(x,y,z)$ and for all $c' \subseteq c \cup e$ with $|c'| = |z|$, $Y^+ : \psi'(b)(c'/z) \in \tau$.

The last rule in this definition is a generalization of the rule for universal requirements in the GF mosaics. It helps ensure that when we construct a model from a saturated set of internally consistent mosaics, a CQ-shaped subformula that is asserted to be false in one mosaic, does not become true in the constructed model.

As before, a GNF X-mosaic $\tau$ is internally consistent if $\tau$ is downward closed and there is no atomic formula $\alpha$ and no $Y,Z \in \{L,R\}$ such that $Y^+ : \alpha \in \tau$ and $Z^+ : \alpha \in \tau$.

As a set of GNF mosaics is said to be saturated if for every mosaic $\tau(e)$ in the set and for every existential requirement $Y^+ : \eta \in \tau$ for $\eta$ of the form $\exists x. \psi(x,b)$, either (i) $Y^+ : \psi(x,b)(a/x) \in \tau$ for some $a \subseteq c \cup e$ (so $\eta$ is fulfilled in $\tau$) or (ii) $\tau$ is linked to some mosaic $\tau'$ in the set via $Y^+ : \eta$ (so $\eta$ is fulfilled in $\tau'$).

These requirements lead to the following key theorem.

Theorem 8. $\varphi_1 \land \neg \varphi_2$ is satisfiable if and only if there is a saturated set of internally consistent GNF mosaics from $M$ that contains a mosaic $\tau$ such that $L^+ : \varphi_1$, $R^- : \varphi_2 \in \tau$.

Using the elimination procedure described earlier, this gives an alternative proof of the $2\text{EXP}TIME$ upper bound for satisfiability of GNF (as stated in Theorem 1).

Extensions of GNF mosaics. We can consider various extensions of these mosaics, to handle input with free variables, relativated quantifiers, and equality. We briefly summarize some of these extensions.

Relativation. For input that uses $U$-relativated quantifiers, a relativated X-mosaic $\tau(e)$ is a collection of $U$-relativated GNF sentences with parameters from $e$. Each relativized mosaic also has a set of distinguished parameters $d \subseteq e$ (which we call the relativized parameters in the mosaic). These parameters $d$ in $\tau$ will be removed from $\theta_X^\tau$ using existential quantification when constructing the mosaic interpolants. Hence, we add further restrictions on the $X$-formulas that use parameters from $d$ to ensure that we can relativize any quantification over $d$. In particular, if $d \in d$ appears in a $X$-formula in an X-mosaic $\tau$ then there must be some relativizer $X^+ : Ud \in \tau$ such that $U \in U$ is an $X$-catom.

Equality. We have ignored the presence of equality in the mosaic construction, but it can be accommodated by normalizing the use of equality in formulas, and then putting additional “equality consistency requirement” on the mosaics.

Constructive interpolation. Using these extensions of the GNF mosaics, we can lift the restrictions placed on $\varphi_1$ and $\varphi_2$ at the beginning of this section, and derive the following constructive interpolation results, subsuming Theorem 5. This proves constructive RIP for GNF, and is a key contribution.

Theorem 9. Let $\varphi_1$ and $\varphi_2$ be GNF (resp., equality-free UNF) formulas over signatures $\sigma_1$ and $\sigma_2$ respectively. If $\varphi_1 \models \varphi_2$ and $|\varphi_1| + |\varphi_2| = n$, then we can construct a DAG representation of a GNF (resp., UNF) interpolant $\theta$ such that

- $\varphi_1 \models \theta$ and $\theta \models \varphi_2$;
- $\text{occ}(\theta) \subseteq \text{occ}(\varphi_1) \cap \text{occ}(\varphi_2)$;
- $\text{free}(\theta) \subseteq \text{free}(\varphi_1) \cap \text{free}(\varphi_2)$;
- $\text{con}(\theta) \subseteq \text{con}(\varphi_1) \cup \text{con}(\varphi_2)$;
- if $\varphi_1$ and $\varphi_2$ are $U$-relativated, for $\varphi_1$ and $\varphi_2$ a distinguished set of unary relations from $\sigma_1 \cup \sigma_2$, then $\theta$ is $U$-relativated (when treating $\top$ and $\bot$ as atomic formulas);
- the DAG representation of $\theta$ is of size at most $2^{\text{occ}(\theta)}$ for some polynomial function $p$ independent of $\varphi_1$ and $\varphi_2$ (and this can be improved to a size of at most $\text{occ}(\theta)$ when $\varphi_1$ and $\varphi_2$ are in GF without equality and the bound on the arity of relations is fixed).

Note that using this interpolation method, we can also construct an $\text{FO}$ interpolant that can be represented by a doubly-exponential sized formula – e.g. by applying the theorem above, and then converting efficiently from a DAG representation to a formula [3].

Beth definability. As a corollary of Theorem 9, we get doubly exponential bounds on the DAG size of explicit GNF definitions coming from Projective Beth Definability. The argument is standard (see [13]).
Formally, an indexed AND/OR tree is a structure over signature $\rho_n$ consisting of the following predicates: 1. Unary predicates IndValOne, IndDepth0, ..., IndDepthn, and a ternary connective IndChild describe the index of one node. Informally, IndDepth0, ..., IndDepthn, and IndChild describe the structure of a binary tree of depth $n$ (with IndDepthn labelling the root), while IndValOne distinguishes the leaves of this tree with value one. Each such tree can be assigned a value, when viewing the leaves as $2^n$ bits of a binary number. 2. Unary predicates InputOne and IsOr along with binary predicate AOChild describe the AND/OR structure, with IsOr representing an OR node and InputOne distinguishing the inputs on the leaves that have value one.

The signatures of $\chi_n$ and $\chi'_n$ extend $\rho_n$ using two copies of a predicate AOValOne for the intermediate value of the circuit computation at each node. Hence $\rho_n$ will be the common signature for the interpolant.

Roughly speaking, the formula $\chi_n$ asserts there is an indexed AND/OR tree of depth $2^n$, and the value (using unary relation AOValOne) of any AND/OR tree in the structure is 1.

Likewise, $\chi'_n$ asserts the value (using unary relation AOValOne') of any AND/OR tree in the structure is 1.

We have $\chi_n \models \chi'_n$, and there is a polynomial function $p$ such that for all $n$, $|\chi'_n| \leq p(n)$. 

### GN bimilation game
In order to prove that there is no “small” GNF interpolant for $\chi_n \models \chi'_n$, we use a variation of GN bimulations [5].

A position in the $k$-width version of the GN bimilation game between $\mathfrak{A}$ and $\mathfrak{B}$ (relative to some signature $\sigma$) is a partial rigid homomorphism $f$ from $\mathfrak{A}$ to $\mathfrak{B}$ or vice versa with $|\text{dom}(f)| \leq k$. This means that $f$ is a partial homomorphism and for any guarded tuple $c \in \text{dom}(f)$, $f(c)$ (the restriction of $f$ to $c$) is a partial isomorphism. We say the active structure is the structure containing dom($f$).

Starting in position $f$, one round of the game consists of the following: (i) Spoiler can restrict to some subset $c \subseteq \text{dom}(f)$, and then the game proceeds from position $f[c]$; or (ii) if $\text{dom}(f)$ is a guarded tuple $c$, then Spoiler can choose to switch structures, and the game proceeds from position $(f[c])^{-1}$; or (iii) if $|\text{dom}(f)| < k$, Spoiler can select an element $e$ such that $e$ is in the active structure but $e \notin \text{dom}(f)$). Duplicator can choose $d$ in the inactive structure such that $f[c \rightarrow d]$ is a partial rigid homomorphism, and then the game proceeds from position $f[c \rightarrow d]$ (Duplicator immediately loses if she is unable to choose a $d$ such that $f[c \rightarrow d]$ is a partial rigid homomorphism).

We write $\mathfrak{A}, a \rightarrow_{k,m} \mathfrak{B}, b$ if Duplicator has a winning strategy in the $k$-width $m$-round guarded negation bimilation game relative to signature $\sigma$ starting from a partial rigid homomorphism $g : a \rightarrow b$. A winning strategy for Duplicator implies agreement between $\mathfrak{A}$ and $\mathfrak{B}$ on certain GNF formulas.

#### Proposition 11
Assume $\mathfrak{A}, a \rightarrow_{k,m} \mathfrak{B}, b$, i.e. Duplicator has a winning strategy in the $k$-width, $m$-round GN bimilation game between $\mathfrak{A}$ and $\mathfrak{B}$ (relative to signature $\sigma$) starting from position $a \rightarrow b$. If $\varphi(x)$ is a DAG representation of a GNF formula over $\sigma$ such that $|\varphi| \leq m$, width$_h(\varphi) \leq k$, and free($\varphi$) = $x$, then $\mathfrak{A} \models \varphi(a) \implies \mathfrak{B} \models \varphi(b)$. 

#### Counterexamples
The specific family of counterexamples that we use to prove Theorem 10 are an inductively defined family of indexed AND/OR trees $\mathfrak{A}_i^n$ and $\mathfrak{B}_i^n$ of increasing depth. The value of $\mathfrak{A}_i^n$ (respectively, $\mathfrak{B}_i^n$) is 1 (respectively, 0), but the trees have some identical subtrees that make them hard to distinguish with “small” GNF sentences.

Roughly speaking, we show that for $i = 2^{m^n} - 1$, Duplicator has a winning strategy in the $1$-round $i$-width GN bimilation game relative to signature $\rho_n$ between the indexed AND/OR trees $\mathfrak{A}_i^n$ and $\mathfrak{B}_i^n$ (starting from the empty homomorphism). By Proposition 11, this is enough to ensure that $\mathfrak{A}_i^n$ and $\mathfrak{B}_i^n$ cannot be distinguished by GNF sentences with a DAG representation of size at most $i$.

Supposing that there is a “small” (size at most $2^{m^n} - 1$) DAG representation of an interpolant $\theta$, for $\chi_n \models \chi'_n$ implies that $\mathfrak{A}_i^n \models \theta_n$ since $\mathfrak{A}_i^n \equiv \chi_n$ by construction. But $\mathfrak{A}_i^n$ and $\mathfrak{B}_i^n$ cannot be distinguished by such “small” formulas, so this means that $\mathfrak{B}_i^n \models \theta_n$, and hence $\mathfrak{B}_i^n \equiv \chi'_n$. This is impossible since $\mathfrak{B}_i^n$ is an indexed AND/OR tree with value 0, not value 1 as asserted by $\chi'_n$.

We now describe in more detail this family of counterexamples $\mathfrak{A}_i^n$ and $\mathfrak{B}_i^n$ for all $i \leq 2^{m^n} - 1$. For notational simplicity in the description below, we write $\mathfrak{A}_i$ and $\mathfrak{B}_i$ instead of $\mathfrak{A}_i^n$ and $\mathfrak{B}_i^n$.

The basic structure of these AND/OR trees is defined recursively as follows, where we write $a$ to denote a single $a$-labelled node, and $\langle T_1, T_2 \rangle$ to denote a tree where the root is labelled $a$ and the children are the roots of $T_1$ and $T_2$:

- $\mathfrak{A}_0 := 1$ and $\mathfrak{B}_0 := 0$;
- for $i > 0$, $\mathfrak{A}_i := \text{AND/OR}(\mathfrak{A}_{i-1}, \mathfrak{A}_{i-1})$, $\text{(OR}(\mathfrak{B}_{i-1}, \mathfrak{B}_{i-1}))$ and $\mathfrak{B}_i := \text{AND/OR}(\mathfrak{B}_{i-1}, \mathfrak{B}_{i-1})$, $\text{(OR}(\mathfrak{A}_{i-1}, \mathfrak{B}_{i-1}))$.

Evaluating $\mathfrak{A}_i$ (respectively, $\mathfrak{B}_i$) gives value 1 (respectively, 0).

A separate index tree (binary tree of depth $n$) is then attached to each node in the trees described above, where the index correctly describes the depth of this node in the AND/OR tree.

We view these indexed AND/OR trees as structures over the signature $\rho_n = \sigma_n \cap \sigma'_n$. In particular, this means that there are no internal computations using AOValOne or AOValOne'.

Let $h : b_i \rightarrow a_i$ be a position (partial rigid homomorphism) in the $k$-width GN bimilation game between some $\mathfrak{A}_i$ and $\mathfrak{B}_i$. We can assume that $h$ is restricted to elements in the AND/OR tree (rather than the elements in the index trees). This is without loss of generality since $h$ can always be chosen to preserve the depth of positions, and the local structure of the index trees is identical in $\mathfrak{A}_i$ and $\mathfrak{B}_i$.

As mentioned earlier, we want to show that Duplicator has a winning strategy in certain GN bimilation games between $\mathfrak{B}_i$ and $\mathfrak{A}_i$. In order to prove this inductively, we need to introduce the notion of “safe” positions $h$, which represent good starting positions for Duplicator.

We first introduce some additional notation. We write $\mathfrak{B}_i(b)$ for the restriction of $\mathfrak{B}_i$ to the subtree rooted at $b$. For $i > 0$, let $C_i$ (respectively, $D_i$) denote the set of nodes in $\mathfrak{A}_i$ (respectively, $\mathfrak{B}_i$) at depth $2$ (these are the roots of the $\mathfrak{A}_{i-1}$ and $\mathfrak{B}_{i-1}$ subtrees of $\mathfrak{A}_i$ and $\mathfrak{B}_i$). We write $b_i$ for the restriction of $b_i$ to those elements appearing in $\mathfrak{B}_i(b)$.

We now define inductively what it means for $h$ to be $(i, k, m)$-safe. We say a strategy in the $m$-round game is $(i, k, m)$-safe if it uses only $(i, k, m')$-safe positions where $m'$ is the number of remaining moves in the game. We say $h : b_i \rightarrow a_i$ is $(i, k, m)$-safe if it satisfies the following properties:

(S1) for all $b \in b_i$, depth$(b) = \text{depth}(h(b));$
(S2) if $i > 0$, then there exists a mapping $f : D_i \rightarrow C_i$ such that for all $b, b_i \in D_i$,
- if $b, b_i$ are siblings, then $f(b), f(b_i)$ are siblings;
- if $b \in b_i$ is the parent of $b_i$, then $h(d)$ is the parent of $f(b)$;
- if $d \in b$ appears in $\mathfrak{B}_i(b)$, then $h(d)$ appears in $\mathfrak{A}_i(f(b));$
- $\mathfrak{B}_i(b), b_i \rightarrow_{k,m} \mathfrak{A}_i(f(b)), h(b)$ via a strategy for Duplicator that is $(i - 1, k, m)$-safe.

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Likewise, we say a position $b : a \mapsto b$ is $(i, k, m)$-safe if the conditions $(S1)$ and $(S2)$ hold with $\mathcal{B}_i$, $\mathcal{A}_i$, exchanged with $\mathcal{A}_i$, $a, C_i$, respectively.

Observe that the empty partial rigid homomorphism $h$ from $\mathcal{B}_k$ to $\mathcal{A}_k$ is $(i, k, m)$-safe: $(S1)$ vacuously holds, and $(S2)$ holds since ($i > 0$) we can choose $f$ such that sibling relationships are preserved, and $\mathcal{A}_{i-1}$ (respectively, $\mathcal{B}_{i-1}$) subtrees in $\mathcal{B}_i$ are mapped to $\mathcal{A}_{i-1}$ (respectively, $\mathcal{B}_{i-1}$) subtrees in $\mathcal{A}_i$. Similarly for the empty partial rigid homomorphism from $\mathcal{A}_i$ to $\mathcal{B}_i$.

More importantly, we can show that Duplicator has a winning $(i, k, m)$-safe strategy when starting from an $(i, k, m)$-safe position.

**Lemma 12.** For all $k, i \leq 2^{2^n-1}$, and $m \leq i$,

- if $b \mapsto a$ is $(i, k, m)$-safe then $\mathcal{A}_i, a \Delta \mathcal{B}_i, b \Rightarrow \mathcal{A}_i, a$ via a strategy for Duplicator that is $(i, k, m)$-safe;
- if $a \mapsto b$ is $(i, k, m)$-safe then $\mathcal{A}_i, a \Delta \mathcal{B}_i, b$ via a strategy for Duplicator that is $(i, k, m)$-safe.

This implies that for $i = 2^{2^n-1}$, Duplicator has a winning strategy in the $i$-round width GN bisimulation game relative to signature $\rho_n$, between the indexed AND/OR trees $\mathcal{A}_i^n$ and $\mathcal{B}_i^n$ (starting from the empty homomorphism). As argued above, this can be combined with Proposition 11 to prove the lower bound stated in Theorem 10.

**Guarded fragment.** A variant of this construction inspired by [12] can be used to get a lower bound for the size of interpolants of GF validities.

**Theorem 13.** There is a polynomial function $p$ and a family of GF sentences $\chi_n \models \varphi_n$ with relations of unbounded arity (respectively, bounded arity) such that $|\chi_n| + |\varphi_n| \leq p(n)$, and there is no GNF interpolant $\theta_n$ for $\chi_n \models \varphi_n$, with size at most $2^{2^n}$ (respectively, size at most $2^n$), even when $\theta_n$ is represented by a DAG.

**Beth definability.** A similar argument can be used to prove doubly-exponential lower bounds on the DAG-size of GNF explicit definitions coming from Beth Definability.

7. Preservation Theorems

In this section, we consider some preservation theorems for GNF and GF that follow from the constructive interpolation results of Theorem 9. We remark that the results in this section hold when preservation and equivalence is considered over finite structures only, as well as over general structures.

We first consider the analog of the Łoś-Tarski theorem. A formula $\varphi$ is preserved under extensions if $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \models \varphi$ together imply $\mathfrak{B} \models \varphi$ (see, e.g., [9, 22]). It was shown in [7] that every GNF formula preserved under extensions has an equivalent existential GNF formula, which is a GNF formula where no quantifier is in the scope of a negation symbol. We can use relativized GNF interpolation to get a bound on the size of this existential formula. This contrasts with the result of Dawar et. al. [9] that for general first-order formulas there is no elementary bound for the Łoś-Tarski theorem.

**Corollary 14 (Analogue of Łoś-Tarski Theorem).** Let $\varphi$ be a GNF formula over signature $\sigma$ with $|\varphi| = n$. If $\varphi$ is preserved under extensions, then we can construct a DAG representation of an equivalent existential GNF formula $\varphi'$ such that $|\varphi'| \leq 2^{2p(n)}$ for some polynomial function $p$ independent of $\varphi$.

**Proof.** We follow Otto [22]. Let $\mathfrak{U} := \{U^1, U^2\}$ for distinct unary predicates $U^1, U^2$ not occurring in $\sigma$.

Since $\varphi$ is preserved under extensions, we have

$$\exists y. (U^1 y \land \neg U^2 y) \land \bigwedge_{t \in U^1, t} U^1 t \land \varphi \models \varphi'$$

where $\varphi$ denotes $\varphi$ with each $R \in \sigma$ replaced with $R'$, $\varphi'$ denotes $\varphi$ with each $R \in \sigma$ replaced with $R'$.

Replacing every occurrence of some $U^2 z$ in $\varphi'$ by $z = z$ results in a GNF formula $\varphi'$ over $\sigma$ that is equivalent to $\varphi$, and such that any quantifier occurs within the scope of an even number of negations. By pushing the negations to the atomic level (and potentially duplicating guards to cover any negated atoms), we have a doubly-exponential size DAG representation of an existential GNF formula.

Similarly, we can use the Lyndon interpolation result to prove a relationship between monotonicity and positivity. We say a formula $\varphi$ is monotone if it is preserved when a structure is modified via adding tuples to the interpretation of some relation. We say a formula $\varphi$ is positive if all relation symbols (except possibly equality) occur positively (within the scope of an even number of negations). A formula is domain-independent (see [1]) if it depends only on the interpretations of relation symbols.

**Corollary 15 (Monotone=Positive).** Let $\varphi$ be a GNF formula over signature $\sigma$ with $|\varphi| = n$ such that $\varphi$ is monotone. Then we can construct a DAG representation of an equivalent positive GNF formula $\varphi'$ such that $|\varphi'| \leq 2^{2p(n)}$ for some polynomial function $p$ independent of $\varphi$. Furthermore, if $\varphi$ is domain-independent, then $\varphi'$ can be taken to be positive and in existential GNF.

**Proof.** Since $\varphi$ is monotone, we have

$$\bigwedge_{R \in \sigma} \neg \exists y. (R^1 y \land \neg R^2 y) \land \varphi \models \varphi'$$

where $\varphi'$ denotes $\varphi$ with each $R \in \sigma$ replaced with $R'$.

Replacing Theorem 9 yields an interpolant only using relations of the form $R^2$ for $R \in \sigma$. Moreover, these relations must occur positively. Replacing each $R^2 \times R$ yields $\varphi'$.

The above argument is modified in the domain-independent case by using additional predicates for the domains. More precisely, we relativize the antecedent and consequent of the entailment (1) by fresh unary predicates $U^1$ and $U^2$, respectively; and we add $\bigwedge_{R \in \sigma, i \in [1, 2]} \neg \exists x. (\exists y \in U^1 x \land R^2 x \land \neg \exists y \in U^1 x)$ as a conjunct to the antecedent. Applying Theorem 9 as before, and replacing each $R^2 \times R$, as well as $U^2 z$ by $z = z$, yields the desired formula $\varphi'$.

**Guarded fragment.** We now turn to effective preservation for GF. Recall that Andréka et. al. [2] prove a Łoś-Tarski theorem for GF, but with no bound. Here we state a constructive version.
of $Q$ inductively from the leaves of the tree up. The size of containing $C$ conjunctive query that describes the restriction of $c$ factor which is at most exponential in the size of the schema. Note indeed, the guarded unravelling of $M$ proceeds by applying Corollary 14 to obtain a DAG representation.

We sketch the argument for the boolean case. The proof hence implies $(as a formula) is at most doubly exponential in the size of $\chi$. Let $\Lambda_{ij}$ be the set of queries in GF of the form $\exists y. \bigwedge A_{ij}$, where $\bigwedge A_{ij}$ is obtained by identifying variables in $\chi''$, and then adding on at most $2^{|\chi|}$ additional positive atoms. We claim that $\chi'$ is equivalent to $\bigvee_{Q \in A_{ij}} Q$. Since the set of indices $i$ is doubly-exponential in the size of $\chi'$ and each $A_i$ has size doubly-exponential in $\chi'$, this gives the desired bound. Clearly each query in $A_i$ implies $\chi''$ and hence implies $\varphi$. On the other hand, consider a model $M$ satisfying $\varphi$. Given any model $M$ there is another structure $M^*$ agreeing with $M$ on all GF sentences that has a guarded tree decomposition. Indeed, the guarded unravelling of $M$ (see, e.g., [2]) gives such an $M^*$. $M^*$ must satisfy $\varphi$ and therefore must satisfy some $\chi''$ via a homomorphism $h$. Let $C$ be the image of $h$ and $Q_C$ be the conjunctive query that describes the restriction of $M^*$ to the bags containing $C$ – such a query can be written in GF by constructing it inductively from the leaves of the tree up. The size of $Q_C$ is at most the size of $\chi''$ times the maximal size of bags – the latter being a factor which is at most exponential in the size of the schema. Note that every element $c \in C$ corresponds to exactly one variable $y_c$ of $Q_C$. We let $Q'_C$ extend $Q_C$ by adding, for every negated atom in $\chi''$, the result of replacing each variable $x_i$ by $y_{h(x_i)}$. Then $Q'_C$ also holds in $M^*$ and $Q'_C \in GF$. Hence $Q'_C$ holds in $M$ and is in $A_i$, which completes the argument.

We get similar bounds for characterizing monotone domain-independent GF formulas.

**Theorem 17.** Let $\varphi$ be a GF formula over signature $\sigma$ with $|\varphi| = n$. If $\varphi$ is monotone and domain-independent then we can construct an equivalent formula $\varphi''$ that is positive and is in existential GF, and such that $|\varphi''| \leq 2^{2^{2^{p(n)}}}$ for a polynomial function $p$ independent of $\varphi$.

**Proof.** Again we focus on the case where $\varphi$ is boolean.

By Corollary 15, $\varphi$ is equivalent to a positive and existential GF formula $\chi'$ whose DAG representation has doubly-exponential size. As above, we can convert this to a union $\chi'' = \bigvee_{i} \chi''_i$, with $\chi''_i$ in this case being a conjunctive query, but possibly including guarded inequalities. As before, this results in a doubly-exponential blow up in the overall size, but only a singly-exponential blow up in the size of each CQ.

Again letting $A_i$ be the set of queries in GF of the form $\exists y. \bigwedge A_{ij}$, where $\bigwedge A_{ij}$ is obtained by identifying variables in $\chi''_i$ and then adding on at most $2^{|\chi|}$ additional positive atoms, we can show that $\chi'$ is equivalent to $\bigvee_{Q \in A_i} Q$.

8. Conclusions

We have shown that GNF is a rich logic that admits interpolation and preservation theorems with elementary complexity, in sharp contrast to full first-order logic. In doing this, we developed the mosaic method for use in interpolation. In future work, we will look at revisiting the applications of GNF interpolation to databases (as in [6]) in light of these results.

The bounds for interpolation are tight for GNF and UNF. We leave open the question of tightness of our bounds for preservation, as well as the tightness of interpolation for GF with restricted arity and equality. We also do not know the status of many other preservation theorems for GNF, such as preservation under surjective homomorphism, and the status of monotone=positive for non-domain independent GF formulas.

