Effective interpolation and preservation in guarded logics

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CSL-LICS 2014
Vienna, Austria
Some decidable fragments of first order logic

<table>
<thead>
<tr>
<th>ML</th>
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<th>tree-like model property</th>
<th>Craig interpolation</th>
<th>Loś-Tarski preservation</th>
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Some decidable fragments of first order logic

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Some decidable fragments of first order logic

```
∃x.α(xy) ∧ ψ(xy)
∀x.α(xy) → ψ(xy)
```

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Some decidable fragments of first order logic

- **Constrain number of variables**
  - $\exists x. \alpha(xy) \land \psi(xy)$
  - $\forall x. \alpha(xy) \rightarrow \psi(xy)$

- **Constrain quantification**

- **Constrain negation**
  - $\exists x. \psi(xy)$
  - $\alpha(xy) \land \neg \psi(xy)$

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Interpolation

$$\varphi \models \psi$$
Interpolation

\[ \varphi \models \chi \models \psi \]

only uses relations in both \( \varphi \) and \( \psi \)
∃xyz(Txyz ∧ Rxy ∧ Ryz ∧ Rzx) ⊨ ∃xy(Rxy ∧ ((Sx ∧ Sy) ∨ (¬Sx ∧ ¬Sy)))

“there is a T-guarded
3-cycle using R”
\[ \exists xyz (Txyz \land Rxy \land Ryz \land Rzx) \models \exists xy (Rxy \land ((Sx \land Sy) \lor (\neg Sx \land \neg Sy))) \]

“there is a \( T \)-guarded 3-cycle using \( R \)”
Interpolation example

\[ \exists xyz (Txyz \land Rxy \land Ryz \land Rzx) \iff \exists xy (Rxy \land ((Sx \land Sy) \lor (\neg Sx \land \neg Sy))) \]

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Interpolation example

\[ \exists xyz (Txyz \land Rxy \land Ryz \land Rzx) \models \exists xy (Rxy \land ((Sx \land Sy) \lor (\neg Sx \land \neg Sy))) \]

“there is a \(T\)-guarded 3-cycle using \(R\)”

Interpolant \(\chi := \exists xyz (Rxy \land Ryz \land Rzx)\)

“there is a 3-cycle using \(R\)”
Interpolation example

\[ \exists x y z \,(T_{xyz} \land R_{xy} \land R_{yz} \land R_{zx}) \models \exists x y \,(R_{xy} \land (S_x \land S_y) \lor (\lnot S_x \land \lnot S_y)) \]

“there is a \(T\)-guarded 3-cycle using \(R\)”

GNF interpolant \(\chi := \exists x y z \,(R_{xy} \land R_{yz} \land R_{zx})\)

“there is a 3-cycle using \(R\)”
Interpolation

\[ \varphi \vdash \chi \vdash \psi \]

only uses relations in both \( \varphi \) and \( \psi \)

**Theorem** (Barany+Benedikt+ten Cate ’13)

Given GNF formulas \( \varphi \) and \( \psi \) such that \( \varphi \vdash \psi \), there is a GNF interpolant \( \chi \) (but model theoretic proof implies no bound on size of \( \chi \)).
Theorem (Barany+Benedikt+ten Cate ’13)

Given GNF formulas \( \varphi \) and \( \psi \) such that \( \varphi \models \psi \), there is a GNF interpolant \( \chi \) (but model theoretic proof implies no bound on size of \( \chi \)).

Even when input is in GF, no idea how to compute interpolants (or other rewritings related to interpolation and preservation).
Interpolation

\[ \phi \models \chi \models \psi \]

only uses relations in both \( \phi \) and \( \psi \)

**Theorem (Barany+Benedikt+ten Cate ’13)**

Given GNF formulas \( \phi \) and \( \psi \) such that \( \phi \models \psi \), there is a **GNF interpolant** \( \chi \) (but model theoretic proof implies no bound on size of \( \chi \)).

