Deciding the weak definability of Büchi definable tree languages

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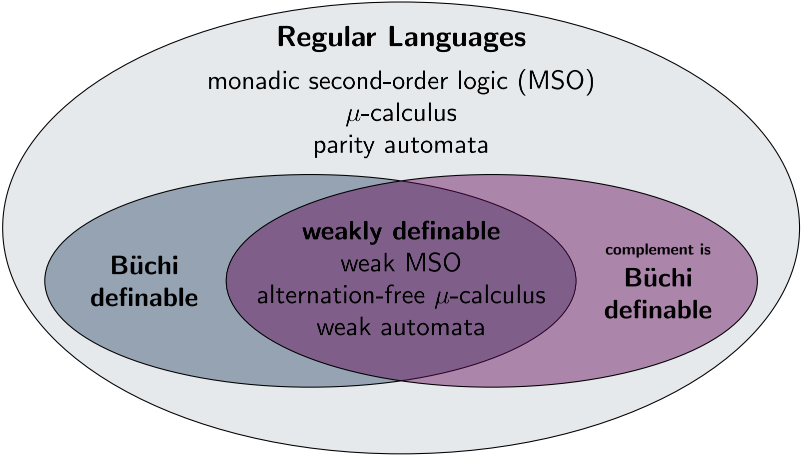
CSL 2013
Turin, Italy

Regular languages of infinite trees



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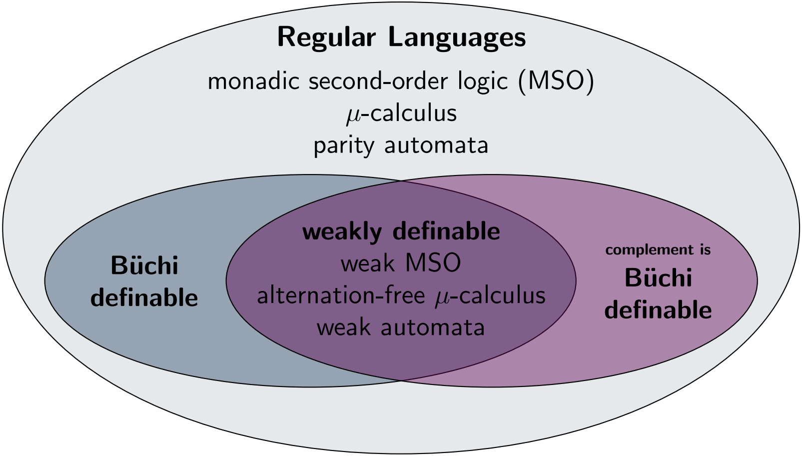

Weak automaton $\langle \mathbb{A}, Q, q_0, \delta, F \rangle$

alternating Büchi automaton such that
no cycle visits both accepting and non-accepting states

Weak definability problem

Weak definability decision problem

INPUT: parity automaton $\mathcal{U}$
OUTPUT: `YES` if there exists weak automaton $\mathcal{W}$ with $L(\mathcal{W}) = L(\mathcal{U})$,
`NO` otherwise

Theorem [Niwiński+Walukiewicz '05]

The weak definability problem is decidable if $L(\mathcal{U})$ is **deterministic**.

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Theorem

The weak definability problem is decidable if $\mathcal{U}$ is **Büchi**.

Cost automata

Finite state automaton $\mathcal{A}$

+ finite set of counters (initialized to 0, values range over $\mathbb{N}$)
+ counter operations on transitions (increment $\mathtt{i}$, reset $\mathtt{r}$, no change ε)

Semantics

$[\![\mathcal{A}]\!]$: structures $\to \mathbb{N} \cup \{\infty\}$

$$[\![\mathcal{A}]\!](s) := \min\{n : \exists \text{ accepting run of } \mathcal{A} \text{ on } s \text{ with counter values at most } n\}$$

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Boundedness with respect to language K (written $[\![\mathcal{A}]\!] \approx \chi_K$)

$[\![\mathcal{A}]\!] \approx \chi_K$ if there is bound $n \in \mathbb{N}$ such that $[\![\mathcal{A}]\!](s) \leq n$ if $s \in K$ and $[\![\mathcal{A}]\!](s) = \infty$ if $s \notin K$

Reduction to boundedness

Many problems for a regular language L have been reduced to deciding $\approx$ for special types of cost automata

- **Finite power property**
  [Simon '78, Hashiguchi '79]

  is there some n such that $L^* = \{\epsilon\} \cup L^1 \cup L^2 \cup \cdots \cup L^n$?