References

A. Mosaic Method and Interpolation using GF Mosaics

In this section, we present details of the GF mosaics defined in Section 4. The purpose of this section is to demonstrate (in the simpler context of GF, rather than GNF) the proof techniques used to prove soundness and completeness of the mosaic method, and how to extract GNF interpolants from it.

A.1 Soundness and completeness of GF mosaic method (Proof of Theorem 3)

We now give detailed proofs of the soundness and completeness of the GF mosaic method.

We must first fix some additional definitions.

We say a formula $X : \psi$ is $X$-safe in $\tau$ if $\text{par}(\psi)$ is $X$-guarded in $\tau$.

Fix parameters $c$. Given a structure $\mathfrak{M}$ over $\sigma_1 \cup \sigma_2 \cup c$ such that $c$ is guarded by some $X$-atom, let $v^Y_\mathfrak{M}$ be the unique $Y$-mosaic over $c$ such that $v^Y_\mathfrak{M} := \{ Y : \psi, Y \in c \}$ and $\mathfrak{M} \models \psi$. The $X$-mosaic $\tau := v^X_\mathfrak{M} \cup \{ \bar{X} : \psi, \bar{X} \in v^Y_\mathfrak{M} \}$ is called the $X$-mosaic $\tau$ of $c$ in $\mathfrak{M}$.

We say an $X$-mosaic $\tau$ over $c$ is realizable if there is some structure $\mathfrak{M}$ such that $\tau$ is the $X$-mosaic of $c$ in $\mathfrak{M}$.

For convenience in the following proofs, we also restate here the definition of linked GF mosaics (updated to use the notion of $Y$-safety).

Let $\tau$ be a GF $X$-mosaic over parameters $c$ with $Y^+ : \eta \in \tau$ for the form $\exists \alpha, \psi(\alpha \beta)$. We say $\tau'$ is linked with $\tau$ via $Y^+ : \eta$ if

- $(E1)$ $\tau'$ is a GF $Y$-mosaic over parameters $ab$ such that $a \cap c = \emptyset$;
- $(E2)$ $\tau' = \alpha \beta(ab)[a][x] \in \tau'$ and $\eta \in \tau'$;
- $(E3)$ for all $Y \in \psi(\beta)(b) \in \tau$;
- $(E4)$ for all $Y \in \psi(\beta)(b) \in \tau$ such that $\bar{Y} : \psi(\beta)(b)$ is $Y$-safe in $\tau'$, $\bar{Y} : \psi(\beta)(b) \in \tau'$;
- $(E5)$ for all $Z : \psi(\beta)(b) \in \tau'$, $Z : \psi(\beta)(b) \in \tau$.

We start with two propositions relating realizability to our coherence requirements for GF mosaics (this will be used for soundness).

Proposition 18. If $\tau$ is realizable, then $\tau$ is internally consistent.

Proof. Let $\mathfrak{M}$ be a structure realizing the $X$-mosaic $\tau(c)$. By definition, $Z : \psi \in \tau$ implies $\mathfrak{M} \models \psi$.

This means it is not possible for there to be some $Y : \alpha$ and $Z : \neg \alpha$ in $\tau$.

The downward closure properties must also be satisfied. For instance, consider the last downward closure property. Assume $Y : \forall \alpha, \psi(\alpha \beta) \rightarrow \psi(\alpha \beta)[a][x] \in \tau$, and there is $c \subseteq e$ such that $Z : \alpha(\beta)[a][x] \in \tau$. We know that $\mathfrak{M} \models \forall \alpha, \psi(\alpha \beta) \rightarrow \psi(\alpha \beta)[a][x]$. If $Y = X$, then this is enough to ensure that $Y : \psi(\beta)(b) \in \tau$. If $Y = X$, then it remains to show that $\mathfrak{M} \models \psi(\beta)(b) \in \tau$.

If $Y = Z$, then $Z : \alpha(\beta)[a][x] \in \tau$ implies that $\text{par}(\alpha(\beta)[a][x] \in \tau)$ is $X$-guarded in $v^X_\mathfrak{M}$, so $Y : \psi(\beta)(b) \in \tau$ is $X$-safe in $v^X_\mathfrak{M}$.

If $Y = Z$, then $\alpha$ appears positively in an $X$-formula (since $Y = X$) and negatively in a $X$-formula (since $Y = X$). Making use of the fact that occurrences are preserved in moving to mosaics, this means that $\alpha$ is an $X$-catom, so $\text{par}(\psi(\beta)[a][x] \in \tau)$ is $X$-guarded by $\alpha(\beta)[a][x] \in v^X_\mathfrak{M}$, and $Y : \psi(\beta)(b) \in \tau$ is $X$-safe in $v^X_\mathfrak{M}$.

It is straightforward to check the other downward closure properties.

Proposition 19. If $\psi$ is realizable, then there is a saturated set $S \subseteq M$ of realizable mosaics that includes $\psi$.

Proof. We inductively construct sets $S_i$, ensuring at each stage $i$ that $S_i$ contains $\psi$ and other realizable mosaics, and that any existential requirement in $S_{i-1}$ is fulfilled in $S_i$. This is enough to ensure that the set $S := \bigcup S_i$ is a saturated set of realizable mosaics that includes $\psi$.

At stage $i = 1$, we set $S_1 := \{ \psi \}$, which is realizable by assumption.

At stage $i > 1$, if every existential requirement in $S_{i-1}$ is fulfilled, then set $S_i := S_{i-1}$.

Otherwise, consider some $X$-mosaic $\tau(c)$ in $S_{i-1}$ with $Y : \eta \in \tau$ for the form $\exists \alpha, \psi(\alpha \beta) \land \psi(\alpha \beta)$ for which there is no $a \subseteq c \cup e$ with $Y : (\alpha(\beta)[a][\alpha][x] \in \tau$, and for all $\tau'$ such that $\tau' \rightarrow Y \rightarrow \tau'$, $\tau' \not\in S_{i-1}$.

Let $\mathfrak{M}$ be a structure realizing $\tau$. Since $\mathfrak{M} \models \tau$, there must be some elements $a' \subseteq a \subseteq M$, parameters $\alpha \in P$ such that $\alpha \cap c = \emptyset$, and an expansion $\mathfrak{M}'$ of $\mathfrak{M}$ with the interpretation $\alpha^{\mathfrak{M}'} := a'$ such that $\mathfrak{M}' \models \alpha(ab) \land \psi(\alpha \beta)$.

This is possible since $|a| \leq \text{width}(\varphi_L \land \neg \varphi_R)$ and $|P| = 2 \cdot \text{width}(\varphi_L \land \neg \varphi_R)$.

Let $\tau'$ be the $X$-mosaic of $ab$ in $\mathfrak{M}'$. This means $\tau'$ is realizable.

We also claim that $\tau' \rightarrow Y \rightarrow \tau''$. $(E1)$ is satisfied by construction. By the choice of $\alpha^{\mathfrak{M}'}$, we have $\mathfrak{M}' \models \alpha(ab) \land \psi(\alpha \beta)$, so $Y : (\alpha(ab) \land \psi(\alpha \beta) \in \tau''$ as required by $(E2)$.

We must now check that formulas with shared parameters $b$ are preserved between $\tau$ and $\tau''$ as described by $(E3)$–$(E5)$.

$(E3)$ and $(E4)$ follow from the definition of realizable mosaics, and the fact that $\mathfrak{M}$ and $\mathfrak{M}'$ are identical with respect to formulas that use only parameters from $b$.

It remains to check $(E5)$. Assume there is some $Z : \psi'(b) \in \tau'$.

We have $\mathfrak{M}' \models \psi'$ since $\tau'$ is realized by $\mathfrak{M}'$. Recall that $\tau$ is an $X$-mosaic, $\tau'$ is a $Y$-mosaic, and $\psi'$ is a $Z$-formula. We proceed by cases depending on the provenances $X$, $Y$, and $Z$. The result is immediate if $Z = X$. Otherwise, if $Z = \bar{X}$, it remains to show that $\mathfrak{M} \models \psi'(b)$ is $Y$-safe in $\tau$.

Assume $Z = \bar{X}$ and $Y = X$. Since $Z = \bar{X}$ and $Z : \psi' \in \tau'$ implies $\text{par}(\psi') = b' \subseteq b$ is $Y$-guarded in $\tau'$ by some $Y : \exists \beta, \beta(y)b') \in \tau'$ where $\beta$ is a $Y$-catom. But this means $\mathfrak{M}' \models \exists \beta, \beta(y)b')$ and $\mathfrak{M}' \models \exists \beta, \beta(y)b')$. Moreover, $X = Y$, so we can conclude that $Y : \exists \beta, \beta(y)b') \in \tau$ and hence $\mathfrak{M} \models \psi'(b) \in \tau$.

Assume $Z = \bar{X}$ and $Y = X$. Since $Y : \eta$ in $\tau$ uses $b$ and $Y = \bar{X}$, we know that $b$ is $Y$-guarded in $\tau$. But this means $\mathfrak{M} \models \psi'(b)$ is $Y$-safe in $\tau$. 

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We have constructed a realizable Y-mosaic \( \tau' \) such that \( \tau \rightarrow_{\eta, \eta} \tau' \) fulfils the existential requirement \( Y : \eta \in \tau \), so we add \( \tau' \) to \( S \). Repeating this procedure for all such \( \tau \) (and all such \( Y : \eta \in \tau \)) in \( S_{k-1} \), we will end up with a set \( S \) of realizable mosaics such that any existential requirement in \( S_{k-1} \) is fulfilled in \( S \).

We now aim towards proving a proposition useful for completeness (a sort of converse of Proposition 19). In order to do this, we first prove two lemmas about the transfer of formulas between linked mosaics.

**Lemma 20** (Backwards transfer property). Let \( \tau_0 \rightarrow \cdots \rightarrow \tau_k \) be linked mosaics such that every \( \tau_i \) includes parameters \( b \). If \( X : \psi(b) \in \tau_k \), then \( X : \psi(b) \in \tau_0 \).

**Proof.** We proceed by induction on \( k \). The base case for \( k = 0 \) is immediate.

Otherwise, assume \( k > 0 \). Consider \( \tau_{k-1} \rightarrow_{\eta, \eta} \tau_k \). By definition (E5), \( X : \psi(b) \in \tau_k \) implies \( X : \psi(b) \in \tau_{k-1} \) (since \( b \) is shared by \( \tau_{k-1} \) and \( \tau_k \)), so we can apply the inductive hypothesis to \( \tau_0 \rightarrow \cdots \rightarrow \tau_{k-1} \) to get the result.

**Lemma 21** (Forward universal transfer property). Let \( \tau_0 \rightarrow \cdots \rightarrow \tau_k \) be linked mosaics such that every \( \tau_i \) includes parameters \( b \). If \( Z : \forall x. \alpha(xb) \rightarrow \psi(xb) \in \tau_0 \) and \( Z' : \exists x. \alpha(xb) \in \tau_k \), then \( Z : \forall x. \alpha(xb) \rightarrow \psi(xb) \in \tau_k \).

**Proof.** We proceed by induction on \( k \). If \( k = 0 \), then the result is immediate.

Assume \( k > 0 \), with \( \tau_0 \) an \( X \)-mosaic and \( \tau_1 \) a \( Y \)-mosaic. In order to apply the transfer to \( \tau_0 \), we need to show that \( Z : \forall x. \alpha(xb) \rightarrow \psi(xb) \in \tau_1 \).

- If \( Z = Y \), then \( Z : \forall x. \alpha(xb) \rightarrow \psi(xb) \in \tau_1 \) by (E3).
- If \( Z = \tilde{Y} \) and \( Y = X \), then \( b \) is \( X \)-guarded in \( \tau \) by some \( X : \exists y. \beta(yb) \in \tau_0 \). By (E3), \( X : \exists y. \beta(yb) \in \tau_1 \), so \( Z : \forall x. \alpha(xb) \rightarrow \psi(xb) \) is \( X \)-safe in \( \tau_1 \), and hence in \( \tau_1 \).
- If \( Z = \tilde{Y} \) and \( Y = X \), then \( Z' : \exists x. \alpha(xb) \in \tau_1 \) by the backwards transfer property (Lemma 20). If \( Z' = \tilde{Y} \), then this means \( b \) must be \( Y \)-guarded in \( \tau_1 \). If \( Z' = Y \), then \( Z' : \exists x. \alpha(xb) \in \tau_1 \) witnesses the fact that \( b \) is \( Y \)-guarded in \( \tau_1 \). In either case, we can apply (E4) to get \( Z : \forall x. \alpha(xb) \rightarrow \psi(xb) \in \tau_1 \).

**Proposition 22.** Let \( S \subseteq M \) be a saturated set of internally consistent mosaics. Then for all \( \tau \in S \), there is a structure \( \mathfrak{M} \) such that \( \mathfrak{M} \models \psi \) for all \( \psi \in \tau \) with \( \exists \bar{x} \phi \) in \( \mathfrak{M} \).

**Proof.** Let \( \tau \in S \) be a mosaic containing \( L : \varphi_L, R : \neg \varphi_R \). Note that \( \tau \) is internally consistent by the definition of \( \mathfrak{M} \). Our goal is to construct a model \( \mathfrak{M} \) witnessing the satisfiability of \( \varphi_L \land \neg \varphi_R \) using the mosaics in \( S \) as building blocks.

We do this by building inductively a tree decomposition \( T \) of \( \mathfrak{M} \), together with a map \( f \) from nodes \( v \) in the tree decomposition to a mosaic in \( S \). We also think of \( f \) as a map from the elements in a node \( v \), to the parameters in the corresponding mosaic.

At stage 0, let \( T_0 \) consist of a single node \( v_0 \) containing a copy of the parameters in \( \tau \) and constants \( e \), and set \( f(v_0) = \tau \).

At stage \( i \), consider the set \( S = \{(v, Y : \eta) : v \text{ is a leaf in } T_{i-1} \text{ and } Y : \eta(f(b)) \in f(v) \text{ is not fulfilled in } f(v) \text{ for any form } \exists \bar{x} \alpha \land \psi(f(b))\} \).

For each \((v, Y : \eta) \in S \), we know that there is some \( \tau' \in S \) such that \( f(v) \rightarrow_{\eta, \eta} \tau' \) (since \( S \) is saturated). We construct \( T_i \) extending \( T_{i-1} \) according to the following procedure: for each \((v, Y : \eta) \in S \), construct a child \( v' \) of \( v \) with \( T(v') = a \cup b \cup e \) for new elements \( a \) and \( b \), and define \( f(v') = \tau' \) such that \( Y : \alpha(f(ab)) \land \psi(f(ab)) \in \tau' \).

Let \( T \) be the limit of this process, and let \( \mathfrak{M} \) be the structure with elements \( M = \bigcup_{v \in T} f(v) \), and such that \( \mathfrak{M} \models \alpha(a) \) iff there exists \( v \) such that \( a \) appears in \( f(v) \) and \( \alpha(f(a)) \in f(v) \).

We must now show that for all \( v \) in \( T \) and for all \( a \) in \( T \) there is \( Z : \varphi(f(a)) \in f(v) \), then \( \mathfrak{M} \models \varphi(a) \). The proof is by induction on the structure of \( \psi \).

- Assume \( Z : \psi \) is an atom. Then \( \mathfrak{M} \models \psi \) by construction.
- Assume \( Z : \psi \) is the negation of an atom \( \alpha(a) \). Suppose for the sake of contradiction that \( \mathfrak{M} \models \alpha(a) \). Then there is some \( w \) such that \( \alpha \subseteq T(w) \), \( X : \alpha(f(a)) \in f(w) \), and parameters \( f(a) \) appear in \( f(w') \), for all \( w' \) on the path between \( v \) and \( w \). Using the backwards transfer property (Lemma 20), there is a node containing both \( Z : \neg \alpha(f(a)) \) and \( X : \alpha(f(a)) \), which contradicts internal consistency of mosaics in \( S \).
- Assume \( Z : \psi \) is of the form \( \psi_1 \lor \psi_2 \). Then by internal consistency of \( v \), we must have \( Z : \psi_1 \in f(v) \) or \( Z : \psi_2 \in f(v) \), so we can apply the inductive hypothesis to get the desired result.
- Assume \( Z : \psi \) is of the form \( \psi_1 \land \psi_2 \). Then by internal consistency of \( v \), we must have \( Z : \psi_1 \in f(v) \) and \( Z : \psi_2 \in f(v) \), so we can apply the inductive hypothesis to get the desired result.
- Assume \( Z : \psi \) is of the form \( \forall x. \alpha(x, f(b)) \rightarrow \psi'(x, f(b)) \), and assume for the sake of contradiction that there is some \( a \in M \) such that \( \mathfrak{M} \models \alpha(ab) \land \neg \psi'(ab) \). Then there must be some \( w \) such that \( ab \subseteq T(w) \), \( X : \alpha(f(ab)) \in f(w) \), and parameters \( f(b) \) appear in \( f(w') \), for all \( w' \) on the path between \( v \) and \( w \). By internal consistency, \( X : \exists x. \alpha(x, f(b)) \in f(w) \).

Using the backwards transfer property (Lemma 20), there is a node \( v' \subseteq v \) such that \( v' \) and \( w \) are on the same branch in \( T \) and \( Z : \forall x. \alpha(x, f(b)) \rightarrow \psi'(x, f(b)) \in f(v') \).

If \( w < v' \) (respectively, \( v' < w \)), then the backwards transfer property in Lemma 20 (respectively, forwards universal transfer property in Lemma 21) means that \( Z : \forall x. \alpha(x, f(b)) \rightarrow \psi'(x, f(b)) \in f(w) \). Within \( f(w) \), internal consistency implies that \( \psi'(f(ab)) \in f(w) \) as well, a contradiction.
• Assume \( Z : \phi \) is of the form \( \exists x.\alpha(x) \land \psi'(x) \). If \( Z : \alpha(f(a)) \land \psi'(f(a)) \in f(v) \) for some \( a \), then we can apply the inductive hypothesis and we are done. Otherwise, there must be some child \( w \) of \( v \) such that \( f(v) \rightarrow_z \phi \). If \( a \subseteq T(w) \) such that \( Z : \alpha(f(a)) \land \psi'(f(a)) \in f(w) \). The result easily follows from the inductive hypothesis applied to \( Z : \alpha(f(a)) \land \psi'(f(a)) \in f(w) \).

We are now ready to prove Theorem 3. For soundness we must show:

If \( \varphi_{L} \land \neg \varphi_{R} \) is satisfiable, then there is a saturated set of internally consistent GNF mosaics from \( M \) that contains some \( \tau \) such that \( L : \varphi_{L}, R : \neg \varphi_{R} \in \tau \).

Since \( \varphi_{L} \land \neg \varphi_{R} \) is satisfiable, there is a model \( M \) of \( \varphi_{L} \land \neg \varphi_{R} \). Let \( \tau \) be the L-mosaic (equivalently, R-mosaic) of \( \emptyset \) in \( M \), i.e. the set of sentences from \( cl_{L}(\emptyset) \cup cl_{R}(\emptyset) \) that are true in \( M \). Since \( \varphi_{L} \land \neg \varphi_{R} \) holds in \( M \), \( \tau \) contains \( L : \varphi_{L} \) and \( R : \neg \varphi_{R} \). Since \( \tau \) is a realizable mosaic, Proposition 19 implies that there is a saturated set \( S \) of realizable mosaics including \( \tau \). Moreover, by Proposition 18, \( S \) must contain only internally consistent mosaics, as desired.

For completeness of the GF mosaic method, we must show:

If there is a saturated set of internally consistent mosaics from \( M \) containing \( \tau \) such that \( L : \varphi_{L}, R : \neg \varphi_{R} \in \tau \), then \( \varphi_{L} \land \neg \varphi_{R} \) is satisfiable.

This follows immediately from Proposition 22.

A.2 Bound on number of GF mosaics (Proposition 2)
Recall the statement.

There is a polynomial function \( p \) (independent of \( \varphi_{L} \) and \( \varphi_{R} \)) such that if \( |\varphi_{L}| + |\varphi_{R}| = n \), then there are at most \( 2^{2p(n)} \) mosaics in \( M \). Moreover, for a fixed bound on the arity of relations, there are at most \( 2^{p(n)} \) mosaics in \( M \).

Let \( l := |e| \) and \( m := \text{width}(\varphi_{L} \land \neg \varphi_{R}) \). Note that \( l, m \leq n \).

We first calculate the size of \( cl_{L}(P) \) (recall that \( P \) is a fixed set of parameters of size \( 2m \)). There are at most \( n \) subformulas of \( \varphi_{Y} \), but \( cl(\varphi_{Y}) \) also contains new formulas formed by taking some atom and quantifying out some subset of the \( m \) variables. This means that the size of \( cl(\varphi_{Y}) \) is at most \( 2n \). Each formula in \( cl(\varphi_{Y}) \) has at most \( m \) variables that can be mapped to parameters from \( P \) or constants from \( e \), so the size of \( cl_{L}(P) \) is \( M := 2n2^{m} \).

Every mosaic in \( M \) is labelled L or R and contains a subset of the formulas in \( cl_{L}(P) \cup cl_{R}(P) \). Hence, \( |M| \leq 2 \cdot 2^{2M} \leq 2^{2p(n)} \) for some polynomial function \( p \) that is independent of \( \varphi_{L} \) and \( \varphi_{R} \).

In the special case that the arity \( m \) is bounded, then \( M \) is polynomial in \( n \), and \( |M| \leq 2^{p(n)} \) for some polynomial function \( p \).

A.3 Construction of GF interpolation for GF (additional details for Lemma 4)
We recall the statement and definition of the mosaic interpolants given already Section 4, and then sketch the proof of correctness for one of the inductive cases.

For each GF X-mosaic \( \tau(e) \in N'' \) (the set of eliminated mosaics), we can construct a DAG representation of a formula \( \theta_{X} \) such that

\[
\begin{align*}
\text{(Imp)} & \quad \tau(e) \models \theta_{X} \land \theta_{X}' \models \neg \tau(e); \\
\text{(Occ)} & \quad \text{occ}(\theta_{X}) \subseteq \text{occ}(\varphi_{L}) \cap \text{occ}(\varphi_{R}) \text{ if } X = L, \\
& \quad \text{occ}(-\theta_{X}) \subseteq \text{occ}(\varphi_{L}) \cap \text{occ}(\varphi_{R}) \text{ if } X = R; \\
\text{(Par)} & \quad \text{par}(\theta_{X}) \subseteq \text{c}; \\
\text{(GN)} & \quad \theta_{X}' \text{ is in GNF, even when parameters are viewed as free variables.}
\end{align*}
\]

Moreover, if \( n = |\varphi_{L}| + |\varphi_{R}| \) then there is a DAG representation of \( \Theta_i := \{ \theta_{X} : \tau \text{ is a } Z\text{-mosaic in } N_i \} \) such that \( |\Theta_i| \leq i \cdot 2^{p'(n)} \) for the fixed arity case), where \( p' \) is some polynomial function independent of \( \varphi_{L} \) and \( \varphi_{R} \).

**Proof.** We proceed by induction on \( i \), the stage at which \( \tau \) was eliminated. We use \( \top \) (respectively, \( \bot \)) as an abbreviation for the GNF sentence \( \exists x. x = x \) (respectively, \( \exists x. (x = x \land \neg (x = x)) \)).

During the interpoint construction, we will need to include guards in order to remain in GNF, so we introduce some notation to help with this. For any tuple \( b \) of parameters that are \( X\)-guarded in \( \tau \), we define a formula \( \text{gdd}_{X}(b) \). If \( b = \emptyset \), then set \( \text{gdd}_{X}(b) := \top \). Otherwise, there is some formula \( X : \exists \alpha, \beta(x) \in \tau \) for some \( X\)-atom \( \beta \), and we set \( \text{gdd}_{X}(b) := \exists x. \beta(x) \) (note that \( x \) can be empty, in which case this is just an atomic formula). We will also write \( \text{gdd}_{X}(\beta(x) \land \neg \psi) \) to indicate some formula \( \exists x. (\beta(x) \land \neg \psi) \).

**Base case.** Consider an X-mosaic \( \tau(e) \in N_i \). Then \( \tau \) has an internal inconsistency, and we consider the following cases:

• Assume one of these conditions holds for \( Y \in \{ L, R \} \):
  - \( Y : \alpha(b), Y : \neg \alpha(b) \in \tau \); 
  - \( Y : \alpha(a \beta) \in \tau, \text{ but } \exists x. \alpha(\beta(x)) \notin \tau \); 
  - \( Y : \exists \alpha. \alpha(\beta) \in \tau, \text{ but } Y : \exists \beta. \alpha(\beta(x)) \notin \tau \); 
  - \( Y : \psi_1 \land \psi_2 \in \tau, \text{ but } (Y : \psi_1 \notin \tau \text{ or } \psi_2 \notin \tau) \); 
  - \( Y : \psi_1 \lor \psi_2 \in \tau, \text{ but } (Y : \psi_1 \notin \tau \text{ and } \psi_2 \notin \tau) \); 
  - \( Y : \exists x. \beta(x) \in \tau \).
\[ Y : \alpha(xb)[a/x], Y : \forall x. \alpha(xb) \rightarrow \psi(xb) \in \tau, \text{ but } Y : \psi(xb)[a/x] \notin \tau. \]

If \( Y = X \), then \( \theta^c_X := \perp \). If \( Y = \bar{X} \), then \( \theta^c_X := \top \).

\[ \bullet \text{ If } X = \alpha(b), X : \alpha(b) \in \tau, \text{ then } \theta^c_X := \alpha(b). \]

\[ \bullet \text{ If } X = -\alpha(b), x : \alpha(b) \in \tau, \text{ then } \theta^c_X := \neg \alpha(b). \]

\[ \bullet \text{ If } X : \alpha(xb)[a/x], \bar{X} : \forall x. \alpha(xb) \rightarrow \psi(xb) \in \tau, \text{ but } \bar{X} : \psi(xb)[a/x] \notin \tau, \text{ then } \theta^c_X := \alpha(xb)[a/x]. \]

\[ \bullet \text{ If } \bar{X} : \alpha(xb)[a/x], X : \forall x. \alpha(xb) \rightarrow \psi(xb) \in \tau, \text{ but } X : \psi(xb)[a/x] \notin \tau, \text{ then } \theta^c_X := \neg \alpha(xb)[a/x]. \]

\[ \text{Inductive case. Now consider an } X\text{-mosaic } \tau(c) \in N^c \setminus N^c_{i-1}. \text{ This means there is some } Y : \eta = Y : \exists x. \alpha(xb) \land \psi(xb) \in \tau \text{ that is not fulfilled in } \tau, \text{ and if } \tau \rightarrow Y, \eta \text{ then } \tau' \in N^c_{i-1}. \]

Assume \( \text{par}(\psi) = b \subseteq c \), and fix some \( a \in P \setminus c \).