**Theorem (Constructive interpolation for GNF)**

Given GNF formulas \( \phi \) and \( \psi \) such that \( \phi \models \psi \), we can construct a **GNF interpolant** \( \chi \) of doubly exponential DAG-size (in size of input).
A mosaic $\tau(a)$ for $\varphi$ is a collection of subformulas of $\varphi$ over some guarded set $a$ of parameters.

$\tau_1(ab)$
- $Raa$
- $\neg Sa$
- $\exists z (Rbz \land Sz)$
- $Sb$
- $Rba$
- ...

$\tau_2(bc)$
- $Sb$
- $\neg Rbb$
- $Rbc \land Sc$
- $Rcb$
- $Sc$
- ...

$\tau_3(d)$
- $Sd$
- $\neg Sd$
- $\exists z (Ryz \land Sz)$
- $\forall z (Rdz)$
- $Rdd \lor Sd$
- ...

Mosaics
A mosaic $\tau(a)$ for $\varphi$ is a collection of subformulas of $\varphi$ over some guarded set $a$ of parameters.

\[ \tau_1(ab) \]
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\[ \ldots \]

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\[ \neg Rbb \]
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\[ Sc \]
\[ \ldots \]

\[ \tau_3(d) \]
\[ \text{Internally inconsistent} \]
\[ (\text{e.g. } Sd \land \neg Sd) \]
A mosaic $\tau(a)$ for $\varphi$ is a collection of subformulas of $\varphi$ over some guarded set $a$ of parameters.

Internally consistent mosaics are windows into a (guarded) piece of a structure.
Mosaics can be **linked** together to fulfill an existential requirement if they agree on all formulas that use only shared parameters.

\[
\exists z (Rbz \land Sz)
\]
Linking mosaics

Mosaics can be linked together to fulfill an existential requirement if they agree on all formulas that use only shared parameters.

\[ \exists z (Rbz \land Sz) \]
Linking mosaics

Mosaics can be **linked** together to fulfill an existential requirement if they agree on all formulas that use only shared parameters.

\[
\exists z (R b z \land S z)
\]

We say a set \( S \) of mosaics is **saturated** if every existential requirement in a mosaic \( \tau \in S \) is fulfilled in \( \tau \) or in some linked \( \tau' \in S \).
Mosaics

Fix some set $P$ of size $2 \cdot \text{width}(\varphi)$ and let $\mathcal{M}_\varphi$ be the set of mosaics for $\varphi$ over parameters $P$. The size of $\mathcal{M}_\varphi$ is doubly exponential in the size of $\varphi$.

**Theorem**

$\varphi$ is satisfiable iff there is a saturated set $S$ of internally consistent mosaics from $\mathcal{M}_\varphi$ that contains some $\tau$ with $\varphi \in \tau$. 
Mosaics

Fix some set $P$ of size $2 \cdot \text{width}(\phi)$ and let $\mathcal{M}_\phi$ be the set of mosaics for $\phi$ over parameters $P$. The size of $\mathcal{M}_\phi$ is doubly exponential in the size of $\phi$.

**Theorem**

$\phi$ is satisfiable iff there is a saturated set $S$ of internally consistent mosaics from $\mathcal{M}_\phi$ that contains some $\tau$ with $\phi \in \tau$.

$$S = \{ \tau_1, \tau_2, \tau_3, \tau_4 \}$$
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$$S = \{ \tau_1, \tau_2, \tau_3, \tau_4 \}$$
Mosaic elimination algorithm for satisfiability testing

\[ M_\varphi \]

\[ \tau_1 \quad \tau_2 \quad \tau_3 \quad \tau_4 \quad \tau_5 \quad \tau_6 \quad \tau_7 \]

Theorem \( \varphi \) is satisfiable iff there is some mosaic \( \tau \in M' \) with \( \varphi \in \tau \).
Stage 1.
Eliminate mosaics with internal inconsistencies.
Mosaic elimination algorithm for satisfiability testing

Stage 1.
Eliminate mosaics with internal inconsistencies.