- **Star-height problem**
  [Hashiguchi '88, Kirsten '05, Colcombet+Löding '08]

  given n, is there a regular expression for L with at most n nestings of Kleene star?

- **Parity-index problem**
  [reduction in Colcombet+Löding '08, decidability open]

  given $i < j$, is there a nondeterministic parity automaton for L which uses only priorities $\{i, i+1, \ldots, j\}$?

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- **Star-height problem**
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  given $i < j$, is there a nondeterministic parity automaton for L which uses only priorities $\{i, i+1, \ldots, j\}$? — cost-parity

Reduction to boundedness (example)

Finite power property decision problem

INPUT: Finite state automaton $\mathcal{A}$ over finite words with $L = L(\mathcal{A})$

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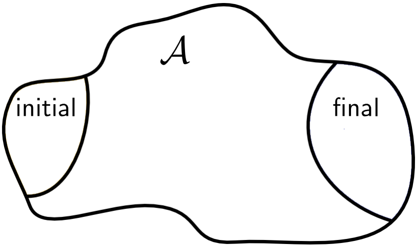

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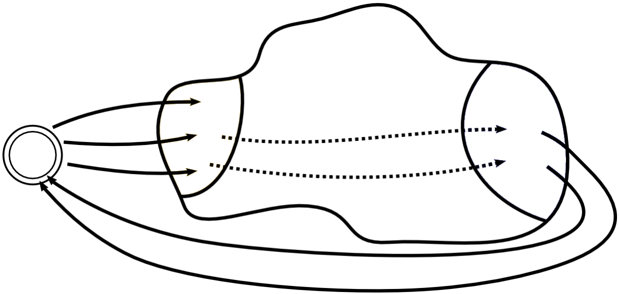

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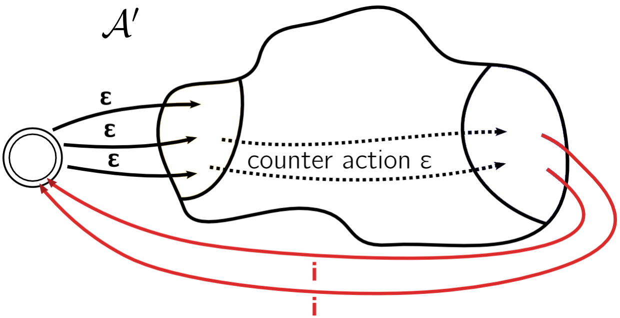
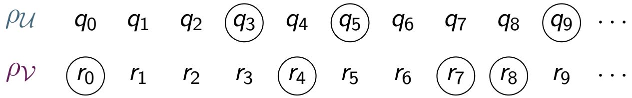


Finite power property holds iff $[\![\mathcal{A}']\!] \approx \chi_{L^*}$

Block counting

Given nondeterministic Büchi automata $\mathcal{U}$ and $\mathcal{V}$

- fix some tree t and let $\rho_{\mathcal{U}}$ and $\rho_{\mathcal{V}}$ be runs of $\mathcal{U}$ and $\mathcal{V}$ on t, with accepting states marked with ◯



- divide each branch in the composed run into **blocks** containing accepting state for $\mathcal{V}$ followed by accepting state for $\mathcal{U}$

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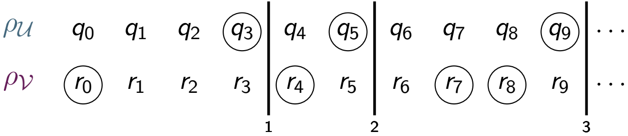

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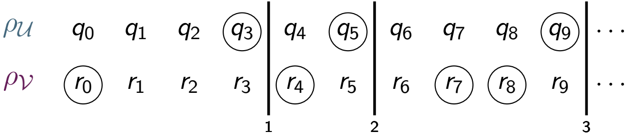

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Theorem [Rabin '70]

If there are at least $m = |Q_{\mathcal{U}}| \cdot |Q_{\mathcal{V}}|$ blocks on every branch in the composed run, then $L(\mathcal{U}) \cap L(\mathcal{V}) \neq \emptyset$.