We introduce some additional notation. For \( Z \in \{L, R\} \), we define \( \tau^Z \) to be the \( Z\)-mosaic that results from taking the \( Z\)-formulas in \( \tau \) that only use parameters from \( b \), and then unifying with the \( Z\)-formulas in \( \tau \) whose parameters are contained in \( b \) and are \( Z\)-caged in \( \tau \). For mosaics \( v, v' \) and \( Z \in \{L, R\} \), we write \( v'(ab) \supseteq_Z v \) if \( v' \) uses only parameters from \( ab \) and is obtained from \( v \) by adding only \( Z\)-formulas (so the \( Z\)-formulas in \( v \) and \( v' \) are identical).

If \( Y = X \), then define \( \theta^c_X \) to be

\[ \bigvee_{v \subseteq X \tau^c_X} \exists x. \left( \bigwedge_{v \in X \tau^c_X} \theta^c_X[x/a] \right). \]

We now begin the proof of correctness, starting with (Imp). Assume there is a model \( M \) for \( \tau_X \). Since \( X : \eta \in \tau \), this means that \( M \models \exists x. \alpha(xb) \land \psi(xb) \), so there are elements \( a' \in M \) and an expansion \( M' \) of \( M \) with the interpretation \( a'_{\text{new}} := a' \) such that \( M' \models \alpha(b) \land \psi(ab) \).

Take \( v' \) to be the union of \( \tau^c_X \) and the set of formulas \( X : \psi \in \text{cl} \psi(ab) \) such that \( M' \models \psi \). It can be checked that \( \tau \rightarrow X, \eta \). By construction, \( M' \models v' \). Moreover, for any choice of \( v'(ab) \supseteq_X v', M' \models \tau^c_X \) (since the \( X\)-formulas are identical in \( v' \) and \( v' \)). By the inductive hypothesis, this means that \( M' \models \theta_X^c \). Hence, \( M' \models \bigwedge_{v(ab) \supseteq_X v' \tau^c_X} \theta^c_X \) and \( M \models \theta^c_X \).

Now assume that there is a model \( M \) of \( \theta^c_X \). Then there is some \( v'(ab) \supseteq_X \tau^c_X \) with \( \tau \rightarrow X, \eta \) \( v' \), elements \( a' \in M \), and an expansion \( M' \) of \( M \) with the interpretation \( a'_{\text{new}} := a' \), such that \( M' \models \bigwedge_{v(ab) \supseteq X v' \tau^c_X} \theta^c_X \).

Note that for all \( v'(ab) \supseteq_X v' \) such that \( \tau \rightarrow X, \eta \), \( M' \models v' \) and consequently, the inductive hypothesis implies \( M' \models \neg v' \). In particular, consider \( v' := \psi \cup S' \) where \( S' \) is the set of formulas \( \bar{X} : \psi \in \text{cl} \psi(ab) \) such that \( M' \models \psi \), \( \text{par}(v') \cap a \neq \emptyset \), and \( \psi : X\text{-safe in } v' \) as required in an \( X\)-mosaic. It is clear that \( v'(ab) \supseteq_X v' \) and \( \tau \rightarrow X, \eta \), so \( M' \models \neg v' \) as described above.

Since \( M' \models \neg v' \), there is some conjunct \( \chi \) in \( v' \) such that \( M' \models \neg \chi \). Consider some conjunct \( \chi(ab) \) in \( v' \) that actually uses some parameters from \( a \). Then by choice of \( S' \), \( M' \models \chi(ab) \), so this formula cannot witness the fact that \( M' \models \neg \chi \). This means there must be some conjunct \( \chi(b) \) in \( v' \) that only uses parameters from \( b \) such that \( M' \models \neg \chi(b) \). If \( \bar{X} : \chi(b) \in \tau \), then \( \bar{X} : \chi(b) \in \tau \) by the definition of \( \tau \rightarrow X, \eta \). Likewise, if \( \bar{X} : \chi(b) \notin \tau \), then it must be the case that \( \text{par}(\chi) \) is \( X\)-caged in \( \tau' \), so \( \bar{X} : \chi(b) \notin \tau \) (otherwise it would contradict the definition of \( \tau \rightarrow X, \eta \)). In either case, this means \( \chi(b) \) must be a conjunct in \( \tau_X^c \). Hence, \( M \models \neg v' \).

For (Occ), recall that for every atomic relation \( S \) in \( \psi \) (where \( \psi \) is not the equality relation), \( (S, +) \in \text{occ}(\phi) \) (respectively, \( (S, -) \in \text{occ}(\phi) \)) if \( S \) occurs positively (respectively, negatively) in \( \phi \). The desired occurrence of constants and the polarity of relations follows from the inductive hypothesis and the fact that the polarity of occurrences of relations are preserved when mosaicing.

The last thing to prove is (Par) and (GN). The inductively defined interpolants \( \theta^c_X \) are in GNF and only use parameters in \( \tau ' \), i.e. parameters from \( ab \). Any parameters from \( a \) in \( \theta^c_X \) are removed in \( \theta^c_X \) using existential quantification. Overall, this means that (Par) and (GN) hold for \( \theta^c_X \).

If \( Y = \bar{X} \), then define \( \theta^c_X \) to be

\[ \text{gdd}^c_X(b) \land \neg \left( \bigvee_{v' \subseteq X \tau^c_X} \exists x. \left( \bigwedge_{v \in X \tau^c_X} \theta^c_X[x/a] \right) \right). \]

(\emph{Imp}) is shown by arguing that \( \tau^c_X \models -\theta^c_X \) and \( -\theta^c_X \models -\tau^c_X \).

The main difference is that we need to ensure the negation is guarded in order to prove (GN). Using the inductive hypothesis, it can be shown that inside the negation, the formula satisfies (GN), and only uses parameters from \( b \) (any parameters from \( a \) are removed using quantification). Because \( \bar{X} : \eta \) is an \( X\)-formula using \( b \) in an \( X\)-mosaic, we know that \text{gdd}^c_X(b) exists. This is enough to conclude that \( \theta^c_X \) is in GNF, even when the parameters are viewed as free variables.

\textbf{Size of interpolants.} Finally, we seek to bound the size of the shared DAG representation of all \( \theta^c_X \) coming from \( \tau \in N^c_i \).

A routine calculation shows that the number of nodes and edges added to the shared DAG representation at each stage \( i \) is polynomial in the number of mosaics in \( N^c_i \). Since the number of mosaics in \( N^c_i \) is at most doubly exponential in \( n \) (respectively, singly exponential
in \( n \) in the fixed arity case) by Proposition 2, this implies that there is some polynomial function \( p' \) independent of \( \varphi_L \) and \( \varphi_R \) such that \(|\Theta_I| \leq i \cdot 2^{p'(n)}\) (respectively, \(|\Theta_I| \leq i \cdot 2^{p'(n)}\)).

**Quasi-interpolants.** We can derive the quasi-interpolant result (Theorem 6). Recall the statement:

For GF sentences \( \varphi_L \models \varphi_R \) there is a GF sentence \( \theta \) such that \( \varphi_L \models \theta \) and \( \theta \models \varphi_R \) and \( \theta \) only uses relations in the common signature or relations occurring as a guard for some quantification in \( \varphi_L \) or \( \varphi_R \).

We modify the inductive case of Lemma 4 as follows:

Assume there is an \( X \)-mosaic \( \tau(c) \in \mathcal{N}_i \setminus \mathcal{N}_{i-1} \), with \( Y : \eta = Y : \exists x. \alpha(xb) \land \psi(xb) \in \tau \), and such that \( \tau' \in \mathcal{N}_{i-1} \) for all \( \tau \rightarrow_{Y,\eta} \tau' \).

If \( Y = X \), then define \( \theta_X' \) to be

\[
\bigvee_{\psi'(ab) \supseteq X} \exists x. \left( \alpha(xb) \land \bigwedge_{\tau' \rightarrow_{X,\eta} \psi'} \theta_X'(x/a) \right).
\]

If \( Y = X \), then define \( \theta_X' \) to be

\[
gdd_X(b) \land \neg \bigvee_{\psi'(ab) \supseteq X} \exists x. \left( \alpha(xb) \land \bigwedge_{\tau' \rightarrow_{X,\eta} \psi'} \theta_X'(x/a) \right).
\]

In other words, we just use the guard that is in the formula \( \exists x. \alpha(xb) \land \psi(xb) \) that is guiding the interpolant construction at that stage.

Because of property (Par), each \( \theta_Y \) can only use parameters from \( ab \). After \( a \) is removed by quantification, this means that \( \alpha(xb) \) guards \( \theta_Y(x/a) \) for all \( \tau'(ab) \) with \( \tau \rightarrow_{Y,\eta} \tau' \).

**B. Relativized GNF Mosaics**

In this section, we describe GNF mosaics in more detail. Recall that in Section 5, we defined GNF mosaics for GF sentences without equality and with ordinary (rather than relativized) quantifiers. For space considerations, we omit the proofs for those mosaics. Instead, we now define a more general form of GNF mosaics for relativized GNF formulas (still without equality), and give full proofs of soundness and completeness for this more complicated case. Note that we overload all of the terminology and notation that was used for the GF mosaics and and GNF mosaics defined in the body.

**B.1 Weak GN-normal form**

The mosaics described in this section will use formulas that are in weak GN-normal form, a slight relaxation of the GN-normal form defined in the body. Formally, weak GN-normal form \( \phi \) are formulas that can be generated using the following recursive definition:

\[
\phi ::= \bigvee_i \exists x. \Lambda_j \psi_{ij} \quad \psi ::= R t \mid \alpha(x) \land \phi(x) \mid \alpha(x) \land \neg \phi(x)
\]

where \( R \) is either a relation symbol or the equality symbol, \( \alpha(x) \) is a guard with \( \text{free}(\alpha) = x \), and \( t \) is a vector over variables and constants.

The idea is that the grammar generates UCQ-shaped formulas, where each conjunct is either an atom, an answer-guarded UCQ-shaped formula, or a guarded negation of a UCQ-shaped formula. Note that a formula in GN-normal form is in trivially in weak GN-normal form.

Given a formula \( \phi \) in GN-normal form or weak GN-normal form, we define \( \text{size}_{\text{UCQ}}(\phi) \) (respectively, \( \text{rank}_{\text{UCQ}}(\phi) \)) to be the maximum size (respectively, maximum number of conjuncts \( \psi_i \)) of any UCQ-shaped subformula \( \exists x. \Lambda_i \psi_i \) of \( \phi \) for non-empty \( x \). For the purposes of the CQ-rank, note that formulas \( \alpha(x) \land \phi(x) \) and \( \alpha(x) \land \neg \phi(x) \) are treated as single conjuncts in a UCQ-shaped subformula.

We write \( \text{guards}(\phi) \) to denote the set of atoms that appear as guards of a negation in \( \phi \) (where \( \phi \) is in GNF, weak GN-normal form, or GN-normal form).

Any GF sentence can be converted to weak GN-normal form with an exponential blow up in overall size, and preserving the polarity of relations.

**Lemma 23.** Let \( \phi \) be a GNF formula. We can construct an equivalent \( \phi' \) in GN-normal form such that

- \( |\phi'| \) is exponential in \( |\phi| \);
- \( \text{width}(\phi') \leq |\phi| \);
- \( \text{size}_{\text{UCQ}}(\phi') \leq |\phi| \);
- \( \text{rank}_{\text{UCQ}}(\phi') \leq |\phi| \);
- \( \text{occ}(\phi') = \text{occ}(\phi) \);
- \( \text{guards}(\phi') = \text{guards}(\phi) \).

**Proof.** Use the transformations described in [5] to convert to GN-normal form.

The advantage of weak GN-normal form over GN-normal form is GF sentences can be converted to weak GN-normal form with only a polynomial blow-up.

**Lemma 24.** Let \( \phi \) be a GF sentence. We can construct an equivalent \( \phi' \) in weak GN-normal form such that
\begin{itemize}
  \item $|\phi'|$ is polynomial in $|\phi|$;
  \item width($\phi'$) = width($\phi$);
  \item rank$_{CQ}(\phi') = 1$;
  \item occ($\phi'$) = occ($\phi$).
\end{itemize}

**Proof.** The idea is that, for every subformula of $\phi$, we can find the innermost guard from a guarded quantification, and copy this guard to that subformula. The resulting sentence is in weak GN-normal form. Note that this result only holds for GF sentences (not GF formulas).

Formally, fix a signature $\sigma$, and let $w$ be the maximum arity of any relation in $\sigma$. We prove a slightly stronger statement:

For every GF formula $\phi(y) := \alpha(y) \land \psi(y)$ over signature $\sigma$, we can construct $\phi'(y) := \alpha(y) \land \psi'(y)$ in weak GN-normal form such that $|\psi'| \leq (w|\psi|^2)^2$, width($\psi'$) = width($\psi$), rank$_{CQ}(\psi') = 1$.

We proceed by structural induction on $\psi$.

- Assume $\psi$ is atomic. Then set $\psi' := \psi$.
- Assume $\psi = \eta \land \phi$. Then set $\psi' := \psi$.
- Assume $\psi = \phi_1 \land \phi_2$. Inductively, construct $\phi'_i$ for $\phi_i := \alpha \land \psi_i$, and set $\psi' := (\alpha \land \phi'_1) \land (\alpha \land \phi'_2)$. Then $|\psi'| \leq (w|\psi|^2 + (w + |\psi_1|^2) + (w + |\psi_2|)^2 + 1 \leq (w(|\psi_1| + |\psi_2| + 1))^2 = (w|\psi|^2)^2$. Although $(\alpha \land \phi'_1) \land (\alpha \land \phi'_2)$ is a CQ-shaped subformula with two conjuncts, it does not contribute to the CQ-rank, since there is no quantification.

Similarly for $\psi$ of the form $\psi_1 \lor \psi_2$.

- Assume $\psi = \exists x. \beta(x) \land \eta(x)$. Inductively, construct $\eta'$ for $\beta(x) \land \eta'(x)$ and set $\psi' := \exists x. \beta(x) \land \eta'(x)$. Then $|\psi'| \leq (w|\psi|^2 + 2w + 3 \leq (w(|\psi| + 3))^2 \leq (w|\psi|^2)^2$.

\[\square\]

### B.2 Relativized GNF mosaics

Fix formulas $\varphi_L$ and $\varphi_R$ over signatures $\sigma_L$ and $\sigma_R$ that are in weak GN-normal form and do not use equality. We assume that free($\varphi_L$) = free($\varphi_R$) = $z$ (which may be the empty set if the input were sentences), and that these variable names are never used in any quantification in the formulas.\(^1\) From now on, we view $z$ as parameters (so par($\varphi_L$) = par($\varphi_R$) = $z$). Let $e := \text{con}(\varphi_L) \cup \text{con}(\varphi_R)$.

We also assume $\varphi_L$ and $\varphi_R$ are in U-relativized GNF, for $U$ a distinguished set of unary relations from $\sigma_L \cup \sigma_R$. As shorthand, we write $\exists x. U(x) \land \psi(x)$ for a formula with relativized quantification of the form $\exists x. U_1 x_1 \land \cdots \land U_k x_k \land \psi(x)$ where $x = x_1 \cdots x_k$ and $U = U_1 \cdots U_k$ with $U \subseteq U$.

**Specializations.** The mosaics must contain additional information about CQ-shaped subformulas of the form $\exists x. U(x) \land \bigwedge_{i \leq k} \psi_i(x)$. Consider a weak GN-normal form formula $\phi(y)$ of the form $\exists x. U(x) \land \bigwedge_{i \leq k} \psi_i(x)$. A (relativized) specialization of $\phi$ is a formula $\phi'$ obtained from $\phi$ by the following operations:

- select a subset $z$ of $x$ (call variables from $yz$ the inside variables and variables from $x \setminus z$ the outside variables);
- select a partition $x_1, \ldots, x_k$ of the outside variables, with the property that for every $\psi_j$, either $\psi_j$ has no outside variables or all of its outside variables are contained in some partition element $x_i$;
- let $U_0$ be the conjunction of atoms from $U(x)$ using only inside variables, and let $U_j$ be the conjunction of atoms from $U(x)$ using outside variables in $x_j$;
- let $\chi_j$ be the conjunction of the $\psi_i$ using only inside variables, and let $\chi_j$ be the conjunction of the $\psi_i$ using outside variables and satisfying par($\psi_i$) $\subseteq x_j y z$;
- set $\phi'(y z)$ to be $U_0(y z) \land \chi_0(y z) \land \bigwedge_{j \in \{1, \ldots, k\}} \exists x_j. U_j(x_j) \land \chi_j(x_j y z)$.

Roughly speaking, each specialization describes a way in which the original CQ-shaped formula could be satisfied in some tree decomposition.

We say a specialization $U_0(x) \land \chi_0(y z) \land \bigwedge_{j \in \{1, \ldots, k\}} \exists x_j. U_j(x_j) \land \chi_j(x_j y z)$ is non-trivial if either there are no outside variables (thus the specialization is only $U_0 \land \chi_0$, or $\chi_0$ is non-empty), or the partition of the outside variables is non-trivial.

**Lemma 25.** Let $\phi(y)$ be a formula of the form $\exists x. U(x) \land \bigwedge_{i \leq k} \psi_i(x)$. Given a structure $\mathfrak{M}$ and a tree decomposition $T$ of $\mathfrak{M}$, if there exists a node $v$ with $b \subseteq T(v)$ and $\mathfrak{M}, b \models \phi(y)$, then there is a non-trivial specialization $\phi'(y z)$ of $\phi$ and a node $w$ with a tuple $c$ such that $\mathfrak{M}, T(w) \models c \models \phi'(y z)$. We say $\phi'$ is the specialization of $\phi$ defined by $\mathfrak{M}$ and $b$.

**Proof.** There must be some tuple $a = a_1 \ldots a_m$ of elements (corresponding to $x = x_1 \ldots x_m$) such that $\mathfrak{M}, ab \models U(x) \land \bigwedge_{i \leq k} \psi_i(x)$. For a node $w$, let $w_1$ denote the parent of $w$ in $T$, and $w_2, \ldots, w_k$ denote the children of $w$ in $T$. We say $ab$ is contained in direction $j = 1$ if $ab$ appears in the tree resulting from removing from $T$ the subtree rooted at $w$, and is contained in direction $j > 1$ if $ab$ appears in the subtree rooted at $w_j$.

If there is a node $w$ in the tree decomposition $T$ with $ab \subseteq T(w)$, then it is easy (take the specialization where all variables $x$ are inside variables).

\(^1\)We remark that considering only formulas where free($\varphi_L$) = free($\varphi_R$) is not actually a restriction for satisfiability or interpolation. Indeed, given U-relativized weak GN-normal form formulas $\varphi_L'(xz)$ and $\varphi_R'(yz)$, we can existentially quantify out the free variables of $\varphi_L'$ that are not free variables of $\varphi_R'$, and we can universally quantify out the free variables of $\varphi_R'$ that are not free variables of $\varphi_L'$, introducing a fresh guard relation and fresh relativizers as necessary. That is, $\varphi_L' \models \varphi_R' \text{ iff } \exists x. U_L x_1 \land \cdots \land U_L x_j \land \varphi_L'(xz) \models R z \land \exists y. (U_R y_1 \land \cdots \land U_R y_k \land \varphi_R'(yz))$ where $x = x_1 \ldots x_j, y = y_1 \ldots y_k, R$ is a fresh predicate of arity $|z|$, and $U_L$ and $U_R$ are fresh unary predicates.
Otherwise, there is a node $w$ in $T$ with $b \subseteq T(w)$ and such that $ab$ is not contained in any direction from $w$ (if not, then starting at the node containing $b$, we could eventually reach a node $w'$ with $ab \subseteq T(w')$, which we are assuming is not possible).

Let $z$ be the tuple of variables from $x$ corresponding to elements in $c := T(w) \cap a$ (i.e., $x_i \in z$ iff $a_i \in T(w)$). Take $yz$ to be the inside variables, corresponding to elements $bc$ (the elements inside $T(w)$).

Let $O$ be the nonempty set of elements from $a$ that are not in $T(w)$ (i.e., the elements that correspond to outside variables). We partition $O$ into $O_1, \ldots, O_k$ such that $O_j$ is the set of elements from $c$ that are contained in direction $j$. Because $a \in O$ do not appear in $T(w)$, $O_1, \ldots, O_k$ partition $O$. This induces a partition of the outside variables.

Taking the resulting specialization $\phi'$, we have $\mathfrak{M}, bc \models \phi'(yz)$.

In the other direction, every specialization of $\phi$ logically implies $\phi$.

**Lemma 26.** Let $\phi(y)$ be a formula of the form $\exists x. U(x) \land \bigwedge_i \psi_i(x,y)$. For all structures $\mathfrak{M}$ and for all specializations $\phi'(yz)$ of $\phi$, if $\mathfrak{M}, bc \models \phi'(yz)$, then $\mathfrak{M}, b \models \phi(y)$.

**Closure.** The closure $cl(X^p : \varphi)$ is the smallest set $C$ of formulas containing $X^p : \varphi$ and such that:
- if $X^+ : \alpha(y) \land \neg \psi(y) \in C$, then $X^+ : \alpha(y), X^+ : \psi(y) \in C$;
- if $X^t : \alpha(y) \land \psi(y) \in C$, then $X^+ : \alpha(y), X^+ : \psi(y) \in C$;
- if $X^p : \bigwedge_i \psi_i(y) \in C$ or $X^p : \bigvee_i \psi_i(y) \in C$, then $X^p : \psi_i(y) \in C$ for all $i$;
- if $X^p : \exists x. U(x) \land \psi(x,y) \in C$, and $U_o(z) \land \chi(yz) \land \bigwedge_i \psi_i(x,y) \land \chi_k(x_iyz)$ for $U_0(z) = U_1 z_1 \land \cdots \land U_j z_j$ is a specialization of $\exists x. U(x) \land \psi(x,y)$, then $X^p : \exists z_i \in C$ for all $i \in \{1, \ldots, j\}$ and $X^p : \chi(yz) \land \bigwedge_i \exists x_i U_j(x_i) \land \chi_k(x_i, yz) \in C$;
- if $X^p : \alpha(yz) \in C$ for $z = \{z_1, \ldots, z_k\}$ and $X^+ : U_1 z_1 \land \cdots \land U_k z_k \land \alpha(yz) \in C$.

Given parameters $c$, we let $cl_1(c)$ (respectively, $cl_0(c)$) consist of formulas $X^p : \psi$ from $cl(L^+ : \varphi_L)$ (respectively, $cl(R^- : \varphi_R)$) with the free variables replaced by parameters from $c$ or constants from $c$, and such that the resulting formulas has parameters contained in $c$. Recall that we are viewing any variables $z$ that were free in $\varphi_X$ as parameters, so these will never be replaced by other parameters. For instance, consider a formula $L^+ : \psi(yz') \in cl(L^+ : \varphi_L)$, where $z' \subseteq z$ are parameters corresponding to original free variables. Then a possible substitution would be $L^+ : \psi(yz') \in cl(L^+ : \varphi_L)$, where $y \subseteq u \subseteq e$, and $L^+ : \psi(yz') \in cl_1(c)$ if $z' \subseteq c$.

Each formula in $cl_1(c)$ is labelled with a provenance $L$ or $R$ and a polarity or $-$ to indicate whether the formula is built from a subformula of $\varphi_L$ or $\neg \varphi_R$, and whether this subformula occurred positively or negatively (i.e., whether it occurred within the scope of an even or odd number of negations). Given a formula $\psi$ and a polarity $p$, we write $p\psi$ to mean $\psi$ (respectively, $\neg \psi$) if $p$ is $+$ (respectively, $-$).

The formulas in $cl_1(c)$ respect the polarity of relations, guards, and constants in the original formulas.

**Proposition 27.** For all $\tau \subseteq cl_1(c) \cup cl_0(c),
- if $L^p : \psi \in \tau$, then $\neg \psi \in \tau$;
- if $\neg L^p : \psi \in \tau$, then $\neg \neg \psi \in \tau$.