Stage $i + 1$.
Eliminate mosaics with existential requirements that can only be fulfilled using mosaics eliminated in earlier stages.
Mosaic elimination algorithm for satisfiability testing

Stage 1.
Eliminate mosaics with internal inconsistencies.

Stage $i + 1$.
Eliminate mosaics with existential requirements that can only be fulfilled using mosaics eliminated in earlier stages.

Continue until fixpoint $\mathcal{M}'$ reached. The set $\mathcal{M}'$ is a saturated set of internally consistent mosaics.

Theorem

$\varphi$ is satisfiable iff there is some mosaic $\tau \in \mathcal{M}'$ with $\varphi \in \tau$. 
Mosaics for interpolation

Assume $\varphi_L \models \varphi_R$.

**Idea:** Construct interpolant from proof that $\varphi_L \land \neg \varphi_R$ is unsatisfiable.
Mosaics for interpolation

Assume \( \varphi_L \models \varphi_R \).

**Idea:** Construct interpolant from proof that \( \varphi_L \land \neg \varphi_R \) is unsatisfiable.

Consider mosaics for \( \varphi_L \land \neg \varphi_R \).

Annotate each mosaic and each formula with a **provenance** \( L \) or \( R \).

\[
\begin{align*}
L : \tau_1(ab) & \quad \begin{align*}
L : Raa \\ R : \neg Sa \\ R : \exists z(Rbz \land Sz) \\ R : \neg Rbb \\ L : Sb \\ R : Rba \\
& \quad \ldots
\end{align*}
\end{align*}
\]

\[
\begin{align*}
R : \tau_2(bc) & \quad \begin{align*}
L : Sb \\ R : \neg Rbb \\ R : Rbc \land Sc \\ R : Rbc \\ L : Rcb \\ R : \exists z(Rbz \land Sz) \\ R : Sc \\
& \quad \ldots
\end{align*}
\end{align*}
\]

\[
\begin{align*}
L : \tau_3(d) & \quad \begin{align*}
L : Sd \\ R : \neg Sd \\ R : Rdd \land Sd \\ R : \exists z(Ryz \land Sz) \\ L : \forall z(Rdz) \\ L : Rdd \lor Sd \\
& \quad \ldots
\end{align*}
\end{align*}
\]

Linking must respect the provenance annotations.
Assume $\varphi_L \models \varphi_R$.

Test satisfiability of $\varphi_L \land \neg \varphi_R$ using mosaic elimination.
Assume $\phi_L \models \phi_R$.

Test satisfiability of $\phi_L \land \neg \phi_R$ using mosaic elimination.

Assign a \textit{mosaic interpolant} $\theta_\tau$ to each eliminated mosaic $\tau$ such that $\tau_L \models \theta_\tau$ and $\theta_\tau \models \neg \tau_R$.

Mosaic interpolants $\theta_\tau$ describe why the mosaic $\tau$ was eliminated.
Mosaics for interpolation

Assume $\varphi_L \vDash \varphi_R$.

Test satisfiability of $\varphi_L \land \neg \varphi_R$ using mosaic elimination.

Assign a **mosaic interpolant** $\theta_\tau$ to each eliminated mosaic $\tau$ such that $\tau_L \vDash \theta_\tau$ and $\theta_\tau \vDash \neg \tau_R$.

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Mosaic interpolants $\theta_\tau$ describe why the mosaic $\tau$ was eliminated.

Theorem

An interpolant $\chi$ for $\varphi_L \models \varphi_R$ of at most doubly exponential DAG-size can be constructed from the mosaic interpolants.
Stronger interpolation theorems

Challenge: ensure interpolant $\chi$ is in GNF (and satisfies other properties)
**Stronger interpolation theorems**

**Challenge:** ensure interpolant $\chi$ is in GNF (and satisfies other properties)

**Solution:** place further restrictions on the formulas in the mosaics
Stronger interpolation theorems

Challenge: ensure interpolant $\chi$ is in GNF (and satisfies other properties)
Solution: place further restrictions on the formulas in the mosaics

Lyndon interpolation: $\chi$ respects polarity of relations
A relation $R$ occurs positively (respectively, negatively) in $\chi$ iff $R$ occurs positively (respectively, negatively) in both $\varphi_L$ and $\varphi_R$.