Weak automaton construction [Kupferman+Vardi '99]

Given nondeterministic Büchi automata $\mathcal{U}$ and $\mathcal{V}$ with $L(\mathcal{U}) = \overline{L(\mathcal{V})}$

Construct weak automaton $\mathcal{W}$ such that $L(\mathcal{W}) = L(\mathcal{V})$

- Adam selects transition from $\Delta_{\mathcal{U}}$
- Eve selects transition from $\Delta_{\mathcal{V}}$ and direction

Adam $\rho_{\mathcal{U}}$	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	$\cdots$
Eve $\rho_{\mathcal{V}}$	r_0	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	$\cdots$

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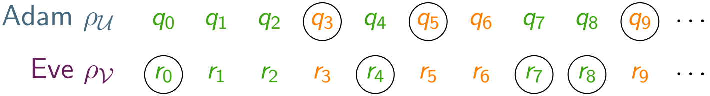

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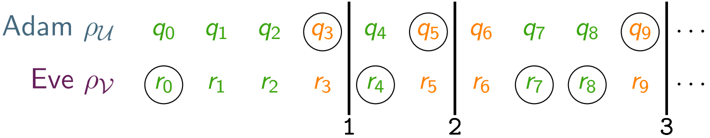
- Adam selects transition from $\Delta_{\mathcal{U}}$
- Eve selects transition from $\Delta_{\mathcal{V}}$ and direction
- Accept/reject depending on occurrences of ◯
- Store the block number in the state, up to value $m := |Q_{\mathcal{U}}| \cdot |Q_{\mathcal{V}}|$

  once m blocks have been witnessed, stabilize in rejecting state

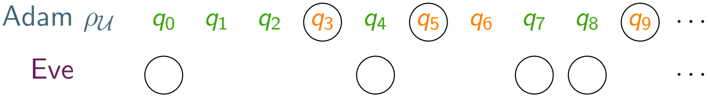


Reduction of weak definability to boundedness

Given nondeterministic Büchi automaton $\mathcal{U}$

Construct cost automaton $\mathcal{Q}$ s.t. $[\![\mathcal{Q}]\!] \approx \chi_{\overline{L(\mathcal{U})}}$ iff $L(\mathcal{U})$ is weakly definable

- Adam selects transition from $\Delta_\mathcal{U}$
- Eve selects direction and guesses whether to output ◯
- Accept/reject depending on occurrences of ◯



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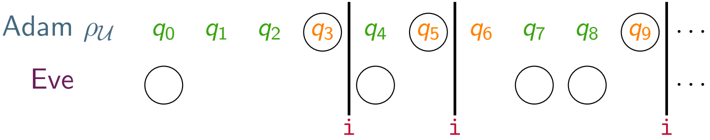
- Adam selects transition from $\Delta_{\mathcal{U}}$
- Eve selects direction and guesses whether to output ◯
- Accept/reject depending on occurrences of ◯
- Store the block number in the counter



Decidability of $\approx$ for cost automata

Decidability of $\approx$ for cost automata over infinite trees is open in general, but is known in some special cases...

Theorem [Kuperberg+VB '11]

The boundedness relation $\approx$ is decidable for **quasi-weak cost automata** over infinite trees.

Quasi-weak cost automaton

alternating cost-Büchi automaton such that in any cycle with both accepting and non-accepting states, there is a counter which is incremented but not reset

Deciding weak definability for Büchi input

Theorem

Given Büchi automaton $\mathcal{U}$, we can construct a quasi-weak cost automaton $\mathcal{Q}$ such that the following are equivalent:

- $L(\mathcal{U})$ is weakly definable;
- $[\![\mathcal{Q}]\!] \approx \chi_{\overline{L(\mathcal{U})}}$.

$\downarrow$

Theorem [Kuperberg+VB '11]

The boundedness relation $\approx$ is decidable for quasi-weak cost automata.

$\downarrow$

Theorem

Given Büchi automaton $\mathcal{U}$, it is decidable whether $L(\mathcal{U})$ is weakly definable.

Conclusion

Cost automata can be used to help prove the decidability of definability problems for regular languages of infinite trees.

- The **weak definability problem** is decidable when the input is a Büchi automaton.
- The **co-Büchi definability problem** is decidable when the input is a parity automaton.

Open question

Is $\approx$ decidable for larger classes of cost automata over infinite trees?