**Guards and relativizers.** We say $\alpha$ is an $L$-atom (respectively, $R$-atom) if $\alpha, + \in \text{occ} (\varphi_L)$ (respectively, $\alpha, - \in \text{occ} (\varphi_R)$). Likewise, we say $\alpha$ is an $L$-catom (respectively, $R$-catom) if $\alpha, + \in \text{occ} (\varphi_L)$ (respectively, $\alpha, - \in \text{occ} (\varphi_R)$). We say $\alpha$ is an $X$-guard (respectively, $X$-guard) if $\alpha$ is an $X$-atom (respectively, $X$-catom) and $\alpha \in \text{guards}(\varphi_L) \cup \text{guards}(\varphi_R)$. We say $U$ is an $X$-relativizer (respectively, $X$-relativizer) if $U$ is an $X$-atom (respectively, $X$-catom) and $U \in U$. Let $\tau$ be a collection of formulas from $cl_1(c) \cup cl_0(c)$ and let $\mathfrak{c} \subseteq c$ be a distinguished subset of these parameters. We say $b \in c$ is $X$-caterlated in $\tau$ if there is some $X^+ : U b \in \tau$ where $U \in U$ and $U$ is an $X$-atom. We say $b$ is $X$-caterlated if every $b \in c$ is $X$-caterlated.

We say $\tau \subseteq \mathfrak{c}$ is $X$-guarded in $\tau$ if $|b| \leq 1$, or if there is some $X^+ : \exists x. U(x) \land \alpha(\alpha b) \in \tau$ where $\alpha$ and $U$ are $X$-atoms, $\alpha \in \text{guards}(\varphi_L) \cup \text{guards}(\varphi_R)$, $\alpha \subseteq U \subseteq U'$, $b' \supseteq b$, and $b' \cap c$ is $X$-caterlated. Note that this definition is different than in the GF mosaics because of technical requirements when proving soundness and completeness of this GNF mosaic method. In particular, unary relations are always considered to be $X$-caterlated, which allows $X$-relativizers to move freely between linked mosaics (which is important for completeness).

We say a formula $\tilde{X}^p : \psi$ is $X$-safe in $\tau$ if:
- $X^p = \tilde{X}^+ \land \text{par} (\psi) \cap d \in \tau$ is $X$-caterlated in $\tau$ and $\text{par} (\psi)$ is $X$-caterlated in $\tau$, or
- $X^p = \tilde{X}^- \land \text{par} (\psi) \cap d \in \tau$ is $X$-caterlated in $\tau$.

We emphasize that these definitions (X-guarded, X-caterlated, and X-safety) depend on the distinguished set of parameters $d$ in $\tau$, so when we talk about, e.g., being $X$-caterlated in $\tau$, we will assume that $\tau$ has some distinguished set of parameters $d$.

**Relativized GNF mosaics.** Let $\mathfrak{c}$ be a tuple of parameters such that $|\mathfrak{c}| \leq \text{width}(\varphi_L \land \neg \varphi_R)$. A (relativized) GNF X-mosaic $\tau$ for $\varphi_L \land \neg \varphi_R$ using parameters $\mathfrak{c}$ is a subset of $\Gamma (\mathfrak{c}) := cl_1(c) \cup cl_0(c)$ with some distinguished parameters $d \subseteq c$ (which we call the relativized parameters in $\tau$) and satisfying the requirement that if $\tilde{X}^p : \psi \in \tau$, then $\tilde{X}^p : \psi$ is $X$-safe in $\tau$ (with respect to $d$). We write $\tau (c)$ to emphasize that $\tau$ uses parameters $c$.

Let $\tau$ be a GNF $X$-mosaic over parameters $c$ and relativizing $d$. Assume $Y^+ : \eta \in \tau$ for an existential requirement of the form $\exists x. \psi (x \mathfrak{b})$. Then $\tau$ is linked to $\tau'$ via $Y^+ \vdash \eta$ (written $\tau \rightarrow_{Y^+ \eta} \tau'$) if:
- $\tau'$ is a GNF $Y$-mosaic over parameters $ab$ relativizing $a \cup (b \cap d)$ (respectively, $a$) if $Y = X$ (respectively, $Y = \tilde{X}$), and with $a \cap c = \emptyset$;
- $Y^+ : \psi (x \mathfrak{b}) [a/x] \in \tau'$;
- for all $Y^p : \psi'(\mathfrak{b}) \in \tau$, $Y^p : \psi'(\mathfrak{b}) \in \tau'$;
- for all $\tilde{Y}^p : \psi'(\mathfrak{b}) \in \tau$ such that $\tilde{Y}^p : \psi'(\mathfrak{b})$ is $Y$-safe in $\tau'$, $\tilde{Y}^p : \psi'(\mathfrak{b}) \in \tau'$;
- for all $Z^p : \psi'(\mathfrak{b}) \in \tau'$, $Z^p : \psi'(\mathfrak{b}) \in \tau'$.
As in the GF mosaics, the maximum amount of information about shared parameters is passed through a link, and no new information about shared parameters can be added.

Fix some set \( P \) of parameters of size \( 2 \cdot \text{width}(\phi_L \land \neg \phi_R) \) that includes the parameters \( \text{par}(\phi_L) = \text{par}(\phi_R) \) (the parameters corresponding to the original free variables in the input formulas). Let \( \mathcal{M} \) be the set of GNF mosaics over parameters \( P \).

Coherence. A GNF \( X \)-mosaic \( \tau(c) \) is downward closed if the following properties are satisfied:

- if \( Y^+ : \alpha(b) \land \neg \psi(b) \in \tau \), then \( Y^- : \alpha(b) \in \tau \) and \( Y^- : \psi(b) \in \tau \);
- if \( Y^+ : \alpha(b) \land \neg \psi(b) \in \tau \) and \( Z^+ : \alpha(b) \in \tau \), then \( Y^+ : \psi(b) \in \tau \);
- if \( Y^+ : \bigwedge_i \psi_i \in \tau \) (respectively, \( Y^- : \bigvee_i \psi_i \in \tau \)), then \( Y^+ : \psi_i \in \tau \) (respectively, \( Y^- : \psi_i \in \tau \)) for all \( i \);
- if \( Y^+ : \bigwedge_i \psi_i \in \tau \) (respectively, \( Y^- : \bigvee_i \psi_i \in \tau \)), then \( Y^+ : \psi_i \in \tau \) (respectively, \( Y^- : \psi_i \in \tau \)) for some \( i \);
- if \( Y^+ : \exists x. \psi(xb) \in \tau \), and there is some specialization \( U_0(e') \land \chi_0(bc') \land \bigwedge_i \exists x_k U_i(x_k) \land \chi_k(x_k bc') \) such that \( U_0(e') = U_{1C1} \land \cdots \land U_{jCj} \land Z^+_i : U_i \in \tau \) for all \( i \in \{1, \ldots, j\} \), then \( Y^- : \chi_0(bc') \land \bigwedge_i \exists x_k U_i(x_k) \land \chi_k(x_k bc') \in \tau \).

The last rule in this definition is a generalization of the rule for universal statements in the GF mosaics. It helps ensure that when we construct a model from a collection of linked, internally consistent mosaics, a CQ-shaped subformula that is asserted to be false in one mosaic, does not become true in the constructed model.

As before, a GNF \( X \)-mosaic \( \tau \) is internally consistent if \( \tau \) is downward closed and there is no atomic formula \( \alpha \) and no \( Y, Z \in \{L, R\} \) such that \( Y^+ : \alpha \in \tau \) and \( Z^- : \alpha \in \tau \).

A set of relativized GNF mosaics is said to be saturated if for every mosaic \( \tau(c) \) in the set and for every existential requirement \( Y^+ : \eta \in \tau \) for \( \eta \) of the form \( \exists x. \bigwedge_i U_i(x) \) (or \( \bigwedge_i \exists x \bigwedge_i U_i(x) \)), either (i) \( Y^+ : U_i[x/a] \in \tau \) for all \( i \in \{1, \ldots, j\} \) and \( Y^+ : \psi(xb) \in \tau \) for some \( a \subseteq c \cup e \) (so \( \eta \) is fulfilled in \( \tau \)) or (ii) \( \tau \) is linked to some mosaic \( \tau' \) in the set via \( Y^+ : \eta \) (so \( \eta \) is fulfilled in \( \tau' \)).

These coherence requirements lead to the following key theorem.

**Theorem 28.** \( \phi_L \land \neg \phi_R \) is satisfiable if and only if there is a saturated set of internally consistent GNF mosaics from \( \mathcal{M} \) that contains a mosaic \( \tau \) such that \( L^\tau : \phi_L \land \neg \phi_R \in \tau \).

We give the proof in the next section.

We first state some intuition behind what the distinguished, relativized parameters represent in a saturated set of internally consistent mosaics. Roughly speaking, the relativized parameters \( d \) in an \( X \)-mosaic represent parameters that were introduced due to some existential requirement in an \( X \)-formula (and hence were originally \( X \)-relativized). This helps explain why for linked mosaics \( \tau \) and \( \tau' \) over \( Y^+ : \exists x. \psi(xb) \in \tau \), the “new” parameters \( a \) used in \( \tau' \) always become part of the relativized parameters in the \( Y \)-mosaic \( \tau' \).

If we have a series of mosaics over all \( X \) existential requirements, we want to remember all of these \( X \)-relativized parameters. As soon as we link to a \( X \)-mosaic, however, then we can forget about these \( X \)-relativized parameters, and concentrate on the new \( X \)-relativized parameters. Thus, the set of distinguished relativized parameters \( b \cap d \) is carried through a link between \( X \)-mosaics, but not carried through a link between an \( X \)-mosaic and a \( X \)-mosaic. This forgetfulness is acceptable, because in the interpolation construction for the \( X \)-mosaic \( \tau' \), we will only quantify out (and hence need relativizers for) the most recently introduced \( Y \)-parameters.

### B.3 Soundness and completeness of GNF mosaic method (Proof of Theorem 28)

We now give the detailed proofs of the soundness and completeness of the GNF mosaic method, in the terms of the definitions given in the previous subsection.

We start with some additional definitions.

Fix parameters \( c \) and some \( d \subseteq c \). Given a structure \( \mathcal{M} \) over \( \sigma_L \cup \sigma_R \cup c \) such that \( |c| \leq \text{width}(\phi_L \land \neg \phi_R) \), let \( \nu^\mathcal{M} \) be the unique \( Y \)-mosaic over \( c \) relativizing \( d \) such that \( \nu^\mathcal{M} : Y^+ : \psi | Y^- : \psi \in c_1(c) \) and \( \exists x. \psi(xb) \), where \( \mathcal{M} \models \psi(xb) \) means \( \mathcal{M} \models \psi \) (respectively, \( \mathcal{M} \models \neg \psi \) if \( p = + \) (respectively, \( p = - \)). The \( X \)-mosaic \( \tau(c) := \nu^\mathcal{M} \sqcup \{ X^+ : \psi | X^- : \psi \in c_1(c) \} \) and \( \psi \in \mathcal{M} \models X^- \) is called the \( X \)-mosaic \( \tau(c) \) relativizing \( d \) in \( \mathcal{M} \).

We say an \( X \)-mosaic \( \tau \) over \( c \) relativizing \( d \) is realizable if there is some structure \( \mathcal{M} \) such that \( \tau \) is \( X \)-mosaic of \( c \) relativizing \( d \) in \( \mathcal{M} \).

For convenience in the following proofs, we also state the definition of linked GNF mosaics. Let \( \tau \) be a GNF \( X \)-mosaic over parameters \( c \) and relativizing \( d \) with \( Y^+ : \eta \in \tau \) for \( \eta \) of the form \( \exists x. \psi(xb) \). We say \( \tau' \) is linked with \( \tau \) via \( Y^+ : \eta \) if

- \( \eta' \) is a GNF \( Y \)-mosaic using parameters \( ab \), with \( a \cap e = \emptyset \), and relativizing \( a \cup (b \cap d) \) (respectively, \( a \) if \( Y^+ = X^p \));
- \( Y^+ : \psi(xb) | a \cup [e] \in \tau' \);
- \( Y^+ : \psi'(b) \in \tau' \);
- \( \forall y^+ : \psi'(b) \in \tau' \); and \( Y^- : \psi'(b) \in \tau' \);
- \( Z^+ : \psi'(b) \in \tau' \);
- \( Z^+ : \psi'(b) \in \tau' \).

We start with two propositions relating realizability to our coherency requirements for GNF mosaics (this will be used for soundness).

**Proposition 29.** If \( \tau \) is realizable, then \( \tau \) is internally consistent.

**Proof.** Let \( \mathcal{M} \) be a structure realizing the \( X \)-mosaic \( \tau(c) \) relativizing \( d \). By definition, \( Z^+ : \psi \in \tau \) implies \( \exists x. \psi(xb) \).

This means it is not possible for there to be some \( Y^+ : \alpha \) and \( X^- : \alpha \) in \( \tau \).

We must check that the downward closure properties are satisfied in \( \tau \). We test them here for ease of reference.

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2 An alternative approach would be to annotate every parameter with a provenance and carry information about the parameters through a link. We could then place restrictions on L-formulas that use R-parameters (and vice versa).
If $Y = \bar{X}$, then it is straightforward to check that these downward closure properties hold.

For instance, consider (i). If $Y^+ : \alpha(b) \land \neg \psi(b) \in \tau$, then $2\mathfrak{M} \models \alpha(b) \land \neg \psi(b)$. Since $2\mathfrak{M} \models \alpha(b)$ and $2\mathfrak{M} \models \neg \psi(b)$ and $Y^+ : \alpha(b) \in c_{1Y}(c)$ and $Y^- : \psi(b) \in c_{1Y}(c)$, this is enough to ensure that $Y^+ : \alpha(b), Y^- : \psi(b) \in \tau$.

Note that for (v), we make use of Lemma 26.

If $Y = \bar{X}$, then we use a similar argument, but in order to conclude that the required formulas appear in $\tau$, we must also show that these formulas are $X$-safe in $v_{X0}^\tau$.

For instance, consider (ii). Assume $Y^- : \alpha(b) \land \neg \psi(b) \in \tau$ and $Z^+ : \alpha(b) \in \tau$. Then we know that $2\mathfrak{M} \models \alpha(b)$ and $2\mathfrak{M} \models \alpha(b) \rightarrow \psi(b)$, hence $2\mathfrak{M} \models \psi(b)$ in $v_{X0}^\tau$. It remains to show that this formula is $X$-safe in $v_{X0}^\tau$.

If $Y = Z$, then $Z^+ : \alpha(b) \in \tau$ implies that $Z^+ : \alpha(b)$ is $X$-safe in $v_{X0}^\tau$, so $Y^+ : \psi(b)$ is also $X$-safe in $v_{X0}^\tau$.

If $Y = Z$, then $\alpha$ appears positively in an $X$-formula (since $Z = X$), and negatively in a $\bar{X}$-formula (since $Y = \bar{X}$). Making use of the fact that polarities and occurrences are preserved in moving to mosaics, this means that $\alpha$ is an X-cugard. Moreover, $Y^- : \alpha(b) \land \neg \psi(b) \in \tau$ implies that $b \land d$ is $X$-cuguard in $\tau$. This means $b$ is $X$-cuguarded by $b \land \alpha(b)$ in $\tau$, so $Y^+ : \psi(b)$ is $X$-safe in $v_{X0}^\tau$. Now consider the make of the fact that polarities and occurrences are preserved in moving to mosaics.

Assume for the sake of contradiction that $Y^- : \chi(c) \land \bigwedge_{k \in \{1, \ldots, l\}} \exists x_k U_{\tau}(x_k) \land \chi_k(\bar{x}_k b c)$, but this would mean $2\mathfrak{M} \models U_{\tau}(c) \land \chi(c) \land \bigwedge_{k \in \{1, \ldots, l\}} \exists x_k U_{\tau}(x_k) \land \chi_k(\bar{x}_k b c)$, since $2\mathfrak{M} \models U_{\tau}(c)$. By Lemma 26, every specification of a formula implies the original formula, contradicting the fact that $2\mathfrak{M} \models \neg \exists x. U_{\tau}(x) \land \psi(x)$.}

\begin{proposition} If $v$ is realizable, then there is a saturated set $S \subseteq M$ of realizable mosaics that includes $v$.\end{proposition}

\begin{proof}
We inductively construct sets $S_i$, ensuring at each stage $i$ that $S_i$ contains $v$ and other realizable mosaics, and that any existential requirement in $S_{i-1}$ is fulfilled in $S_i$. This is enough to ensure that the set $S := \bigcup S_i$ is a saturated set of realizable mosaics that includes $v$.

At stage $i = 1$, we set $S_1 := \{v\}$, which is realizable by assumption.

At stage $i > 1$, if every existential requirement in $S_{i-1}$ is fulfilled, then set $S_i := S_{i-1}$.

Otherwise, consider some $X$-mosaic $\tau(c)$ relativizing $d$ in $S_{i-1}$ with $Y^+ : \eta \in \tau$ for some of the form $\exists x. \psi(x b)$ and $\exists x. \psi(x b)$ for which there is no $a \subseteq c \cup \{x\}$ with $Y^+ : \psi(x b) \mid \exists x. \psi(x b)$ and $\exists x. \psi(x b)$. But this would mean $2\mathfrak{M} \models U_{\tau}(c) \land \chi(c) \land \bigwedge_{k \in \{1, \ldots, l\}} \exists x_k U_{\tau}(x_k) \land \chi_k(\bar{x}_k b c)$, since $2\mathfrak{M} \models U_{\tau}(c)$. By Lemma 26, every specification of a formula implies the original formula, contradicting the fact that $2\mathfrak{M} \models \neg \exists x. U_{\tau}(x) \land \psi(x)$.}

\end{proof}

We must now check that formulas with shared parameters $b$ are preserved between $\tau$ and $\tau'$ as described by (E3)–(E5).

We then follow by the definition of realizable mosaics, and the fact that $2\mathfrak{M}$ and $2\mathfrak{M}'$ are identical with respect to formulas that use only parameters from $b$.

It remains to check (E5). Assume there is some $Z^+ : \psi(b) \in \tau$. We have $2\mathfrak{M} \models \psi(b)$ since $\tau$ is realized by $2\mathfrak{M}'$. Recall that $\tau$ is an $X$-mosaic, $\tau'$ is an $Y$-mosaic, and $\psi'$ is a $Z^+$-formula. The result is immediate if $Z^+ = X^0$. Otherwise, if $Z^+ = \bar{X}^0$ it suffices to show that $Z^+ : \psi'(b)$ is $X$-safe in $\tau$.

\begin{itemize}
\item $Z^+ = \bar{X}^0$ and $X = Y$. Then $\par(\psi') \cap d$ must be $X$-cuguarded in $\tau'$, and hence $X$-cuguarded in $\tau$. Moreover, if $p = +$, then $\bar{X}^0 : \psi'(b) \in \tau'$ and $\tau'$ is an $X$-mosaic, $\par(\psi')$ must be $X$-cuguarded in $\tau'$ by some $X^+ : \exists w. U_{\tau}(w) \land \beta(w b')$ where $a' \subseteq a$ and $\par(\psi') \subseteq b' \subseteq b$ such that parameters in $a' \cap (a \cap b \cap d)$ are $X$-cuguarded in $\tau'$. But this means that $2\mathfrak{M} \models \exists w. U_{\tau}(w) \land U_{\tau}^+(x) \land \beta(w b') \in \tau'$. Since the parameters in $b' \cap d$ are $X$-cuguarded in $\tau$, $\par(\psi') \subseteq b$ is $X$-safe in $\tau$.
\item $Z^+ = \bar{X}^0$ and $X \neq Y$. Then $\par(\psi')$ must be $X$-cuguarded in $\tau$ and $\par(\psi') \cap d$ must be $X$-cuguarded in $\tau$. Since $Y^+ : \eta \in \tau$ with $\par(\eta) = b$ and $X \neq Y$. This is enough to ensure that $X^0 : \psi' \in X$-safe in $\tau$.
\end{itemize}

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We have constructed a realizable $Y$-mosaic $\tau'$ such that $\tau \rightarrow_{Y+\eta} \tau'$, so we add $\tau'$ to $S_i$. Repeating this procedure for all such $\tau$ (and all such $Y^+ : \eta \in \tau$) in $S_{i-1}$, we will end up with a set $S_i$ of realizable mosaics such that any existential requirement in $S_{i-1}$ is fulfilled in $S_i$. \hfill \square

We now aim towards proving a proposition useful for completeness (a sort of converse of Proposition 30). In order to do this, we first prove lemmas about the transfer of formulas between linked mosaics.

**Lemma 31 (Backwards transfer property).** Let $\tau_0 \rightarrow \cdots \rightarrow \tau_k$ be linked mosaics such that every $\tau_i$ includes parameters $b$. If $Z^+ : \psi(b) \in \tau_k$, then $X^+ : \psi(b) \in \tau_0$.

*Proof.* We proceed by induction on $k$. The base case for $k = 0$ is immediate.

Otherwise, assume $k > 0$. Consider $\tau_{k-1} \rightarrow \tau_k$. By definition (E5), $Z^+ : \psi(b) \in \tau_k$ implies $Z^+ : \psi(b) \in \tau_{k-1}$ (since $b$ is shared by $\tau_{k-1}$ and $\tau_k$), so we can apply the inductive hypothesis to $\tau_0 \rightarrow \cdots \rightarrow \tau_{k-1}$ to get the result. \hfill \square

**Lemma 32 (Forward relativizer transfer property).** Let $\tau_0 \rightarrow \cdots \rightarrow \tau_k$ be linked mosaics such that every $\tau_i$ includes parameters $b$. If $Z^+ : Ub \in \tau_0$ for $b \in b$, then $Z^+ :Ub \in \tau_k$.

*Proof.* We proceed by induction on $k$. The base case for $k = 0$ is immediate.

Otherwise, assume $k > 0$. Consider $\tau_0 \rightarrow \tau_1$ where $\tau_0$ is an $X$-mosaic relativizing $d$ and $\tau_1$ is a $Y$-mosaic.

- If $Z = Y$, then (E3) implies that $Z^+ : Ud \in \tau_1$.
- If $Z = Y$ and $Y = X$, then $\psi(b) \in \tau_0$ implies that $b \cap d$ is $X$-relativized in $\tau_0$.
- If $b \notin d$, then $Y^+ :Ub \in \tau_0$.

Otherwise, if $b \in d$, then $b$ is $X$-relativized by some $X^+ : Ub \in \tau_0$. Since $X^+ : Ub$ is trivially $Y$-safe in $\tau_1$, (E3) ensures that $X^+ : Ub \in \tau_1$. Hence, $b$ is $Y$-relativized in $\tau_1$. Moreover, $b$ is trivially $Y$-guarded (since there is only one parameter). This means that $Y^+ :Ub \in \tau_1$.

Hence, in either case, we can apply (E4) to ensure that $Z^+ : Ub \in \tau_1$.

- If $Z = Y$ and $Y = X$, then it is trivially $Y$-safe: it is $Y$-guarded because there is only one parameter, and it does not need to be $Y$-relativized (since no shared parameters are relativized when moving from an $X$ to $X$ mosaic). Hence, we can apply (E4) to get $Z^+ :Ub \in \tau_1$.

Since we have guaranteed that $Z^+ :Ub \in \tau_1$, we can apply the inductive hypothesis to $\tau_1 \rightarrow \cdots \rightarrow \tau_k$ to get the result. \hfill \square

**Lemma 33 (Forward negative transfer property).** Let $\tau_0 \rightarrow \cdots \rightarrow \tau_k$ be linked mosaics such that every $\tau_i$ includes parameters $b$. If $Z^+ : \psi(b) \in \tau_0$, then $Z^+ : \psi(b) \in \tau_k$.

*Proof.* We proceed by induction on $k$. If $k = 0$, then the result is immediate.

Otherwise, assume $k > 0$. Consider $\tau_0 \rightarrow \tau_1$ where $\tau_0$ is an $X$-mosaic relativizing $d$ and $\tau_1$ is a $Y$-mosaic. In order to apply the inductive hypothesis, we must show that $Z^+ : \psi(b) \in \tau_1$.

- If $Z = Y$, then $Z^+ : \psi(b) \in \tau_1$ by (E3).
- If $Z = Y$ and $Y = X$, then $\psi(b) \in \tau_0$ implies that $b \cap d$ is $X$-relativized in $\tau_0$.
- By the previous lemma, we know that $b \cap d$ is $X$-relativized in $\tau_1$ (i.e. we know that all of the $X$-relativizers in $\tau_0$ are in $\tau_1$).

This is enough to ensure that $Y^+ : \psi(b) \in \tau_1$, so (E4) guarantees that $Z^+ : \psi(b) \in \tau_1$.

- If $Z = Y$ and $Y = X$, then it is trivially $Y$-safe: no parameters need to be $Y$-relativized (since no shared parameters are relativized when moving from an $X$ to $X$ mosaic). Hence, we can apply (E4) to get $Z^+ : \psi(b) \in \tau_1$.

Since $Z^+ :Ub \in \tau_1$, we can apply the inductive hypothesis to $\tau_1 \rightarrow \cdots \rightarrow \tau_k$ to get the result. \hfill \square

We are now ready to show that whenever there is a set of saturated mosaics, we can get a satisfying model (completeness).

**Proposition 34.** Let $\tau \subseteq \mathcal{M}$ be a saturated set of internally consistent mosaics. Then for all $\tau(c) \in \mathcal{S}$, there is a model $\mathfrak{M}$ over $\sigma_L \cup \sigma_R \cup \epsilon$ such that for all $X^+ : \psi \in \tau$, $\mathfrak{M} \models p\psi$.