Relativized interpolation: $\chi$ respects quantification pattern
If the quantification in $\varphi_L$ and $\varphi_R$ is relativized to a distinguished set of unary predicates $U$, then $\chi$ is $U$-relativized.
I.e. quantification is of the form $\exists x \ (Ux \land \psi(xy))$ for $U \in U$
Effective preservation theorems

φ is **monotone** if $A \models \varphi$ implies that $A' \models \varphi$ for any $A'$ obtained from $A$ by adding tuples to the interpretation of some relation.

φ is **positive** if every relation appears within the scope of an even number of negations.

**Corollary** (Monotone = Positive)

If φ is monotone and in GNF, then we can construct an equivalent positive GNF formula $\varphi'$ of doubly exponential DAG-size.
Effective preservation theorems

φ is **monotone** if \( \mathcal{A} \models \phi \) implies that \( \mathcal{A}' \models \phi \) for any \( \mathcal{A}' \) obtained from \( \mathcal{A} \) by adding tuples to the interpretation of some relation.

φ is **positive** if every relation appears within the scope of an even number of negations.

**Corollary (Monotone = Positive)**

If φ is monotone and in GNF, then we can construct an equivalent positive GNF formula \( \phi' \) of doubly exponential DAG-size.

φ is **preserved under extensions** if \( \mathcal{A} \models \phi \) and \( \mathcal{A} \subseteq \mathcal{B} \) implies \( \mathcal{B} \models \phi \).

φ is in **existential GNF** if no quantifier is in the scope of a negation.

**Corollary (Analog of Loś-Tarski)**

If φ is preserved under extensions and in GNF, then we can construct an equivalent existential GNF formula \( \phi' \) of doubly exponential DAG-size.
GNF is an expressive fragment of FO with good computational and model-theoretic properties.

Proved **constructive interpolation and preservation theorems** for GNF.

Adapted **mosaic method** to prove interpolation.
GNF is an expressive fragment of FO with good computational and model-theoretic properties.

Proved constructive interpolation and preservation theorems for GNF.

Adapted mosaic method to prove interpolation.

More in the paper:

Proved matching lower bounds for constructive interpolation results.

Analyzed special cases when input is in GF or the unary negation fragment (UNF).
Mosaic interpolants $\theta_\tau$ satisfy $\tau_L \models \theta_\tau$ and $\theta_\tau \models \neg \tau_R$. They describe why the mosaic $\tau$ was eliminated.

**Stage 1:**

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<tr>
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<th>Implication</th>
<th>Result</th>
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<td>$\theta_\tau := \bot$</td>
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Shape of interpolants

Mosaic interpolants $\theta_\tau$ satisfy $\tau_L \vdash \theta_\tau$ and $\theta_\tau \vdash -\tau_R$. They describe why the mosaic $\tau$ was eliminated.

Stage 1:

\[
\begin{align*}
\text{L : } & Rab & \text{L : } & -Rab & \Rightarrow & \theta_\tau := \bot \\
\text{Internal} & \Rightarrow & \theta_\tau := T \\
\text{inconsistency} & \Rightarrow & \theta_\tau := Rab \\
\text{R : } & Rab & \text{L : } & -Rab & \Rightarrow & \theta_\tau := -Rab
\end{align*}
\]

Stage $i + 1$:

Unfulfilled $\Rightarrow \theta_\tau := \bigvee \exists z \left[ G(bz) \land \psi(bz) \right]$

\[
\bigvee \exists z \left[ \bigwedge_{\tau'' \supseteq \tau'} \theta_{\tau''}(bz) \right]
\]

"there is a mosaic $\tau'$ that can be linked to $\tau$ to fulfil the requirement, but no matter what $R$-formulas are added, the resulting mosaic $\tau''$ has already been eliminated"