*Proof.* Fix $\tau \in \mathcal{S}$. Note that $\tau$ is internally consistent by assumption. Our goal is to construct a model $\mathfrak{M}$ witnessing the satisfiability of the formulas in $\tau$. We will use the mosaics in $\mathcal{S}$ as building blocks.

We build inductively a tree decomposition $T$ of $\mathfrak{M}$, together with a map $f$ from nodes $v$ in the tree decomposition to a mosaic in $\mathcal{S}$. We also think of $f$ as a map from the elements in a node $v$, to the parameters in the corresponding mosaic.

At stage 0, let $T_0$ consist of a single node $v_0$ containing a copy of the parameters in $\tau$ and constants $e$, and set $f(v_0) = \tau$.

At stage $i$, consider the set

$S = \{(v, Y^+ : \eta) : v \text{ is a leaf in } T_i \text{ and } Y^+ : \eta \in f(v) \neq \psi \text{ is not fulfilled in } f(v) \}$

for $\eta$ of the form $\exists x. \psi(x. f(b))$.

For each $(v, Y^+ : \eta) \in S$, we know that there is some $\tau' \in S$ such that $f(v) \rightarrow_{Y^+ \eta} \tau'$ (since $S$ is saturated). We construct $T_i$ by extending $T_{i-1}$ according to the following procedure: for each $(v, Y^+ : \eta) \in S$, we construct a child $v'$ of $v$ with $T(v') = a \cup b \cup e$ for new elements $a$, and define $f(v') = \tau'$ such that $Y^+ : \psi(f(ab)) \in \tau'$.

Let $T$ be the limit of this process, and let $\mathfrak{M}$ be the structure with elements $M = \bigcup_{v \in T} T(v)$, and such that $\mathfrak{M} \models a(e)$ iff there exists $v$ such that $a$ appears in $T(v)$ and $Y^+ : a(f(a)) \in f(v)$.

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We must now show that for all \(v\) in \(T\) and for all \(a\) in \(T(v)\), if \(Z^p : \psi(f(a)) \in f(v)\), then \(\mathcal{M} \models \psi(f(a))\).

The proof is by induction on the structure of \(\psi\).

- Assume \(\psi = \alpha\) is an atom. Then \(\mathcal{M} \models \psi\) by construction.
- Assume \(\psi = \alpha \land \neg \psi'\) (where \(\psi'\) is \(\alpha\) and \(\neg \psi'\)). Suppose for the sake of contradiction that \(\mathcal{M} \not\models \alpha\). Then there is some \(w\) such that \(X^+ : \alpha(a) \in f(w)\) and \(f(a)\) appears in \(f(w')\) for all \(w'\) on the path between \(v\) and \(w\). Using the backwards transfer property (Lemma 31), there must be some node containing both \(Z^+ : \alpha(f(a))\) and \(X^+ : \alpha(f(a))\), which contradicts internal consistency of mosaics in \(\mathcal{M}\).
- Assume \(\psi = \alpha\) is of the form \(\bigvee \psi_i\) or \(\bigwedge \psi_i\). The result follows using internal consistency and the inductive hypothesis.
- Assume \(\psi = \alpha(f(b))\). By internal consistency, \(Z^+ : \alpha(f(b)) \in f(v)\) and \(Z^- : \psi(f(b)) \in f(v)\). The result follows using the inductive hypothesis.
- Assume \(\psi = \alpha(f(b)) \land \neg \psi'\). Suppose for the sake of contradiction that \(\mathcal{M} \models \alpha(f(b)) \land \neg \psi'(f(b))\). Then there is some \(w\) such that \(X^+ : \alpha(f(b)) \in f(w)\) and \(f(b)\) appears in \(f(w')\) for all \(w'\) on the path between \(v\) and \(w\). Using the backwards transfer property (Lemma 31), there must be some node containing both \(Z^+ : \alpha(f(b))\) and \(X^+ : \alpha(f(b))\). By internal consistency, this implies \(Z^+ : \psi'(f(b))\) is also in this node, so \(\mathcal{M} \models \psi'(f(b))\) by the inductive hypothesis, which contradicts \(\mathcal{M} \models \alpha(f(b)) \land \neg \psi'(f(b))\).
- Assume \(\psi = \psi'\). Suppose for the sake of contradiction that \(\mathcal{M} \models \psi\). Then either \(\psi = \psi'\) or \(\mathcal{M} \models \psi'\). Since \(\mathcal{M} \models \psi\) and \(\mathcal{M} \models \psi'\), \(\mathcal{M} \models \psi\).
- Assume \(\psi = \psi'(f(b))\). Let \(\mathcal{M} \models \psi'(f(b))\), and suppose for the sake of contradiction that \(\mathcal{M} \not\models \psi'(f(b))\). Then there is some \(w\) such that \(X^+ : \psi'(f(b)) \in f(w)\) and \(f(b)\) appears in \(f(w')\) for all \(w'\) on the path between \(v\) and \(w\). Using the backwards transfer property (Lemma 31), there must be some node containing both \(Z^+ : \psi'(f(b))\) and \(X^+ : \psi'(f(b))\). By internal consistency, this implies \(Z^+ : \psi'(f(b))\) is also in this node, so \(\mathcal{M} \models \psi'(f(b))\) by the inductive hypothesis, which contradicts \(\mathcal{M} \models \psi'(f(b))\).

By Lemma 25, there is a node \(w\) with \(bc \subseteq T(w)\) and a non-trivial specialization \(\varphi'(yz)\) of \(\psi\) such that

\[
\mathcal{M}, bc \models (U_0(z) \land \chi_0(yz)) \land \bigwedge_{j \in \{1, \ldots, k\}} \exists x_j, U_j(x_j) \land \chi_j(x_j, yz)
\]

By the backward/forward transfer properties (Lemmas 31 and 33), \(Z^- : \psi \in f(w)\).

Let \(U_0(z) = U_1 z_1 \land \cdots \land U_l z_l\). Since \(\mathcal{M} \models U_0(c)\), for each \(i \in \{1, \ldots, l\}\), there is a node \(w_i\) such that \(Z^+ : U_i f(c_i) \in f(w_i)\), and the parameter \(f(c_i)\) is in every mosaic on the path between \(w_i\) and \(w\). By the backward transfer property and forward relativizer transfer properties (Lemmas 31 and 32), \(Z^+ : U_i f(c_i) \in f(w)\).

By internal consistency, \(Z^- : \varphi'(bc, y, z) \in f(w)\).

Likewise, by internal consistency, we have that \(Z^- : U_0(f(c)) \land \chi_0(f(bc)) \in f(w)\) or \(Z^- : \exists x_j, f(x_j) \land \chi_j(x_j, f(bc)) \in f(w)\) for some \(j\). Since it is a non-trivial specialization, we can apply the inductive hypothesis to this formula in \(f(w)\), and get a contradiction. We remark that this is the only case where we need both forward and backward transfer properties, since we need to be able to move \(Z^- : \psi\) (and the relativizing formulas) to the node \(w\) related to the non-trivial specialization.

We are now ready to prove Theorem 28. For soundness we must show:

If \(\varphi_L \land \neg \varphi_R\) is satisfiable, then there is a saturated set of internally consistent GNF mosaics from \(\mathcal{M}\) that contains some \(\tau\) such that \(L^\tau : \varphi_L, R^\tau : \varphi_R \in \tau\).

Let \(z\) be the L-mosaic of \(\tau\) relativizing \(\theta\) in \(\mathcal{M}\), i.e. the set of formulas from \(cL(z)\) that are true in \(\mathcal{M}\), and the set of formulas from \(cL(z)\) that are true in \(\mathcal{M}\) and satisfy the L-safety conditions on R-formulas in an L-mosaic. Since \(\varphi_L \land \neg \varphi_R\) and holds in \(\mathcal{M}\), \(\tau\) contains \(L^\tau : \varphi_L\).

Note that \(\varphi_R\) is trivially L-safe (since it is a negative formula, and there are no relativized parameters), \(\tau\) also contains \(R^- : \varphi_R\). Since \(\tau\) is a realizable mosaic, Proposition 30 implies that there is a saturated set \(S\) of realizable mosaics including \(\tau\). Moreover, by Proposition 29, \(S\) must contain only internally consistent mosaics, as desired.

For completeness of the mosaic method, we must show:

If there is a saturated set of internally consistent mosaics from \(\mathcal{M}\) containing \(\tau\) such that \(L^\tau : \varphi_L, R^\tau : \varphi_R \in \tau\), then \(\varphi_L \land \neg \varphi_R\) is satisfiable.

This follows immediately from Proposition 34.

### B.4 Bound on number of relativized GNF mosaics

Given \(U\)-relativized GNF formulas \(\varphi_L\) and \(\varphi_R\) in weak GN-normal form, we can prove a doubly exponential upper bound for the number of relativized GNF mosaics for \(\varphi_L \land \neg \varphi_R\) in \(\mathcal{M}\). The following proposition states this bound more precisely.

**Proposition 35.** There is a polynomial function \(p\) such that \(\varphi := \varphi_L \land \neg \varphi_R\) for \(\varphi_L\) and \(\varphi_R\) in weak GN-normal form without equality, and \(n = |\varphi_L| + |\varphi_R|, w = \text{width}(\varphi), h = \text{rank}_{\text{CQ}}(\varphi)\), then the number of relativized GNF mosaics in \(\mathcal{M}\) for \(\varphi\) is at most \(2^{p(n)} 2^{w^{(h+1)}}\).

**Proof.** Fix \(\varphi := \varphi_L \land \neg \varphi_R\) with \(n = |\varphi_L| + |\varphi_R|, w = \text{width}(\varphi), h = \text{rank}_{\text{CQ}}(\varphi)\). Let \(l := |\text{con}(\varphi)|\). Note that \(w, h, l \leq n\).

We first calculate the size of \(\text{cl}(Y^p : \varphi_Y)\) (where \(p = +\) if \(Y = L\) and \(p = -\) if \(Y = R\)).

There are at most \(2^n\) polarity-labelled subformulas of \(\varphi_Y\). However, \(\text{cl}(Y^p : \varphi_Y)\) contains additional formulas according to the definition of GN closure.

For each positive atomic formula \(\alpha\) in \(Y^p : \varphi_Y\), there are at most \(w\) free variables, and hence at most \(2^w\) ways to choose variables for quantification. There can be at most \(n\) predicates in \(U\), and at most \(w\) ways to relativize the quantified variables using these unary predicates. Hence, for each positive atomic formula \(\alpha\) in \(Y^p : \varphi_Y\), there are at most \(2^{2^n} w^n\) additional formulas that may be added.

For each negative CQ-shaped formula \(\phi\) in \(Y^p : \varphi_Y\), there are at most \(2^n\) CQ-shaped subformulas \(\phi'\) obtained by choosing some subset of the conjuncts in \(\phi\). In each \(\phi'\), there are at most \(2^n\) choices of the inside variables, and \(w^n\) ways to partition the outside variables resulting...
in specializations $\phi''$ of $\phi'$. Notice that these specializations $\phi''$ only have CQ-shaped subformulas resulting from taking some subset of the conjuncts in the original CQ-shaped formula $\phi$, so these specializations of these subformulas have already been accounted for. Hence, for each negative CQ-shaped subformula in $Y^p : \phi_Y$, there are most $2^{2^n w} w^n$ specializations that may be added.

This means that the size of $cl(Y^p : \phi_Y)$ is at most $2n(2^n w^n + 2^{2^n w} w^n)$.

Now recall that $P$ is a fixed set of parameters of size at most $2w$. Each formula in $cl(Y^p : \phi_Y)$ has at most $w$ variables that can be mapped to parameters from $P$ or constants from $e$, so the size of $cl_Y(P)$ is $M := 2n(2^n w^n + 2^{2^n w} w^n)(2w + l)w$.

Every mosaic in $\mathcal{M}$ contains a subset of the formulas in $cl_Y(P)$, is labelled $L$ and $R$, and has some subset of its at most $w$ parameters distinguished as the relativized parameters. Hence, $|\mathcal{M}| \leq 2 \cdot 2^w \cdot 2^{|M|} \leq 2^n(n)^2p(hw)$ for some polynomial function $p$ that is independent of $\varphi_L$ and $\varphi_R$. 

C. Interpolation using Relativized GNF Mosaics

In this section, we present details for the interpolation results stated in Theorem 9.

We first prove a mosaic interpolation lemma (like Lemma 4) for the relativized GNF mosaics defined in Appendix B. We then discuss how to obtain the interpolation results stated in Theorem 9 from these mosaic interpolants.

Throughout this section, we continue to restrict our attention to input formulas $\varphi_L$ and $\varphi_R$ that do not use equality. Please see Appendix D for remarks on how to extend this approach to deal with equality. We remark that even with equality-free input, the interpolants may use equality (for guards of negated formulas with at most one free variable).

C.1 Mosaic interpolants for relativized GNF mosaics

Fix $U$-relativized GNF formulas $\varphi_L$ and $\varphi_R$ without equality as in the previous section. That is, $\varphi_L$ and $\varphi_R$ are $U$-relativized formulas in weak GNF-normal form and without equality over signatures $\sigma_L$ and $\sigma_R$. Moreover, $\text{par}(\varphi_L) = \text{par}(\varphi_R)$.

Let $\mathcal{M}$ be the set of relativized GNF mosaics for $\varphi_L \land \neg \varphi_R$ as defined in Appendix B.

Consider the mosaic elimination procedure over $\mathcal{M}$. We write $N'$ for the set of mosaics for $\varphi_L \land \neg \varphi_R$ that are eliminated using this procedure, and $N_i$ for the set of mosaics that have been removed by stage $i$. By Theorem 28, we know that every mosaic containing $L^+ : \varphi_L$ and $R^- : \varphi_R$ must be in $N'$.

For an $X$-mosaic $\tau(x)$, let

$$\tau_X := \bigwedge_{X \subseteq P, \psi \in \tau} p\psi \land \bigwedge_{X \subseteq P, \psi \in \tau} \neg p\psi$$

and

$$\tau^X := \bigwedge_{X \subseteq P, \psi \in \tau} p\psi \land \bigwedge_{X \subseteq P, \psi \in \tau} \neg p\psi.$$

That is, $\tau_Y$ asserts the truth of the $Y$-formulas in $\tau$, and the negation of the $Y$-formulas that are not in $\tau$ (but satisfy the conditions for being in $\tau$ in terms of $X$-safety if $Y = X$).

We now define mosaic interpolants for every $X$-mosaic $\tau(x) \in N'$.

**Lemma 36.** For each $X$-mosaic $\tau(x) \in N_i$, relativizing $d$, we can construct a DAG representation of a formula $\theta^X_x$ such that

- (Imp) $\tau_x \models \theta^X_x$ and $\theta^X_x \models \neg \tau^X_x$;
- (Occ) $\text{occ}(\theta^X_x) \subseteq \text{occ}(\varphi_L) \cap \text{occ}(\varphi_R)$ if $X = L$;
- $\text{occ}(\neg \theta^X_x) \subseteq \text{occ}(\varphi_L) \cap \text{occ}(\varphi_R)$ if $X = R$;
- $\text{con}(\theta^X_x) \subseteq \text{con}(\varphi_L) \land \text{con}(\varphi_R)$;
- (Par) $\text{par}(\theta^X_x) \subseteq \text{par}(\varphi_L) \land \text{par}(\varphi_R)$ and $d$ is $X$-relativized in $\tau$;
- (Rel) $\theta^X_x$ is in $U$-relativized GNF (when treating $T$ and $\bot$ as atomic formulas), even when parameters are viewed as free variables.

Moreover, if $n = |\varphi_L| + |\varphi_R|$, $w = \text{width}(\varphi_L \land \neg \varphi_R)$, and $h = \text{rank}_{CQ}(\varphi_L \land \neg \varphi_R)$ then there is a DAG representation of $\Theta_i := \{\theta^X_x : \tau(x) \land \neg \psi(x)\}$ where $\psi$ is a $q$-polynomial function independent of $\varphi_L$ and $\varphi_R$.

**Proof.** We fix some notation.

For any tuple $b$ of parameters that is $X$-guarded in $\tau$, we define a formula $\text{gdd}^X_x(b)$. If $b = 0$, then we set $\text{gdd}^X_x(b) := \top$. If $b = b$ (a single parameter), then we set $\text{gdd}^X_x(b) := (b = b)$ (an equality guard). Otherwise, there is some $X^+ : \exists x.(U(x) \land \beta(xb')) \in \tau$ where $\beta$ and $U$ are $X$-catoms, $U \subseteq U$, $b' \supseteq b$, and $b' \in d$ is $X$-relativized, and we set $\text{gdd}^X_x(b) := \exists x.(U(x) \land \beta(xb'))$. In this last case, when we write $\text{gdd}^X_x(b) \land \neg \psi$ we mean the formula $\exists x.(U(x) \land \beta(xb') \land \neg \psi)$.

The proof proceeds by induction on $i$, the stage at which $\tau$ was eliminated. We intersperse the definitions of the interpolants and proofs of correctness.

**Base case.** Assume there is an $X$-mosaic $\tau(x) \in N_i$. Then $\tau$ has an internal inconsistency.

Assume one of the following conditions holds for $Y \in \{L, R\}$:

- $Y^+ : \alpha(b), Y^- : \alpha(b) \in \tau$;
- $Y^+ : \bigwedge_i \psi_i \in \tau$ (respectively, $Y^- : \bigvee_i \psi_i \in \tau$), but $Y^+ : \psi_i \notin \tau$ (respectively, $Y^- : \psi_i \notin \tau$) for some $i$;
- $Y^+ : \bigvee_i \psi_i \in \tau$ (respectively, $Y^- : \bigwedge_i \psi_i \in \tau$), but $Y^+ : \psi_i \notin \tau$ (respectively, $Y^- : \psi_i \notin \tau$) for all $i$;
- $Y^+ : \alpha(b) \land \neg \psi(b) \in \tau$, but $Y^+ : \psi(b) \notin \tau$ or $Y^- : \psi(b) \notin \tau$;
- $Y^+ : \alpha(b) \in \tau$ and $Y^- : \alpha(b) \land \neg \psi(b) \in \tau$, but $Y^+ : \psi(b) \notin \tau$.

These internal inconsistencies produce very simple interpolants. If $X = Y$, then $\theta^X_x := \bot$. If $X \neq Y$, then $\theta^X_x := \top$.

The more interesting cases are when the inconsistency comes from interaction between $L$ and $R$ formulas.
Assume $X^+ : \alpha(b), \bar{X}^- : \alpha(b) \in \tau$. Then we set $\theta^\tau_X := \alpha(b)$. This clearly satisfies (Imp), (Par), and (Rel).

Assume $X^+ : \alpha(b), \bar{X}^- : \alpha(b) \in \tau$ then $\theta^\tau_X := \text{gld}_X(b) \wedge \neg \alpha(b)$. Although $\neg \alpha(b)$ would suffice for (Imp), we need to guard this formula with $\text{gld}_X(b)$ in order to ensure that we remain in GNF. Because $b$ occurs in a $X^+$-formula in the $X$-mosaic $\tau$, $\text{gld}_X(b)$ exists with $X^+ : \text{gld}_X(b) \in \tau$. Hence, (Imp) holds, and $\theta^\tau_X$ is in GNF as required for (Rel). Property (Par) is immediate.

Assume $X^+ : \alpha(b)$ and $X^-$ : $\alpha(b) \wedge \neg \psi(b) \in \tau$, but $\bar{X}^- : \psi(b) \not\in \tau$. Then we set $\theta^\tau_X := \alpha(b)$. (Par) and (Rel) clearly hold. For (Imp), it is clear that $\tau_X \models \theta^\tau_X$. To prove the other part of (Imp), we argue that $\tau_X \models \neg \theta^\tau_X$. Suppose for the sake of contradiction that there is some model $\mathfrak{M}$ of $\tau_X \wedge \alpha(b)$. Since $X^- : \alpha(b) \wedge \neg \psi(b) \in \tau$, this means that $\mathfrak{M} \models \neg (\alpha(b) \wedge \neg \psi(b)) \wedge \alpha(b)$, so $\mathfrak{M} \models \psi(b)$.

By assumption, $X^+ : \psi(b) \not\in \tau$. Notice $X^+ : \alpha(b) \in \tau$ for $\alpha$ appearing positively in an $X$-formula and negatively in a $\bar{X}$-formula. Since occurrences are preserved in moving to mosaics (Proposition 27), this means that $\alpha$ is an $X$-eguard. Likewise, $b \cap d$ is $X$-cretelativized in $\tau$, since $X^- : \alpha(b) \wedge \neg \psi(b) \in \tau$. Hence, $b$ is $X$-eguarded in $\tau$ by $X^+ : \alpha(b)$. We can conclude that $\neg \psi(b)$ is a conjunct in $\tau_X$, so $\mathfrak{M} \models \neg \psi(b)$, a contradiction.

Assuming $X^+ : \alpha(b)$ and $X^- : \alpha(b) \wedge \neg \psi(b) \in \tau$, but $X^+ : \psi(b) \not\in \tau$, a similar argument shows that if we set $\theta^\tau_X := \text{gld}_X(b) \wedge \neg \alpha(b)$ the desired properties are satisfied.

Assume $X^- : \phi \in \tau$ for $\phi$ of the form $\exists x. \psi(xb) \in \tau$. Assume there is some specialization $U_0(c') \wedge \chi_0(bc') \wedge \bigwedge_i \exists x_k U_i(x_k) \wedge \chi_i(x_kbc')$ of $\phi$ such that $U_0(c') = U_1c_1 \cdots U_jc_j$ and $Z^+_i : U_0c_i \in \tau$ for all $i \in \{1, \ldots, j\}$, but $X^- : \chi_0(bc') \wedge \bigwedge_i \exists x_k U_i(x_k) \wedge \chi_i(x_kbc') \not\in \tau$. Set $\theta^\tau_X$ to be the conjunct of $U_i c_i$ for all $i$ such that such that such that $Z_i = X$ (the empty conjunct is $\tau$). Note that (Par) is satisfied since for all $s$ such that $Z_i = c, c_i \in c$ and $X^+ : U_i c_i \in \tau$ witnesses the fact that $c_i$ is $X$-cretelativized in $\tau$ (this follows from Proposition 27 since $U_i$ appears positively in an $X$-formula, and negatively in a $\bar{X}$-formula). (Rel) is trivially satisfied. For (Imp), it is clear that $\tau_X \models \theta^\tau_X$. Now suppose for the sake of contradiction that there is some model $\mathfrak{M}$ such that $\mathfrak{M} \models \neg \theta^\tau_X$. Then $\mathfrak{M} \models U_i c_i$ for all $i$ and $\mathfrak{M} \models \chi_0(bc') \wedge \bigwedge_i \exists x_k U_i(x_k) \wedge \chi_i(x_kbc')$, so the specialization of $\phi$ holds in $\mathfrak{M}$. But it is straightforward to show that every specialization of $\phi$ implies $\phi (\text{Lemma 26})$, a contradiction of $\mathfrak{M} \models \tau_X$.

Inductive case. Now assume there is an $X$-mosaic $\tau(c) \in \mathcal{N}_i \backslash \mathcal{N}_{i-1}$ relativizing $d \subseteq c$. Then there is some $Y^+ : \eta = Y^+ : \exists x. \psi(xb) \in \tau$, and if $\tau \rightarrow_{Y^+ + \eta} \tau'$ then $\tau' \in \mathcal{N}_{i-1}$.

Let $r$ be the parameters that are relativized in any such $\tau'$ (that is, $r$ is $a \cup (b \cap d)$ if $Y = X$, and $r$ is $a$ if $Y = \bar{X}$).

We introduce some additional notation. For $Z \in \{L, R\}$, we define $\tau_{X^+, \eta}^Z$ to be the $Z$-mosaic over parameters $b$ and relativizing $r$ that consists of the union of (i) the $Z$-formulas in $\eta$ that only use parameters from $b$, and (ii) the $\bar{Z}$-formulas in $\tau$ whose parameters are contained in $b$ and are $Z$-safe in the formulas added in step (i). For mosaics $\psi, \psi'$ and $Z \in \{L, R\}$, we write $\psi(ab) \supseteq \psi$ if $\psi$ uses only parameters from $ab$ and is obtained from $\psi$ by adding only $Z$-formulas (so the $\bar{Z}$-formulas in $\psi$ and $\psi'$ are identical). For a $Y$-mosaic $\psi'$, we write $\text{rel}_{X^+, \eta}^Z(\alpha)$ for the conjunction of all $Ua$ such that $U \in \mathcal{U}$ is an $Y$-catom and $Y^+ : Ua \in \psi'$ (so $Y^+ : Ua$ is an $Y'$-cretelativizer in $\psi'$).

Assume $Y = X$. Then let
\[
\psi(ab) \supseteq \tau_{X^+, \eta}^Z \quad \text{s.t.} \quad \tau \rightarrow_{X^+, \eta} \tau' \quad \text{then} \quad \nu(ab) \supseteq \tau_{X^+, \eta}^Z \quad \text{s.t.} \quad \tau \rightarrow_{X^+, \eta} \tau'.
\]

where $a' \subseteq x$ is the subset of $x$ appearing in $\text{rel}_{X^+, \eta}^Z(\alpha)$.

We now begin the proof of correctness, starting with (Imp). Assume there is a model $\mathfrak{M}$ for $\tau_X$. Since $X^+ : \eta \in \tau$, this means that $\mathfrak{M} \models \exists x. \psi(xb)$, so there are elements $a' \in M$ and an expansion $\mathfrak{M}'$ of $\mathfrak{M}$ with the interpretation $a_{X^+, \eta}^Z := a'$ such that $\mathfrak{M}' \models \psi(ab)$.

Take $\nu$ to be the union of $\tau_{X^+, \eta}^Z$ and the set of formulas $X^P : \psi'$ in $cl_X(ab)$ such that $\mathfrak{M}' \models \psi(\nu)$. It is straightforward to check that $\tau \rightarrow_{X^+, \eta} \nu'$. By construction, $\mathfrak{M}' \models \nu' \wedge \neg \psi'(ab)$. Moreover, for any choice of $\nu(ab) \supseteq \nu', \mathfrak{M}' \models \tau_X$ (since the $X$-formulas are identical in $\nu'$ and $\tau'$). By the inductive hypothesis, this means that $\mathfrak{M}' \models \theta^\tau_X$. Hence, $\mathfrak{M}' \models \theta^\tau_X(a) \wedge \bigwedge s.t. \tau \rightarrow_{X^+, \eta} \tau' \quad \text{such that} \quad \nu(ab) \supseteq \theta^\tau_X$.

Now assume that there is a model $\mathfrak{M}$ of $\theta^\tau_X$. Then there is some $\nu(ab) \supseteq \tau_{X^+, \eta}^Z$ with $\tau \rightarrow_{X^+, \eta} \nu' \quad \text{such that} \quad \nu(ab) \supseteq \theta^\tau_X(a)$, and an expansion $\mathfrak{M}'$ of $\mathfrak{M}$ with the interpretation $a_{X^+, \eta}^Z := a'$, such that $\mathfrak{M}' \models \theta^\tau_X(a) \wedge \bigwedge s.t. \tau \rightarrow_{X^+, \eta} \tau'$.
Note that for all $\tau'(ab) \supseteq \chi \chi'$ with $\tau \rightarrow \chi \chi'$, $\mathcal{M}' \models \theta_{\chi}^{\tau'}$ and consequently, the inductive hypothesis implies $\mathcal{M}' \models \neg \tau_{\chi}$. In particular, consider $\tau' := \chi' \cup S' \supseteq \chi \chi'$ where $S'$ is the set of formulas $\bar{X}^p : \psi'$ in $cl_{\chi}^{\chi}(ab)$ such that $\mathcal{M}' \models p\psi'$, $\text{par}(\psi') \cap a \neq 0$, and $X^p : \psi'$ is $X$-safe in $\chi'$. It is clear that $\tau'(ab) \supseteq \chi \chi'$ and $\tau \rightarrow \chi \chi'$, so $\mathcal{M}' \models \neg \tau_{\chi}$ as observed above.

Since $\mathcal{M}' \models \neg \tau_{\chi}$, there is some conjunct $\chi$ in $\tau_{\chi}$ such that $\mathcal{M}' \models \neg \chi$. Consider some conjunct $\chi(ab)$ in $\tau_{\chi}$ that actually uses some parameters from $a$. Then by choice of $S'$, $\mathcal{M}' \models \chi(ab)$, so this formula cannot witness the fact that $\mathcal{M}' \models \neg \tau_{\chi}$. This means there must be some conjunct $\chi(b)$ in $\tau_{\chi}$ that only uses parameters from $b$ such that $\mathcal{M}' \models \neg \chi(b)$. By definition of $\tau \rightarrow \chi \chi'$, $\chi(b)$ must also be a conjunct in $\tau_{\chi}$. Hence, $\mathcal{M}' \models \neg \tau_{\chi}$.

For (Occ), the desired occurrence of constants and the polarity of relations follows from the inductive hypothesis and the fact that occurrences are preserved in moving to mosaics (Proposition 27).

The last thing to prove is (Par) and (Rel). The inductively defined interpolants $\theta_{\chi}^{\tau'}$ only use parameters in $\tau'$. Moreover, any parameters in $a \cup (b \cap d)$ are $X$-crelativized in $\tau'$. By construction, $\tau'$ and $\chi'$ have the same $X$-formulas, so these $X$-crelativizers are also in $\chi'$.

In particular, the $X$-crelativizers for parameters in $a$ appear as conjuncts in $\text{rel}_{\chi}^{\chi}(a)$. This means that any parameters from $a$ in $\theta_{\chi}^{\tau'}$ are removed using existential quantification, and this quantification is $\exists$-relativized by $\text{rel}_{\chi}^{\chi}(a)$.

The remaining parameters in $\theta_{\chi}^{\tau'}$ are from $b$, so $\text{par}(\theta_{\chi}^{\tau'}) \subseteq c$. Moreover, any $X$-relativizers for $b \cap d$ in $\chi'$ must also be in $\tau$ (since $\tau \rightarrow \chi \chi'$ and parameters $b$ occur in both $\tau$ and $\chi'$). Hence, $\text{par}(\theta_{\chi}^{\tau'}) \cap d$ is $X$-crelativized in $\tau$.

Overall, this means that (Par) and (Rel) hold.

Assume $Y = \bar{X}$. Then let $\text{gd}(\bar{X}, \chi') \wedge \neg \left( \bigvee_{\nu'(ab) \supseteq X \chi\eta^p} \exists x'. \left( \text{rel}_{\chi}^{\chi}(a)[x/a] \wedge \bigwedge_{\kappa \in \tau \rightarrow \chi \chi'} \theta_{\chi}^{\chi}(x(a)) \right) \right)$

where $x' \subseteq x$ is the subset of $x$ appearing in $\text{rel}_{\chi}^{\chi}(a)[x/a]$.

We prove (Imp) by showing $\tau_{\chi'} \models \neg \theta_{\chi}$ and $\tau_{\chi} \models \neg \tau_{\chi}$. Assume there is a model $\mathcal{M}$ for $\tau_{\chi}$. Since $\bar{X}^p : \eta \in \tau$, this means that $\mathcal{M} \models \exists x. \psi(x(b))$, so there are elements $a' \in M$ and an expansion $\mathcal{M}'$ of $\mathcal{M}$ with the interpretation $\theta_{\chi}^{\tau'} := a'$ such that $\mathcal{M}' \models \psi(ab)$. Take $\nu'$ to be the union of $\theta_{\chi}^{\tau'}$ and the set of formulas $\bar{X}^p : \psi'$ in $cl_{\chi}^{\chi}(ab)$ such that $\mathcal{M}' \models p\psi'$. It is straightforward to check that $\tau \rightarrow \chi \chi'$.

By the inductive hypothesis, this means that $\mathcal{M}' \models \theta_{\chi}^{\tau'}$. Hence, $\mathcal{M}' \models \text{rel}_{\chi}^{\chi}(a) \wedge \bigwedge_{\kappa \in \tau \rightarrow \chi \chi'} \theta_{\chi}^{\chi}(x(a))$, so $\mathcal{M}' \models \neg \theta_{\chi}^{\tau'}$.

Now assume that there is a model $\mathcal{M}$ of $\neg \theta_{\chi}^{\tau'}$. If $\mathcal{M} \models \neg \text{gd}(\bar{X}, \chi')$, then it must be the case that $|b| > 1$ (since $\neg \tau$ and $\neg (b = b)$ cannot hold in $\mathcal{M}$). Thus, $\mathcal{M} \models \neg \tau_{\chi}$ since $\bar{X}^p : \eta \in \tau$.

Otherwise, there is some $\nu'(ab) \supseteq \chi \chi'$ with $\tau \rightarrow \chi \chi'$, elements $a' \in M$, and an expansion $\mathcal{M}'$ of $\mathcal{M}$ with the interpretation $\theta_{\chi}^{\tau'} := a'$, such that $\mathcal{M}' \models \text{rel}_{\chi}^{\chi}(a) \wedge \bigwedge_{\kappa \in \tau \rightarrow \chi \chi'} \theta_{\chi}^{\chi}(x(a))$.

Note that for all $\nu'(ab) \supseteq \chi \chi'$ with $\tau \rightarrow \chi \chi'$, $\mathcal{M}' \models \theta_{\chi}^{\tau'}$ and consequently, the inductive hypothesis implies $\mathcal{M}' \models \neg \tau_{\chi}$. In particular, consider $\nu' := \nu' \cup S' \supseteq \chi \chi'$ where $S'$ is the set of formulas $\bar{X}^p : \psi'$ in $cl_{\chi}^{\chi}(ab)$ such that $\mathcal{M}' \models p\psi'$, $\text{par}(\psi') \cap a \neq 0$, and $X^p : \psi'$ is $X$-safe in $\nu'$. It is clear that $\nu'(ab) \supseteq \chi \chi'$ and $\tau \rightarrow \chi \chi'$, so $\mathcal{M}' \models \neg \tau_{\chi}$ as observed above.

Since $\mathcal{M}' \models \neg \tau_{\chi}$, there is some conjunct $\chi$ in $\tau_{\chi}$ such that $\mathcal{M}' \models \neg \chi$. Consider some conjunct $\chi(ab)$ in $\tau_{\chi}$ that actually uses some parameters from $a$. Then by choice of $S'$, $\mathcal{M}' \models \chi(ab)$, so this formula cannot witness the fact that $\mathcal{M}' \models \neg \tau_{\chi}$. This means there must be some conjunct $\chi(b)$ in $\tau_{\chi}$ that only uses parameters from $b$ such that $\mathcal{M}' \models \neg \chi(b)$. By definition of $\tau \rightarrow \chi \chi'$, $\chi(b)$ must also be a conjunct in $\tau_{\chi}$. Hence, $\mathcal{M}' \models \neg \tau_{\chi}$.

For (Occ), the desired occurrence of constants and the polarity of relations follows from the inductive hypothesis and the fact that occurrences are preserved in moving to mosaics (Proposition 27).

As an example, we give the proof if $X = R$ and $\text{gd}(\bar{X}, \chi')$ corresponds to an atomic formula. Let $(\alpha, p) \in \text{occ}(\theta_{\chi}^{\tau'})$ (we abuse notation slightly and write this to mean that the relation $S$ used in $\alpha$ satisfies $(S, p) \in \text{occ}(\theta_{\chi}^{\tau'})$). One possibility is that $(\alpha, p)$ is $(\text{gd}(\bar{X}, \chi'), \neg)$, since $\text{gd}(\bar{X}, \chi')$ occurs negatively in $\neg \theta_{\chi}$. By definition of an $X$-catom for $X = R$, $(\text{gd}(\bar{X}, \chi'), \neg) \in \text{occ}(\varphi_1) \cap \text{occ}(\varphi_2)$. Another possibility is that $(\alpha, p) = (U, +)$ for a conjunct $Ua$ in $\text{rel}_{\chi}^{\chi}(a)$. By definition of a $X$-crelativizer for $X = R$, we must have $(U, +) \in \text{occ}(\varphi_1) \cap \text{occ}(\varphi_2)$. Otherwise, $(\alpha, p)$ comes from $\text{occ}(\theta_{\chi}^{\tau'})$ for some $\tau'$ such that $\tau \rightarrow \chi \chi'$. By the inductive hypothesis, this means $(\alpha, p) \in \text{occ}(\varphi_1) \cap \text{occ}(\varphi_2)$. Overall, this means that $\text{occ}(\neg \theta_{\chi}^{\tau'}) \subseteq \text{occ}(\varphi_1) \cap \text{occ}(\varphi_2)$ so (Occ) holds for $\theta_{\chi}^{\tau'}$.

The last thing to prove is (Par) and (Rel). The inductively defined interpolants $\theta_{\chi}^{\tau'}$ only use parameters in $\tau'$. Moreover, any parameters in $\theta_{\chi}^{\tau'}$ from $a$ are $X$-crelativized in $\tau'$. By construction, $\tau'$ and $\chi'$ have the same $X$-formulas, so these $X$-crelativizers are also in $\chi'$.
In particular, the $\tilde{X}$-crelativizers for parameters in $a$ appear as conjuncts in $\text{rel}^{\psi'}_X(\alpha)$. This means that any parameters from $a$ in $\theta^\psi_X$ are removed using existential quantification, and this quantification is $U$-relativized by $\text{rel}^{\psi'}_X(\alpha)$.

The remaining parameters in the negated subformula of $\theta^\psi_X$ are from $b$. Because $\tilde{X}^+$ is a $\tilde{X}^+$-formula using $b$ in an $X$-mosaic, we know that any parameters in $b \cap a$ are $\tilde{X}$-crelativized in $\tau$ and $b$ is $X$-guarded in $\tau$. This means that $\text{gcd}^\psi_X(b)$ is defined and can guard the parameters in $b$ in the negated subformula.

Overall, this means (Par) and (Rel) hold.

Size of interpolants. Finally, we seek to bound the size of the shared DAG representation $\Theta$, of all $\theta^\psi_X$ coming from $\tau \in N_\ell$, such that $|\Theta| \leq i \cdot 2^{n\ell} \cdot 2^{p\psi(h_\psi)}$ for some polynomial function $q$ independent of $\varphi_L$, and $\varphi_R$.

Let $n = |\varphi_L| + |\varphi_R|$, $w = \text{width}(|\varphi_L \land \neg \varphi_R|)$, and $b = \text{rank}c^\ell_\psi(\varphi_L \land \neg \varphi_R)$. Let $f(n, h, w) = 2^n \cdot 2^p(h_\psi)$ be the bound on the total number of mosaics for $\varphi_L$ and $\varphi_R$ in $M$ given by Proposition 35.

Let $\Theta_0$ be the empty graph. We claim that at each stage $i > 0$ in the interpolant construction, the number of nodes and edges added to $\Theta_{i-1}$ in order to represent the new formulas added in stage $i$ is polynomial in $f(n, h, w)$. The desired bound for the size of the DAG representations follows from this claim.

First consider $i = 1$. To build $\Theta_1$, we start with the graph consisting of nodes labelled by atomic formulas (without provenance labels) from $c\lambda(P) \cup c\mu(P)$, as well as nodes labelled with $\top$ and $\bot$. The number of nodes in this graph is at most $f(n, h, w)$. Constructing a formula $\theta^\psi_X$ for $\tau \in N_\ell$ results in a formula with at most $w + 2$ non-atomic subformulas (for a formula of the form $\exists x. \beta(xb) \land \neg \alpha(b)$), so $w + 2$ new nodes may be needed in the DAG representation in addition to the nodes for atomic formulas. Since $|N_\ell| \leq f(n, h, w)$, this means the total size of $\Theta_1$ is quadratic in $f(n, h, w)$.

Now consider $i > 1$. For each $\tau \in N_\ell$, the formula $\theta^\psi_X$ results in formulas with (at most) the following new subformulas:

- $w$ subformulas from existentially quantified variables in $\text{gcd}$,
- three subformulas from the guarded negation,
- $f(n, h, w)$ subformulas from the disjunction over $\psi'$, each with $w$ existential subformulas, $w$ subformulas from rel, and $f(n, h, w)$ subformulas from the conjunction over $\tau'$.

Since $|N_\ell| \leq f(n, h, w)$, this means that the number of new nodes and edges added to the DAG representation of $\Theta_{i-1}$ in order to construct the DAG representation of $\Theta_1$ is polynomial (cubic) in $f(n, h, w)$.

C.2 Constructive interpolation (Proof of Theorem 9 for equality-free formulas)

We now prove Theorem 9, with the additional restriction that the input formulas do not use equality.

Recall the statement:

Let $\varphi_L$ and $\varphi_R$ be GNF (respectively, UNF) formulas without equality over signatures $\sigma_L$ and $\sigma_R$, respectively. If $\varphi_L \models \varphi_R$ and $|\varphi_L| + |\varphi_R| = n$, then we can construct a DAG representation of a GNF (respectively, UNF) interpolant $\theta$ such that

- $\varphi_L \models \theta$ and $\theta \models \varphi_R$;
- $\text{occ}(\theta) \subseteq \text{occ}(\varphi_L) \cap \text{occ}(\varphi_R)$;
- $\text{free}(\theta) \subseteq \text{free}(\varphi_L) \cap \text{free}(\varphi_R)$;
- $\text{con}(\theta) \subseteq \text{con}(\varphi_L) \cap \text{con}(\varphi_R)$;
- if $\varphi_L$ and $\varphi_R$ are $U$-relativized, for $U$ a distinguished set of unary relations from $\sigma_L$, $\sigma_R$, then $\theta$ is $U$-relativized (when treating $\top$ and $\bot$ as atomic formulas);
- the DAG representation of $\theta$ is of size at most $2^{2^n}$ for some polynomial function $p$ independent of $\varphi_L$ and $\varphi_R$ (and this can be improved to a size of at most $2^{2^n}$ when $\varphi_L$ and $\varphi_R$ are in GF without equality and the bound on the arity of relations is fixed).

We consider various cases, depending on the form of the input formulas $\varphi_L$ and $\varphi_R$. Let $n = |\varphi_L| + |\varphi_R|$. We can always assume that $\text{free}(\varphi_L) = \text{free}(\varphi_R)$ (because, we can existentially quantify out the free variables of $\varphi_L$ that are not free variables of $\varphi_R$, and we can universally quantify the free variables of $\varphi_R$ that are not free variables of $\varphi_L$, introducing a fresh guard relation if necessary).

Relativized GNF input in weak GN-normal form. Assume $\varphi_L$ and $\varphi_R$ are in $U$-relativized GNF formulas without equality, and are already in weak GN-normal form. Moreover, assume $\text{free}(\varphi_L) = \text{free}(\varphi_R) = z$.

Let $\varphi_L := \varphi'_L$ and $\varphi_R := \varphi'_R$.

Use the mosaic elimination procedure over the set $M$ of relativized GNF mosaics for $\varphi_L \land \neg \varphi_R$. Then construct mosaic interpolants for the eliminated mosaics $N'' \subseteq M$ using Lemma 36, and let

$$\theta := \bigvee_{\tau(z) \subseteq L \colon \varphi_L} \bigwedge_{s.t., \tau' \subseteq C\mathcal{N}} \theta^\psi_X,$$

where $N''$ is the set of eliminated mosaics and we write $\psi'(z) \supseteq \psi$ if $\psi'$ is over parameters $z$ and relativizing 0, and is obtained from $\psi$ by adding only Z-formulas.

We claim $\theta$ is a GNF interpolant for $\varphi_L \models \varphi'_R$ satisfying the properties in Theorem 9.

We first prove $\varphi'_L \models \theta \models \varphi'_R$.

Assume there is a structure $\mathfrak{M}$ such that $\mathfrak{M} \models \varphi'_L$. Let $\tau(z) \subseteq L \colon \varphi_L$ be the set of formulas from $c\lambda(z)$ that are true in $\mathfrak{M}$. For any $\tau'(z) \subseteq \tau$, $\mathfrak{M} \models \theta^\psi_X$. By Lemma 36, this means that $\mathfrak{M} \models \theta^\psi_X$. Overall, $\mathfrak{M} \models \theta$.

Now assume there is a structure $\mathfrak{M}$ such that $\mathfrak{M} \models \theta$. Then there is some $\tau(z) \subseteq L \colon \varphi_L$ such that $\mathfrak{M} \models \land_{s.t., \tau' \subseteq C\mathcal{N}} \theta^\psi_X$. Let $S'$ be the set of formulas $R^\psi : \psi$ in $c\lambda(z)$ such that $\mathfrak{M} \models p\psi$, and $R^\psi : \psi$ is $L$-safe in $\tau$. Consider the L-mosaic over $z$ (relativizing 0) such that
\( \tau' := \tau \cup S' \cup \{ R^- : \varphi_R \} \). By completeness of the mosaic system (Proposition 34, which is part of Theorem 28), every mosaic containing \( L^+ : \varphi_L \) and \( R^- : \varphi_R \) must be in \( N^\tau \). Hence, \( \tau'(z) \geq_R \tau \) and \( \tau' \in N^\tau \), so \( \mathfrak{M} \models \tau'_R \). By Lemma 36, this implies that \( \mathfrak{M} \models \neg \tau_R \). But every formula \( R \psi \in S' \) was chosen such that \( \mathfrak{M} \models \psi \). The only way that \( \mathfrak{M} \models \neg \tau_R \) is if \( \mathfrak{M} \models \varphi_R \).

Since each \( \theta_R \) is in \( U \)-relativized GNF by Lemma 36, this formula is in \( U \)-relativized GNF, with \( T \) and \( \bot \) Replacing \( T \) and \( \bot \) by \( \exists x. x = x \) and \( \exists x. (x = x \land \neg(x = x)) \), respectively, keeps the formula in GNF, but the resulting formula is technically no longer \( U \)-relativized. Note that it is not necessarily in GN-normal form or weak GN-normal form.

The other properties about the polarity of relations and constants follow from Lemma 36.

Let if \( n = |\varphi_L| + |\varphi_R| \), \( w = \text{width}(\varphi_L \land \neg \varphi_R) \), and \( h = \text{rank}_{CQ}(\varphi_L \land \neg \varphi_R) \). Since the combined size of the DAG representation of all \( \theta_R \) used in \( \theta \) is at most \( q^2(n)2^{q(hw)} \) (for polynomial function \( q \) given by Lemma 36 that are independent of the input formulas), and there are at most \( (2^q(n)2^{q(hw)})^2 \) new nodes that need to be added to this DAG representation to construct \( \theta \), the overall size of the DAG representation of \( \theta \) is at most \( 2^r(n)2^{r(hw)} \) for some polynomial functions \( r \) independent of the input formulas. Since \( w, h \leq n \), this means that overall, the size of this DAG representation for \( \theta \) is at most doubly exponential in \( n \).

**Relativized GNF input.** Convert the formulas above to weak GN-normal form using Lemma 23, and then argue as in the previous case for this \( U \cup \{ U \} \)-relativized formulas.

Because the conversion to weak GN-normal form keeps the width and CQ-rank bounded by \( n \) and the overall size exponential in \( n \), the size of the DAG representation of the interpolant is still at most doubly exponential in \( n \).

**GNF input with ordinary (unrelativized) quantifiers.** Assume \( \varphi'_L \) and \( \varphi'_R \) are in GNF and have ordinary (unrelativized) quantifiers. We introduce a new unary relation \( U \) that is not in \( \sigma_L \) or \( \sigma_R \). Let \( \varphi'_X \) be the result of relativizing all quantifiers in \( \varphi'_X \) to \( U \). Consider the validity

\[
\bigwedge_{x \in \mathbb{A}} U z \land \bigwedge_{e \in \mathbb{A}'} U e \land \varphi'_L \models \varphi'_R
\]

where \( z := \text{par}(\varphi'_L) = \text{par}(\varphi'_R) \) and \( e := \text{con}(\varphi'_L) \cup \text{con}(\varphi'_R) \). Then apply the interpolation result for this relativized GNF input, which yields a \( U \)-relativized interpolant \( \theta' \) for \( U = \{ U \} \). Replacing all occurrences of \( U x \) in \( \theta' \) with \( T \) results in a GNF interpolant \( \theta \) for the original validity.

**UNF input.** Assume \( \varphi'_L \) and \( \varphi'_R \) are in UNF (and for simplicity that they are already in weak GN-normal form). Observe that in the interpolation construction in Lemma 36, the only guards for negation that are used for an interpolant \( \theta_X \) are equality guards of the form \( x = x \) or \( X \)-guards in \( \tau \). By definition, \( X \)-guards must have appeared as guards in one of the original formulas \( \varphi'_L \) and \( \varphi'_R \). Since the only guards in the original formulas were unary guards, this means that \( \theta_X \) uses only unary negation.

For UNF input that is not in weak GN-normal form, the conversion into weak GN-normal form using Lemma 23 keeps the formula in UNF and does not change the guards, so this argument applies.

**GF input.** Assume \( \varphi'_L \) and \( \varphi'_R \) are in GF. We can convert these formulas to weak GN-normal form formulas \( \varphi_L \) and \( \varphi_R \) as described in Lemma 24, and then argue as in the previous cases. Because the conversion to weak GN-normal form keeps the width the same, CQ-rank 1, and size polynomial in \( n \), this means that the size of the DAG representation for the interpolant is doubly exponential in \( n \) in general, but singly exponential in \( n \) when the width is considered fixed.

### D. Equality

In this section, we describe a way to extend our interpolation results to handle input formulas with equality.

The first thing we show is that any GNF formula with equality can be converted to a form with a very limited use of equality. Let us say that a GNF formula \( \phi \) is equality-normalized if

(i) every occurrence of a relational atomic formula \( R t \) in \( \phi \) appears in conjunction with

\[
\text{all-distinct}(t) := \bigwedge_{t \in \mathbb{A}} (t = t) \land \bigwedge_{t, t' \in \mathbb{A}, t' \neq t} \neg (t = t'),
\]

(ii) whenever equalities are used as guards for negations, then these equality guards are of the form \( x = x \), and

(iii) there are no occurrences of equality in \( \phi \) other than those described in (i) and (ii).

Let \( \phi(x) \) be a GNF formula containing constants \( e \), and let \( \equiv \) be any equivalence relation over \( \{ x, e \} \). We denote by \( \xi_\equiv \) the first-order formula

\[
\bigwedge_{s \in \mathbb{A}} (s = t) \land \bigwedge_{s \in \mathbb{A}'} \neg (s = t),
\]

that is, the conjunction of all equalities and inequalities corresponding to \( \equiv \). Note that this formula does not necessarily belong to GNF as it may contain unguarded inequalities.

**Lemma 37.** Let \( \phi(x) \) be a GNF formula in GN-normal form containing constants \( e \), and let \( \equiv \) be any equivalence relation over \( \{ x, e \} \). We can construct an equality-normalized \( \phi_\equiv(x) \) in GN-normal form such that

- \( \models \forall x (\xi_\equiv(x) \rightarrow (\phi(x) \leftrightarrow \phi_\equiv(x))) \).
- \( \text{width}(\phi_\equiv) \leq p(\text{width}(\phi)) \), where \( k \) is the quantifier rank of \( \phi \) and \( p \) is a polynomial function independent of \( \phi \).
- \( \text{width}(\phi_\equiv) \leq \text{width}(\phi) \).
- \( \text{rank}_{CQ}(\phi_\equiv) \leq \text{rank}_{CQ}(\phi) \cdot (\text{width}(\phi) + |\{ e \} |)^2 \).

Furthermore, \( \text{occ}(\phi_\equiv) \leq \text{occ}(\phi) \), and, if \( \phi \) is \( U \)-relativized, then \( \phi_\equiv \) is as well.

Here, by quantifier rank we mean the maximal nesting depth of quantifiers in the formula.
Proof. We may assume that every equality guard in \( \phi \) is of the form \( x = x \) (if an equality guard is of the form \( x = y \), we can freely replace all occurrences of \( y \) in the subformula by \( x \)). This takes care of condition (ii) in the definition of equality-normalized formulas.

For each \( \equiv \)-equivalence class, we fix an arbitrary representative — a constant whenever possible. We replace all occurrences of each variable and constant in \( x, e \) by the representative of its equivalence class. Next, we replace subformulas of the form \( s = t \), with \( s, t \in \{ x, e \} \), by \( \top \) if \( s = t \) and by \( \bot \) otherwise; finally, we conjoin every relational atom containing distinct \( s, t \in \{ x, e \} \) with \( \neg (s = t) \).

At this point, the conditions (i) and (iii) in the definition of equality-normalized formulas are satisfied for equalities that have a free variables or constants on both sides. It remains only to take care of the equalities that involve a quantified variable.

Consider any subformula of the form

\[
\psi(z) = \exists y \chi(y, z)
\]

We will essentially do a case distinction, for each quantified variable \( y_i \in y \), of the possible values that \( y_i \) may take. More precisely, we replace \( \psi(z) \) by the disjunction, for each map \( f : \{ y \} \to \{ y, z, e \} \) of the formula \( \psi_f \) obtained from \( \psi \) by (i) replacing each \( y_i \in y \) by \( f(y_i) \), (ii) replacing \( y_i = y_j \) by \( \top \) if \( f(y_i) = f(y_j) \) and by \( \bot \) otherwise, (iii) replacing \( y_i = t \) or \( t = y_i \) for \( t \in \{ x, e \} \) by \( \top \) if \( f(y_i) = t \) and by \( \bot \) otherwise (and dropping the quantifiers corresponding to quantified variables that no longer occur in the formula). Finally, we conjoin every relational \( \equiv \)-relation symbol. We say that an involved are equality-normalized. We can do this by applying the mosaic method, as in the previous section, treating equality as an ordinary relation symbol. The entailment

\[
\chi(y, z) \equiv \exists y \chi(y, z)
\]

Furthermore, \( \psi \) participating in all the same facts that \( \psi \) did. This modification of the structure does not affect the truth of the above form we obtain the desired equality-normalized \( \phi' \equiv \psi' \)

In GN-normal form. There is an explosion involved in this procedure, but this explosion is at most exponential in the quantifier rank of \( \phi \) (as can be seen by performing the rewriting step in a bottom-up fashion starting with the innermost quantifiers). Furthermore, even though we introduce disjunctions, the resulting formula is easily seen to be still in GN-normal form. The bounds on the width and CQ-rank of \( \phi' \equiv \psi' \) are immediate from the construction (note that the number of distinct terms occurring in any atomic formula in \( \phi \) is at most \( \text{width}(\phi) + |\{e\}| \)).

Combining Lemma 37 with Lemma 23 (and using also the fact that the formula transformation in the proof of Lemma 23 preserves quantifier rank) we obtain:

**Proposition 38.** Let \( \phi(x) \) be a GNF formula (not necessarily in GN-normal form) containing constants \( e, e' \), and let \( \equiv \) be any equivalence relation over \( \{ x, e \} \). We can construct an equality-normalized \( \phi'_\equiv(x) \) in GN-normal form such that

- \( \phi'_\equiv \) is exponential in \( \phi \);
- \( \text{width}(\phi'_\equiv) \leq |\phi| \);
- \( \text{rank}_{\text{CQ}}(\phi'_\equiv) \leq p(|\phi|) \) for a polynomial function \( p \) independent of \( \phi \).

Furthermore, \( \text{occ}(\phi'_\equiv) \leq \text{occ}(\phi) \), and, if \( \phi \) is \( \cup \)-generalized, then \( \phi'_\equiv \) is as well.

Let \( \varphi_L \) and \( \varphi_R \) be GN formulas with equality, and let \( x \) be the free variables and \( e \) the constants occurring in these formulas. Applying the above lemma, we obtain that \( \varphi_L \models \varphi_R \) holds if and only if, for each equivalence relation \( \equiv \) on \( \{ x, e \} \),

\[
\xi_\equiv \land (\varphi_L)_\equiv \models (\varphi_R)_\equiv.
\]

Moreover, a GN-interpolant for \( \varphi_L \models \varphi_R \) can be constructed by taking the disjunction of GN-interpolants for (2) for all equivalence relations \( \equiv \) (but recall that \( \xi_\equiv \) may not be a GNF-formula). Towards constructing the latter, we can simplify the entailment (2) even further:

**Lemma 39.** The entailment \( \xi_\equiv \land (\varphi_L)_\equiv \models (\varphi_R)_\equiv \) is valid if and only if \( \varphi_L \equiv \models (\varphi_R)_\equiv \) is valid.

**Proof.** One direction is trivial. For the other direction, we proceed by contraposition: let \( \not\exists \tau \models (\varphi_L)_\equiv \land \neg (\varphi_R)_\equiv \). Recall that \( \varphi_L \equiv \) and \( \varphi_R \equiv \) contain only the representative of each \( \equiv \)-equivalence class. We can freely expand \( \not\exists \tau \) so that all other variables and constants denote the same element that is denoted by the representative of their equivalence class. The only potential source of difficulty is that \( \exists \tau \) may interpret two constants or free variables, \( s \) and \( t \), as the same element \( a \), even though they are representatives of different \( \equiv \)-equivalence classes. However, in this case, we can simply “pull \( s \) and \( t \) apart”, that is, we can replace \( a \) by two copies, \( a_1 \) and \( a_2 \), such that \( a_1 \) and \( a_2 \) participate in all the same facts that \( a \) did. This modification of the structure does not affect the truth of \( \varphi_L \equiv \) and the falsity of \( \varphi_R \equiv \), due to the fact that these formulas are equality normalized (as can be shown by a straightforward formula induction). 

We have thus reduced the problem at hand to obtaining interpolants for entailments of the form \( \varphi_L \equiv \models (\varphi_R)_\equiv \), where both formulas involved are equality-normalized. We can do this by applying the mosaic method, as in the previous section, treating equality as an ordinary relation symbol. We say that an X-mosaic (with \( X \in \{ L, R \} \)) is equality-trivial if (i) \( Y^{-} : t = t' \in \tau \) only if \( t \) and \( t' \) are the same term (parameter or constant), and (ii) \( Y^{-} : t = t' \) only if \( t \) and \( t' \) are distinct terms (parameters or constants). A close inspection of the soundness and completeness proofs for the mosaic method show that, for equality-normalized input formulas \( \psi_L \) and \( \psi_R \), we can restrict attention to equality-trivial mosaics. Furthermore, in the completeness proof, the model that is constructed is such that distinct terms are realized by distinct elements of the model. Consequently, whenever an equality-trivial mosaic contains an equality or inequality, this equality or inequality is guaranteed to be realized by every corresponding pair of elements of the model. We can conclude that, for equality-normalized input sentences, the mosaic method constitutes a sound and complete decision procedure.

Let \( M \) be the set of such equality-normalized GN mosaics for \( \psi_L \land \neg \psi_R \) over some set of parameters of size twice the width of \( \psi_L \land \neg \psi_R \).

**Theorem 40.** For equality-normalized GN formulas \( \psi_L \) and \( \psi_R \), \( \psi_L \models \psi_R \) holds if and only if there is a saturated set of internally consistent equality-trivial GN mosaics from \( M \) that contains a mosaic \( \tau \) such that \( L^{+} : \psi_L, R^{-} : \psi_R \in \tau \).

Proposition 38 ensures that the complexity for deciding satisfiability using these mosaics remains in 2\text{EXP}TIME, and the DAG size of the resulting interpolants is at most doubly exponential in the size of the original input. This is the case, even when we consider bounded arity G\text{F} input.
Interpolation follows from this as before: we prove a mosaic interpolation lemma like Lemma 36, but where the construction now ranges only over equality-trivial mosaics. For the purposes of the construction, equality is treated like any other relation.

We remark that, using this method, we can make no claims about the polarity of occurrences of equality in the interpolant – and this is why no such claims are made in our interpolation result, Theorem 9. The conversion to equality-normalized GN-normal form may change the polarity of occurrences of equality and equality may be introduced as unary guards in both polarities in the interpolant. This implies that the polarity of occurrences of equality in the interpolant are unrelated to the polarity of equality in the original formulas (in contrast to the other relations in the common signature, which can only appear in the interpolant in polarity \( p \) if they occurred in both \( \varphi_L \) and \( \varphi_R \) in polarity \( p \)). Indeed, even if the original formulas did not use equality, equality may appear in both polarities in the resulting interpolant due to the introduction of guards for unary negation.

E. Lower Bounds

In this section, we provide details for the lower bound argument described in Section 6.

E.1 Defining indexed AND/OR trees in UNF

Recall from Section 6 that we are interested in defining in UNF indexed AND/OR trees of depth \( 2^{2^n} \) where each node \( x \) in the AND/OR tree is the root of an index tree, with the index equal to the depth of \( x \) in the AND/OR tree.

Let \( \sigma_n \) consist of

- unary predicates IsOr, InputOne, AOValOne, IndValOne, IndDepth_{i,0}, \ldots, IndDepth_{i,n},
- binary predicate AOChild, and
- ternary predicate IndChild.

Signature \( \sigma'_n \) is the same as \( \sigma_n \), except predicate AOValOne is replaced by a different unary predicate AOValOne'. We write \( \rho_n \) for the signature \( \sigma_n \cap \sigma'_n \).

The relations IndValOne, IndDepth_{i,0}, \ldots, IndDepth_{i,n}, IndChild are used to describe index trees. We say a set of nodes is an index tree if the nodes form a tree (considering the relation IndChild, and viewing IndChild \( x_1 \ldots x_n \) as asserting that \( x_1 \) and \( x_n \), respectively, left and right children of \( x \), the labels IndDepth_{i} correctly identify the depth \( i \) of each node in the tree (with the root satisfying IndDepth_{0}), and any two nodes that have a corresponding position in the tree (based on the left/right branching used to reach it) agree on the relation IndValOne. The index of such a tree is the value when viewing the leaves as \( 2^n \) bits of a binary number (where the order of the bits is determined by the branching, and only one bit from each branching type is used).

The other relations are used for the AND/OR tree. InputOne is viewed as a labelling of the AND/OR tree with either 0 or 1 (we will only be interested in these labels at the leaves). If IsOr \( x \) holds (respectively, \( \neg \text{IsOr } x \) holds), then we view this as a labelling of \( x \) with OR (respectively, AND). AOValOne and AOValOne' will be used to describe the intermediate calculations of the value of the AND/OR tree.

We will be interested in structures where the nodes in the AND/OR tree are roots of index trees describing the depth.

We first define the following auxiliary UNF formulas (technically, these should all be indexed by \( n \)). We will use the convention that built-in predicates like IndChild will be written IndChild \( x_1 \ldots x_{2^n} \), whereas our derived auxiliary formulas like IndChild_{d} that have free variables \( x \) and \( x_{\overline{d}} \) will be written IndChild_{d}(x, x_{\overline{d}}).

\[
\text{IndChild}_{d}(x, x_{\overline{d}}): \quad x_{d} \text{ is the } d\text{-child of } x \text{ for } d \in \{0, 1\}.
\]

\[
\exists x_{l}\text{.IndChild } xx_{l}x_{r} \quad \text{if } d = 1
\]

\[
\exists x_{l}\text{.IndChild } xx_{l}x_{d} \quad \text{if } d = 0
\]

\[
\text{Link}(x_{n}, x'_{n}): \quad x_{n} \text{ and } x'_{n} \text{ are leaves in corresponding positions in the same index tree.}
\]

\[
\exists x_{0} \cdots x_{n-1} x'_{0} \cdots x'_{n-1} \left( \text{IndDepth}_{0} x_{0} \land (\text{IndChild}_{d}(x_{0}, x_1) \land \text{IndChild}_{d}(x_{0}, x'_{1})) \lor (\text{IndChild}_{d}(x_{0}, x_1) \land \text{IndChild}_{d}(x_{0}, x'_{1})) \land \left( \bigwedge_{i=1}^{n-1} (\text{IndChild}_{d}(x_{i}, x_{i+1}) \land \text{IndChild}_{d}(x'_{i}, x'_{i+1}) \lor (\text{IndChild}_{d}(x_{i}, x_{i+1}) \land \text{IndChild}_{d}(x'_{i}, x'_{i+1}))) \right) \right)
\]

\[
\text{LinkNext}(x_{n}, y_{n}): \quad x_{n} \text{ and } y_{n} \text{ are leaves in corresponding positions in neighboring index trees.}
\]

\[
\exists x_{0} \cdots x_{n-1} y_{0} \cdots y_{n-1} \left( \text{AOChild}_{d} x_{y_{0}} \land \text{IndDepth}_{0} x_{0} \land \text{IndDepth}_{0} y_{0} \land \left( \bigwedge_{i=0}^{n-1} (\text{IndChild}_{d}(x_{i}, x_{i+1}) \land \text{IndChild}_{d}(y_{i}, y_{i+1}) \lor (\text{IndChild}_{d}(x_{i}, x_{i+1}) \land \text{IndChild}_{d}(y_{i}, y_{i+1}))) \right) \right)
\]
Before($x_n, x'_n$): $x_n$ and $x'_n$ are leaves in the same index tree, with $x_n$ to the left of $x'_n$.

$$\exists x_0 \cdots \exists x_{n-1} \cdot x'_n \cdot \bigvee_{j=0}^{n-1} \left( \text{IndDepth}_0 x_0 \land \text{IndChild}_0 (x_j, x_{j+1}) \land \text{IndChild}_r (x_j, x_{j+1}) \land \bigwedge_{i=0}^{j-1} \left( \text{IndChild}(x_i, x_{i+1}) \lor \text{IndChild}_r (x_i, x_{i+1}) \right) \right) \land \bigwedge_{i=j+1}^{n-1} \left( \text{IndChild}_0 (x_i, x_{i+1}) \lor \text{IndChild}_r (x_i, x_{i+1}) \lor \text{IndChild}_l (x_i, x_{i+1}) \right)$$

IndexedAOTrees: Every node satisfies exactly one IndDepth, AOChild only connects IndDepth_0-nodes, and every IndDepth_0-node is the root of an index tree. Take the conjunction of the following sentences.

$$\neg \exists x \left( \bigwedge_{i=0}^{n} \neg \text{IndDepth}_i x \right) \lor \bigvee_{i,j \in \{0, \ldots, n\}} \left( \text{IndDepth}_i x \land \text{IndDepth}_j x \right)$$

$$\neg \exists x y. \left( \text{AOChild} x y \land (\neg \text{IndDepth}_0 x \lor \neg \text{IndDepth}_0 y) \right)$$

$$\neg \exists x x_r x_r. \left( \text{IndDepth}_0 x \land \text{IndChild} x x_r x_r \right)$$

$$\neg \exists x x_r x_r. \bigvee_{i=0}^{n-1} \left( \text{IndDepth}_i x \land \text{IndChild} x x_r x_r \land (\neg \text{IndDepth}_{i+1} x_1 \lor \neg \text{IndDepth}_{i+1} x_r) \right)$$

$$\neg \exists x. \bigvee_{i=0}^{n-1} \left( \text{IndDepth}_i x \land \neg \exists x_r x_r. \text{IndChild} x x_r x_r \right)$$

$$\neg \exists x x'_n. \left( \text{Link}(x_n, x'_n) \land (\neg \text{IndValOne} x_n \land \neg \text{IndValOne} x'_n) \lor (\neg \text{IndValOne} x_n \land \text{IndValOne} x'_n) \right)$$

IndMin($x_0$): The index tree rooted at $x_0$ is the minimum (all 0’s).

$$\text{IndDepth}_0 x_0 \land \neg \exists x_1 \cdots x_n. \left( \bigvee_{i=0}^{n-1} \left( \text{IndChild}_0 (x_i, x_{i+1}) \lor \text{IndChild}_r (x_i, x_{i+1}) \right) \right) \land \text{IndValOne} x_n$$

IndMax($x_0$): The index tree rooted at $x_0$ is the maximum (all 1’s).

$$\text{IndDepth}_0 x_0 \land \neg \exists x_1 \cdots x_n. \left( \bigvee_{i=0}^{n-1} \left( \text{IndChild}_0 (x_i, x_{i+1}) \lor \text{IndChild}_r (x_i, x_{i+1}) \right) \right) \land \neg \text{IndValOne} x_n$$

AORoot($x_0$): $x_0$ is the root of an index tree with no incoming AOChild-edges.

$$\text{IndDepth}_0 x_0 \land \neg \exists x'. \text{AOChild} x' x_0$$

AOLeaf($x_0$): $x_0$ is the root of an index tree with no outgoing AOChild-edges.

$$\text{IndDepth}_0 x_0 \land \neg \exists x'. \text{AOChild} x_0 x'$$

IndexDepth: Every root of an index tree with no incoming AOChild-edges has minimal index, and for every index tree with non-maximal value $l$, every AOChild-neighboring tree has index $l + 1$.

$$\neg \exists x. \left( \text{AORoot}(x) \land \neg \text{IndMin}(x) \right) \land \neg \exists x. \text{IndDepth}_0 x \land \neg \text{IndValOne} x \land \neg \exists x'. \left( \text{Before}(x, x') \land \neg \text{IndValOne} x' \right) \land \left( \exists y. \left( \text{LinkNext}(x, y) \lor \neg \text{IndValOne} y \right) \land \left( \exists x'y'. \left( \text{LinkNext}(x', y') \land \text{Before}(x', x') \land (\neg \text{IndValOne} x' \land \text{IndValOne} y') \lor (\neg \text{IndValOne} x' \land \neg \text{IndValOne} y') \right) \right) \right)$$

ConsistentAOValues: For all leaves in an AND/OR tree, AOValOne matches InputOne, and for all other nodes in an AND/OR tree, AOValOne matches the evaluation of the AND/OR tree rooted at that node.

$$\neg \exists x. \left( \text{AOLeaf}(x) \land ((\text{InputOne} x \land \neg \text{AOValOne} x) \lor (\neg \text{InputOne} x \lor \text{AOValOne} x)) \right) \land \neg \exists x. \left( \text{IndDepth}_0 x \land \text{IsOr} x \land \text{AOValOne} x \land \neg \exists y. \left( \text{AOChild} x y \land \text{AOValOne} y \right) \right) \land \neg \exists x. \left( \text{IndDepth}_0 x \land \neg \text{IsOr} x \land \neg \text{AOValOne} x \land \exists y. \left( \text{AOChild} x y \land \neg \text{AOValOne} y \right) \right) \land \neg \exists x. \left( \text{IndDepth}_0 x \land \neg \text{IsOr} x \land \neg \text{AOValOne} x \land \neg \exists y. \left( \text{AOChild} x y \land \neg \text{AOValOne} y \right) \right)$$
Note that some unary negations in these UNF formulas are not explicitly guarded by an atomic relation, but it is straightforward to modify the formulas to explicitly guard the negations with relations from $\rho_n$. 

Let $\chi_n, \chi'_n$ be UNF sentences over signatures $\sigma_n$ and $\sigma'_n$, such that

$$\chi_n := \text{IndexedAOTrees} \land \text{IndexIsDepth} \land \text{ConsistentAOValues} \land \\
\exists x.(\text{AORoot}(x)) \land \\
\neg \exists x.(\text{IndDepth}_0 x \land \neg \text{IndMax}(x) \land \neg \exists y.(\text{IndDepth}_0 y \land \text{AOChild} x y)) \land \\
\neg \exists x.(\text{AORoot}(x) \land \neg \text{AOValOne}(x))$$

$$\chi'_n := \neg [\text{ConsistentAOValues}'] \lor \\
\neg \exists x.(\text{AORoot}(x) \land \neg \text{AOValOne'}(x))$$

Note that in UNF we cannot enforce that a structure has an actual tree-shaped part corresponding to an indexed AND/OR tree; we can only pick out parts of the structure that can be viewed as indexed AND/OR trees (more precisely, parts of the structure that are GN bisimilar to such trees). Thus, $\chi_n$ expresses that (up to GN-bisimulation) there is an indexed AND/OR tree in the structure and every indexed AND/OR tree has depth $2^n$. Moreover, $\text{AOValOne}$ describes the internal value computations of every AND/OR tree, and $\text{AOValOne'}$ holds at the root of every such tree. Likewise, $\chi'_n$ expresses that $\text{AOValOne'}$ describes the internal value computations of every AND/OR tree in the structure, and $\text{AOValOne}$ holds at the root of every such tree.

We have $\chi_n \models \chi'_n$, and it is straightforward to check that $|\chi_n| + |\chi'_n|$ is polynomial in $n$.

### E.2 GN bisimulation game and DAG size (Proof of Proposition 11)

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures over signatures containing $\sigma$, and let $a$ and $b$ be elements within $\mathfrak{A}$ and $\mathfrak{B}$, respectively. We write $\mathfrak{A}, a \rightarrow_{g m}^n \mathfrak{B}, b$ if Duplicator has a winning strategy in the $k$-width $m$-round GN bisimulation game relative to signature $\sigma$ starting from a partial rigid homomorphism $g : a \mapsto b$. We write $\mathfrak{A}, a \rightarrow_{g m}^n \mathfrak{B}, b$ if $\mathfrak{A}, a \rightarrow_{g m}^n \mathfrak{B}, b$ and $\mathfrak{B}, b \rightarrow_{g n} \mathfrak{A}, a$.

Recall the statement of Proposition 11:

Assume $\mathfrak{A}, a \rightarrow_{g m}^n \mathfrak{B}, b$ and $g : a \mapsto b$ is a witnessing partial rigid homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Let $\varphi(x)$ be a DAG-represented GNF formula over signature $\sigma$ such that $|\varphi| \leq m$, $\text{width}(\varphi) \leq k$, and $\text{free}(\varphi) = x$. Then $\mathfrak{A} \models \varphi(a)$ implies $\mathfrak{B} \models \varphi(g(a))$.

Assume that $\mathfrak{A}, a \rightarrow_{g n} \mathfrak{B}, b$, and $g : a \mapsto b$ is a witnessing partial rigid homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. We proceed by induction on $m$. The base case is trivial.

Let $m > 0$ and consider some DAG-represented GNF formula $\varphi(x)$ such that $|\varphi| \leq m$, $\text{width}(\varphi) \leq k$, $\text{free}(\varphi) = x$ and $\mathfrak{A} \models \varphi(a)$. We must show that $\mathfrak{B} \models \varphi(b)$.

- Assume $\varphi$ is of the form $\alpha(x)$. The result follows since $g$ is a partial homomorphism.
- Assume $\varphi$ is of the form $\psi_1 \land \psi_2$. For each $i \in \{1, 2\}$, we can consider the partial rigid homomorphism after Spoiler restricts to $a_i \subseteq a$ corresponding to $\text{free}(\psi_i) \subseteq x$. Since this is a valid move for Spoiler, Duplicator has a winning strategy in the $m - 1$ round game from $g' := g |_{a_i}$. Since $|\psi_i| \leq m - 1$ and $\text{width}(\psi_i) \leq k$, the inductive hypothesis implies that $\mathfrak{B} \models \psi_i(g'(a_i))$ for $i \in \{1, 2\}$, so $\mathfrak{B} \models \varphi(b)$. The proof is similar for $\varphi$ of the form $\psi_1 \lor \psi_2$.
- Assume $\varphi$ is of the form $\alpha(x) \land \neg \psi(x')$ where $x' \subseteq x$. Since $\mathfrak{A} \models \varphi(a)$, $\mathfrak{A} \models \alpha(a)$ and hence $a$ is guarded in $\mathfrak{A}$. We also know that $\mathfrak{B} \models \alpha(b)$ since $g$ is a partial homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Let $a'$ be the restriction of $a$ to $x'$. Since $a$ is guarded, $a'$ is guarded in $\mathfrak{A}$. Consider the play where Spoiler restricts to the guarded tuple $a'$, and then switches structures. Since these are valid moves for Spoiler in the game, Duplicator has a winning strategy in the $m - 2$ round game starting from $g' := (g | a_a)^{-1}$. Let $b' = g'(a')$. Assume for the sake of contradiction that $\mathfrak{B} \models \psi(b')$. Since $|\psi| \leq m - 2$, the inductive hypothesis ensures that $\mathfrak{B} \models \psi(b')$ implies $\mathfrak{A} \models \psi(g'(b'))$, so we must have $\mathfrak{A} \models \psi(a')$. But this contradicts the assumption that $\mathfrak{A} \models \varphi(a)$. Therefore, we must have $\mathfrak{B} \models \neg \psi(b')$, so $\mathfrak{B} \models \varphi(b)$.
- Assume $\varphi$ is the form $\exists y.\psi(xy)$. Since $\mathfrak{A} \models \varphi(a)$, there must be some $c$ such that $\mathfrak{A} \models \psi(ac)$. If $c \in \alpha$, then let $g' := g$. Otherwise, let $g'$ be Duplicator’s choice of partial rigid homomorphism after Spoiler adds $c$. This is a valid move in the game, since $\text{width}(\varphi) \leq k$ implies $|\alpha| < k$ and hence $|\alpha| < k$.

In both cases, we must have $\mathfrak{A}, ac \rightarrow_{g m - 1} \mathfrak{B}, bd$ for $d := g'(c)$. Since $|\psi| \leq m - 1$, the inductive hypothesis implies that $\mathfrak{B} \models \psi(bd)$, so $\mathfrak{B} \models \varphi(b)$ as desired.

### E.3 Counterexample using indexed AND/OR trees

We now seek to exhibit indexed AND/OR trees that cannot be distinguished by GNF formulas of certain sizes. We define inductively a sequence of indexed AND/OR trees $\mathfrak{A}_0$ and $\mathfrak{B}_0$ of depth $2^i$ over signature $\rho_0$ (technically, these structures should be indexed by $n$).

The basic structure of these AND/OR tree are defined recursively as follows, where we write $a$ to denote a single $a$-labelled node, and $a(T_1, T_2)$ to denote a tree where the root is labelled $a$, the left child is the root of some tree $T_1$, and the right child is the root of some tree $T_2$:

- Let $\mathfrak{A}_0 := 1$ and $\mathfrak{B}_0 := 0$.
- Let $\mathfrak{A}_i := \text{AND}(\mathfrak{B}_{i-1}, \mathfrak{A}_{i-1}), (\text{OR}(\mathfrak{B}_{i-1}, \mathfrak{A}_{i-1})).$
- Let $\mathfrak{B}_i := \text{AND}(\mathfrak{B}_{i-1}, \mathfrak{A}_{i-1}), (\text{OR}(\mathfrak{B}_{i-1}, \mathfrak{B}_{i-1})).$

Evaluating $\mathfrak{A}_i$ (respectively, $\mathfrak{B}_i$) gives value 1 (respectively, 0).
A separate index tree (binary tree of depth $n$) is then attached to each node in the trees described above, where the index correctly describes the depth of this node in the AND/OR tree.

We view these indexed AND/OR trees as structures over the signature $\sigma_n = \sigma \cap \sigma_n$. In particular, this means that there are no internal computations using AOValOne or AOValOne'. Although the children of nodes in the AND/OR tree are unordered, we will refer to the left (direction I) and right (direction r) child (based on the order of the children as shown above). This is just to aid in the description below.

We write $\mathfrak{B}_i(b)$ for the restriction of $\mathfrak{B}_i$ to the subtree rooted at $b$. For $i > 0$, let $C_i$ (respectively, $D_i$) denote the set of nodes in $\mathfrak{A}_i$ (respectively, $\mathfrak{B}_i$) at depth 2 (these are the roots of the $\mathfrak{A}_{i-1}$ and $\mathfrak{B}_{i-1}$ subtrees of $\mathfrak{A}_i$ and $\mathfrak{B}_i$). We write $\mathfrak{B}_i[b]$ for the restriction of $\mathfrak{B}_i$ to those elements appearing in $\mathfrak{B}_i(b)$.

Let $h : b \mapsto a$ be a position (partial rigid homomorphism) in the $k$-width GN bisimulation game between $\mathfrak{B}_i$ and $\mathfrak{A}_i$. For ease of presentation, we assume that $h$ is restricted to elements in the AND/OR tree (rather than the elements in the index trees). This is without loss of generality since, in our construction below, $h$ will always preserve the depth of positions, and the local structure of the index trees is identical in $\mathfrak{A}_i$ and $\mathfrak{B}_i$.

We define inductively what it means for $h$ to be $(i, k, m)$-safe. We say a strategy in the $m$-round game is $(i, k, m)$-safe if it uses only $(i, k, m)$-safe positions where $m'$ is the number of remaining moves in the game.

We say $h : b \mapsto a$ is $(i, k, m)$-safe if it satisfies the following properties:

(S1) for all $b \in B_i$, depth $(b) = \text{depth}(h(b));$

(S2) if $i > 0$, then there exists a mapping $f : D_i \rightarrow C_i$ such that for all $b, b' \in D_i$,

- if $b, b'$ are siblings, then $f(b), f(b')$ are siblings;
- if $d \in b$ is the parent of $b$, then $h(d)$ is the parent of $f(b)$;
- if $d \in b$ appears in $\mathfrak{B}_i(b)$, then $h(d)$ appears in $\mathfrak{A}_i(f(b));$

$\mathfrak{B}_i(b), b \mapsto k,m^{\mathfrak{B}_i}_{\sigma \sigma_n}[\mathfrak{A}_i(f(b)), h(b)[a]$ via a strategy for Duplicator that is $(i-1, k, m)$-safe.

Likewise, we say a position $h : a \mapsto b$ is $(i, k, m)$-safe if the conditions (S1) and (S2) hold with $\mathfrak{B}_i$, $b, D_i$ exchanged with $\mathfrak{A}_i$, $a, C_i$, respectively.

Observe that the empty partial rigid homomorphism $h : B_i \rightarrow A_i$, $(i, k, m)$-safe: (S1) vacuously holds, and (S2) holds since (if $i > 0$) we can choose $f$ such that sibling relationships are preserved, and $\mathfrak{A}_{i-1}$ (respectively, $\mathfrak{B}_{i-1}$) subtrees in $\mathfrak{A}_i$ are mapped to $\mathfrak{B}_{i-1}$ (respectively, $\mathfrak{A}_{i-1}$) subtrees in $\mathfrak{B}_i$. Similarly for the empty partial rigid homomorphism from $\mathfrak{B}_i$ to $\mathfrak{B}_i$.

More importantly, we can show that Duplicator has a winning $(i, k, m)$-safe strategy in certain games when starting from an $(i, k, m)$-safe position.

**Lemma 41.** For all $i, k \leq 2^{\alpha n-1}$, and $m \leq i$,

- if $b \mapsto a$ is $(i, k, m)$-safe then $\mathfrak{B}_i(b) \mapsto k,m^{\mathfrak{B}_i}_{\sigma \sigma_n}[\mathfrak{A}_i, a]$ via a strategy for Duplicator that is $(i, k, m)$-safe;

- if $a \mapsto b$ is $(i, k, m)$-safe then $\mathfrak{A}_i(a) \mapsto k,m^{\mathfrak{A}_i}_{\sigma \sigma_n}[\mathfrak{B}_i, b]$ via a strategy for Duplicator that is $(i, k, m)$-safe.

**Proof.** We fix $k$ and proceed by induction on $i$ and $m$.

The base cases are covered (for $i = 0$ and $m = 0$, or $i > 0$ and $m = 0$), since Duplicator wins by default when $m = 0$.

Assume $i > 0$ and $m > 0$ and the starting position $h$ is $(i, k, m)$-safe. We assume that the active structure is $\mathfrak{B}_i$, the current position is $h : b \mapsto a$, and (S2) is witnessed by a map $f : D_i \rightarrow C_i$. For each possible move of Spoiler, we describe Duplicator’s move, leading to an $(i, k, m-1)$-safe position $h'$.

By the inductive hypothesis (on $m$), this is enough to ensure that we can extend this to a full winning strategy for Duplicator.

- If Spoiler restricts to elements $b' \subseteq b$, then the new position $h' := h |_{b'}$ is still $(i, k, m)$-safe, and hence $(i, k, m-1)$-safe.

- If Spoiler switches structures to the new position $h' := a \mapsto b$ then $b$ must be a single position or neighboring positions in the AND/OR tree (since $b$ must be guarded). We must show $h'$ is $(i, k, m-1)$-safe. (S1) trivially holds since no new elements have been added. We must define $f' : C_i \rightarrow D_i$, satisfying (S2) for $(i, k, m-1)$.

Assume $b$ corresponds to a single element $b$.

If $b$ is at depth at least 2, then let $d \in D_i$ be the node at depth 2 such that $\mathfrak{B}_i(d)$ contains $b$.

Let $a := h(b)$ and $c := f(d) \in C_i$. We define $f'$ such that $f'(c) := d$. Then $\mathfrak{A}_i(c)$ contains $a$ and $\mathfrak{B}_i(d)$ contains $b$, and we can use the $(i-1, k, m)$-safe strategy from $\mathfrak{B}_i(d), b \mapsto k,m^{\mathfrak{B}_i}_{\sigma \sigma_n}[\mathfrak{A}_i(f(d)), h(b)]$ guaranteed by (S2) for $h$, to see that $\mathfrak{A}_i(c), a \mapsto k,m-1^{\mathfrak{A}_i}_{\sigma \sigma_n}[\mathfrak{B}_i(f'(c)), h'(a)]$ as desired.

The sibling $c'$ of $c$ can be mapped by $f'$ to the sibling $d'$ of $d$ (in fact, it would also work to map to $d$). If the sibling of $c$ corresponds to an $\mathfrak{A}_{i-1}$ subtree, then $\mathfrak{A}_i(c')$ may correspond to an $\mathfrak{A}_{i-1}$ subtree and $\mathfrak{B}_i(d')$ to a $\mathfrak{B}_{i-1}$ subtree. Because dom$(h')$ does not include any elements from these mismatched subtrees, the inductive hypothesis (on $i$) ensures that $\mathfrak{A}_{i-1}, \emptyset \mapsto k,m-1^{\mathfrak{A}_{i-1}}_{\sigma \sigma_n}[\mathfrak{B}_{i-1}, \emptyset]$ using an $(i-1, k, m-1)$-safe strategy, so $\mathfrak{A}_i(c'), \emptyset \mapsto k,m-1^{\mathfrak{A}_i}_{\sigma \sigma_n}[\mathfrak{B}_i(d'), \emptyset]$ using an $(i-1, k, m-1)$-safe strategy as desired.

The remaining elements $c' \in C_i$ can be mapped to elements in positions II and IR in $\mathfrak{B}_i$ such that if $c'$ is the root of an $\mathfrak{A}_{i-1}$ (respectively, $\mathfrak{B}_{i-1}$) subtree in $\mathfrak{A}_i$, then $f'(c')$ is the root of an $\mathfrak{A}_{i-1}$ (respectively, $\mathfrak{B}_{i-1}$) subtree in $\mathfrak{B}_i$. In these subgames, Duplicator clearly has an $(i-1, k, m-1)$-safe winning strategy, since the arenas are isomorphic.

If $b$ is at depth 1, then $h(b)$ is also at depth 1 by (S1). Let $c_1$ and $c_2$ be the children of $h(b)$ in $\mathfrak{A}_i$. Set $f'(c_1)$ and $f'(c_2)$ to be the corresponding children of $b$ in $\mathfrak{B}_i$. As in the previous case, this may result in mapping an $\mathfrak{A}_{i-1}$ subtree in $\mathfrak{A}_i$ to a $\mathfrak{B}_{i-1}$ subtree in $\mathfrak{B}_i$, but the inductive hypothesis (on $n$) ensures that (S2) still holds for these positions.

The remaining elements $c' \in C_i$ can be mapped to elements in positions II and IR in $\mathfrak{B}_i$ such that if $c'$ is the root of an $\mathfrak{A}_{i-1}$ (respectively, $\mathfrak{B}_{i-1}$) subtree in $\mathfrak{A}_i$, then $f'(c')$ is the root of an $\mathfrak{A}_{i-1}$ (respectively, $\mathfrak{B}_{i-1}$) subtree in $\mathfrak{B}_i$. In these subgames, Duplicator clearly has an $(i-1, k, m-1)$-safe winning strategy, since the arenas are isomorphic.
If $b$ is the root in $\mathfrak{B}_i$, then $h(b)$ must be the root in $\mathfrak{A}_i$. Let $f'$ be the map $\mathbf{II} \mapsto \mathbf{II}, \mathbf{Ir} \mapsto \mathbf{Ir}, \mathbf{rl} \mapsto \mathbf{II}, \mathbf{rr} \mapsto \mathbf{Ir}$. Notice that we are ‘cheating’ for nodes in the right subtree of $\mathfrak{A}_i$ by mapping them to the left subtree in $\mathfrak{B}_i$. This clearly satisfies (S2), since we are mapping $\mathfrak{A}_{i-1}$ (respectively, $\mathfrak{B}_{i-1}$) subtrees in $\mathfrak{A}_i$ to $\mathfrak{A}_{i-1}$ (respectively, $\mathfrak{B}_{i-1}$) subtrees in $\mathfrak{B}_i$.

Now assume $b$ is a pair of neighboring elements.

If $b$ are neighboring positions both of depth at least 2, then the argument is similar to the depth 2 case above.

If $b$ spans the root and depth 1, then the argument is similar to the depth 1 case above.

If $b$ spans depth 1 and depth 2, then the argument is similar to the depth 2 case above.

- If Spoiler adds an element $b$ in $\mathfrak{B}_i$, then we must select $a$ in $\mathfrak{A}_i$ such that $h' := h[b \mapsto a]$ is $(i, k, m - 1)$-safe.

  If $b$ is at depth at least 2, then let $b_2$ be the node at depth 2 such that $b$ is in $\mathfrak{B}_i(b_2)$. By (S2), $\mathfrak{B}_i(b_2), b \mid_{2} \xrightarrow{k, m} \mathfrak{A}_i(f(b_2)), h(b \mid_{2})$ using an $(i - 1, k, m)$-safe strategy. Select $a$ based on Duplicator’s move according to this strategy when Spoiler plays $b$ in $\mathfrak{B}_i(b_2)$ (the strategy will give a position in $\mathfrak{A}_i(f(b_2))$, so $a$ should be the element in the corresponding position in $\mathfrak{A}_i$). We know that $h[b \mapsto a]$ is $(i - 1, k, m - 1)$-safe restricted to $\mathfrak{B}_i(b_2)$ and $\mathfrak{A}_i(f(b_2))$. This ensures that (S1) holds. We set $f' := f$.

  We must check that $h' := h[b \mapsto a]$ is a partial rigid homomorphism from $\mathfrak{B}_i$ to $\mathfrak{A}_i$ (not just restricted to the $(i - 1)$-subtrees).

  The delicate situation is if $b = b_2$, and the parent $d$ of $b$ is in $s$, since this means AOChild $d$ holds (recall that AOChild is the successor relation in the AND/OR tree). In that case, (S2) for $h$ ensures that the parent of $f(b)$ is $h(d)$. But in order for $h'$ to be $(i - 1, k, m - 1)$-safe restricted to $\mathfrak{B}_i(b_2)$ and $\mathfrak{A}_i(f(b_2))$, $a$ must be $f(b)$. Since the parent of $a$ is $h(d) = h'(d)$, this ensures that AOChild $h'(d)h'(b)$ holds, so $h'$ is still a partial rigid homomorphism.

  The conditions of (S2) for $(i, k, m - 1)$-safety follow since $h$ was $(i, k, m)$-safe using $f$, and $f' := f$.

  Thus, $h[b \mapsto a]$ is $(i, k, m - 1)$-safe.

  If $b$ is at depth 1, then let $d_1$ and $d_2$ be the nodes at depth 2 that are children of $b$. By (S2), it must be the case that $f(d_1)$ and $f(d_2)$ are children of a single node at depth 1 in $\mathfrak{A}_i$ (though $f(d_1)$ could be equal to $f(d_2)$). Select $a$ to be the parent of $f(d_1)$ and $f(d_2)$. This ensures $h[b \mapsto a]$ is a partial rigid homomorphism (using the fact that if $d' \in b$ is in $\mathfrak{B}_i(d_1)$ then $h(d')$ is in $\mathfrak{A}_i(f(d_1)))$. (S1) still holds since $a$ is a node at depth 1. (S2) still holds with $f' := f$ since no elements in depth at least 2 have changed, and we have respected siblings. Hence, $h[b \mapsto a]$ is $(i, k, m - 1)$-safe.

  If $b$ is at the root, then set $a$ to be the root in $\mathfrak{A}_i$. This is a partial rigid homomorphism since depths of elements are preserved by $h$. (S1) clearly still holds. (S2) holds with $f' := f$. Thus, $h[b \mapsto a]$ is $(i, k, m - 1)$-safe.

\[\square\]

### E.4 Lowerbound on size of GNF interpolants for UNF (Proof of Theorem 10)

We are now ready to prove Theorem 10. The statement here is slightly different than in the body due to the fact that the AND/OR trees $\mathfrak{A}_i$ and $\mathfrak{B}_i$ defined in the previous subsection have depth $2i$ (so technically, the bound has to be adjusted by a factor of 2); of course, this has no impact on the lower bound.

There is a polynomial function $p$ and a family of UNF sentences $\chi_n \models \chi'_n$ such that $|\chi_n| + |\chi'_n| \leq p(n)$, and there is no GNF interpolant $\theta_n$ for $\chi_n \models \chi'_n$ of size at most $2^{2^n}-1$, even when $\theta_n$ is represented via a DAG.

**Proof.** Let $i = 2^{2^{n-1}}$. Then the indexed AND/OR trees $\mathfrak{A}_i$ and $\mathfrak{B}_i$ defined in the previous subsection are of depth $2i = 2^{2^n}$. In particular, note that $\mathfrak{A}_i \models \chi_n$ (when viewed as a $\sigma_n$ structure), but $\mathfrak{B}_i \not\models \chi'_n$ (when viewed as a $\sigma'_n$ structure).

By Lemma 41 (and the observation that the empty partial rigid homomorphism is safe), $\mathfrak{A}_i \not\models_{\mathfrak{B}_i} \mathfrak{B}_i$. Hence, by Proposition 11, $\mathfrak{A}_i$ and $\mathfrak{B}_i$ must agree on all DAG-represented GNF sentences $\phi$ over the signature $\rho_n$ such that $|\phi| \leq i$ and $\text{width}(\phi) \leq i$.

Since $\chi_n \models \chi'_n$, there is a GNF sentence $\theta_n$ over the signature $\rho_n$ such that $\chi_n \models \theta_n$ and $\theta_n \models \chi'_n$.

Assume for the sake of contradiction that $\theta_n$ is of size at most $i$ (which implies that there is also a DAG representation of $\theta_n$ of size at most $i$). Since $\mathfrak{A}_i \models \chi_n$, it must be the case that $\mathfrak{A}_i \models \theta_n$. Since a DAG representation of $\theta_n$ is at most size $i$ and $\text{width}(\theta_n) \leq i$, $\mathfrak{A}_i$ and $\mathfrak{B}_i$ must agree on $\theta_n$, so $\mathfrak{A}_i \models \theta_n$. But $\mathfrak{B}_i \models \theta_n$ implies $\mathfrak{B}_i \models \chi'_n$, which is a contradiction since the value at the root of $\mathfrak{B}_i$ based on the internal calculations using AOValOne' is 0, not 1 as asserted by $\chi'_n$.

\[\square\]

### E.5 Beth definability

We now show that there are UNF sentences $\phi_n$ over a signature $\sigma_n \cup \rho_n \cup \{R\}$ such that $\phi_n$ implicitly defines $R$ over $\sigma_n$, but any sequence of DAG representations of GNF $\phi'_n$ over $\sigma$ that provide explicit definitions for $R$ with respect to $\psi$ must grow doubly-exponentially in the size of $\phi_n$.

Let $\sigma_n$ be as in the previous lower bound, $\rho_n = \sigma_n$, and $R = \text{AOValOne}$. We can implicitly define $\text{AOValOne}$ by:

- there is AND/OR tree of depth $2^{2^n}$
- and every AND/OR tree has depth $2^{2^n}$, and
- $\text{ConsistentAOValues}$.

Assume for the sake of contradiction that there is a GNF formula $\psi(x)$ explicitly defining $\text{AOValOne}$ over signature $\rho_n$, and such that $|\psi(x)| \leq 2^{2^{n-1}}$ (so there is a DAG representation with this size bound as well). Let $i = 2^{2^{n-1}}$ and let $a, b$ be the roots of the structures $\mathfrak{A}_i$ and $\mathfrak{B}_i$ over signature $\sigma_n$, and satisfying the implicit definition above (i.e. such that $\text{AOValOne}$ correctly labels intermediate calculations of the value of the AND/OR tree). We have $\mathfrak{A}_i \models \text{AOValOne} a$, so $\mathfrak{A}_i \models \psi(a)$. But $\mathfrak{A}_i, a \xrightarrow{k, m} \mathfrak{B}_i, b$ for $i = k = m = 2^{2^{n-1}}$ by Lemma 41 (and observing that $a \maps b$ and $b \maps a$ are $(i, k, m)$-safe since $a$ and $b$ are the roots). This means $\mathfrak{B}_i \models \psi(b)$ by Proposition 11, so $\mathfrak{B}_i \models \text{AOValOne} b$, a contradiction.
Since an explicit definition for AOValOne corresponds exactly to an interpolant, the lower bound follows from the lower bound for interpolation.

E.6 Lower bound on size of GNF interpolants for GF (Proof sketch for Theorem 13)

In [12], a family of GF sentences \( \varphi_n \) using equality and using relations of unbounded arity are defined such that every model \( \mathcal{M} \) of \( \varphi_n \) contains a binary tree of depth \( 2^{2n} \). This is described in detail in the proof of Theorem 4.4 in [12] (where it is used to show that satisfiability is \( 2\text{EXP} \)-hard for GF).

We can adapt this construction to define indexed AND/OR trees with small (polynomial in \( n \)) size GF formulas. A technical difference in this approach is that a node in the AND/OR tree is specified by a pair of elements in the structure rather than a single element, and equality is used in the GF formulas.

As before, this allows us to prove a doubly exponential lower bound on the size of GNF interpolants (even represented as a DAG) for GF validities, but it relies on relations of unbounded arity.

For the single exponential lower bound for relations of bounded arity, we can modify the construction in a different way. Instead of attaching index trees to each node in the AND/OR tree, we can introduce new unary relations \( \text{IndBit}_0, \ldots, \text{IndBit}_n \), and require that for every node in the AND/OR tree, the index value (when we view these new predicates as \( n \) bits of a binary number) corresponds to the depth of this node in the AND/OR tree. This allows us to define indexed AND/OR trees of depth \( 2^n \) in GF, which can be used to prove an exponential lower bound on the size of GNF interpolants for GF validities with relations of bounded arity.

F. Preservation Theorems

Details of effective Łoś-Tarski for GF (Theorem 16)

Recall the statement:

Let \( \varphi(x) \) be a GF formula over signature \( \sigma \) with \( |\varphi| = n \). If \( \varphi \) is preserved under extensions, then we can construct an equivalent existential GF formula \( \varphi'(x) \) such that \( |\varphi'| \leq 2^{2p(n)} \) for a polynomial function \( p \) independent of \( \varphi \).

For simplicity, we focus on the case of boolean queries. By Corollary 14, we can find a DAG representation of some GNF existential \( \chi' \) that is equivalent to \( \varphi \) and is of doubly-exponential size. We can convert \( \chi' \) to \( \chi'' \) that is a union of \( \chi''_i \), where each \( \chi''_i \) is a “GNCQ” – a conjunctive query with atomic negation, where every negated atom is guarded. Further, each GNCQ \( \chi''_i \) within \( \chi'' \) is at most exponential in the size of \( \chi' \), and the overall size of \( \chi'' \) (as a formula) is at most doubly exponential in the size of \( \chi' \). Let \( A_i \) be the set of queries in GF of the form \( \exists y, \bigwedge_j A_{ij} \), where \( \bigwedge A_{ij} \) is obtained by identifying variables in \( \chi''_i \) and then adding on at most \( 2^{|\varphi|} \) additional positive atoms. We claim that \( \chi' \) is equivalent to \( \bigvee_i \bigwedge_{Q \in A_i} Q \). Since the size of indices \( i \) is doubly-exponential in the size of \( \chi' \) and each \( A_i \) has size doubly-exponential in \( \chi' \), this gives the desired bound. Clearly each query in \( A_i \) implies \( \chi''_i \) and hence implies \( \varphi \). On the other hand, consider a model \( M \) satisfying \( \varphi \). Given any model \( M \) there is another structure \( M^* \) agreeing with \( M \) on all GF sentences that has a guarded tree decomposition. Indeed, the guarded unravelling of \( M \) (see, e.g., [2]) gives such an \( M^* \). \( M^* \) must satisfy \( \varphi \) and therefore must satisfy some \( \chi''_i \) via a homomorphism \( h \). Let \( C \) be the image of \( h \) and \( Q_C \) be the conjunctive query that describes the restriction of \( M^* \) to the bags containing \( C \) – such a query can be written in GF by constructing it inductively from the leaves of the tree up. The size of \( Q_C \) is at most the size of \( \chi''_i \) times the maximal size of bags – the latter being a factor which is at most exponential in the size of the schema. Note that every element \( c \in C \) corresponds to exactly one variable \( y_c \) of \( Q_C \). We let \( Q'_C \) extend \( Q_C \) by adding, for every negated atom in \( \chi''_i \), the result of replacing each variable \( x_i \) by \( y_{h(x_i)} \). Then \( Q'_C \) also holds in \( M^* \) and \( Q'_C \in GF \). Hence \( Q'_C \) holds in \( M \) and is in \( A_i \), which completes the argument.

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