

Interpolation with decidable fixpoint logics

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Joint work with
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Fixpoint logics

Fixpoint logics give mechanism to express *dynamic, recursive properties*.

Example

binary relation R , unary relation P

“from y , it is possible to R -reach some P -element”

$$[\text{lfp}_{Y,y} . P y \vee \exists z (R y z \wedge Y z)](y)$$

Modal mu-calculus (L_μ)

[Kozen '83]

extension of modal logic
with fixpoints

describes transition systems
(relations of arity at most 2)

decidable satisfiability
(EXPTIME-complete)

tree model property

Some decidable fixpoint logics

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Unary negation fixpoint logic (UNFP)

[Segoufin, ten Cate '11]

fragment of LFP with monadic fixpoints
and negation of formulas with at most
one free variable

describes relational structures
(relations of arbitrary arity)

decidable satisfiability
(2EXPTIME-complete)

tree-like model property
(models of bounded tree-width)

UNFP is **expressive**:

- modal logic and L_μ , even with backwards modalities;
- positive existential FO (i.e. unions of conjunctive queries);
- description logics including \mathcal{ALC} , \mathcal{ALCFIO} , \mathcal{ELI} ;
- monadic Datalog.

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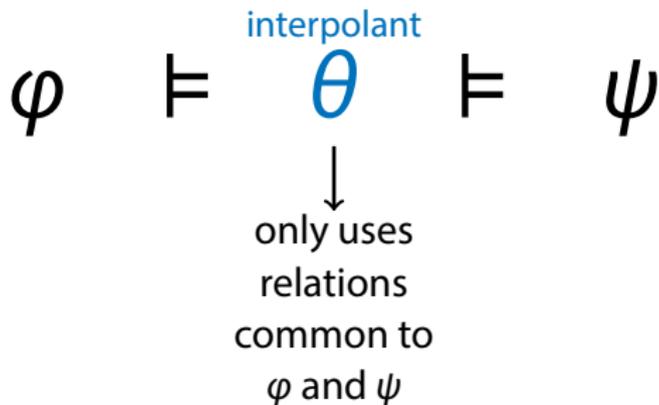
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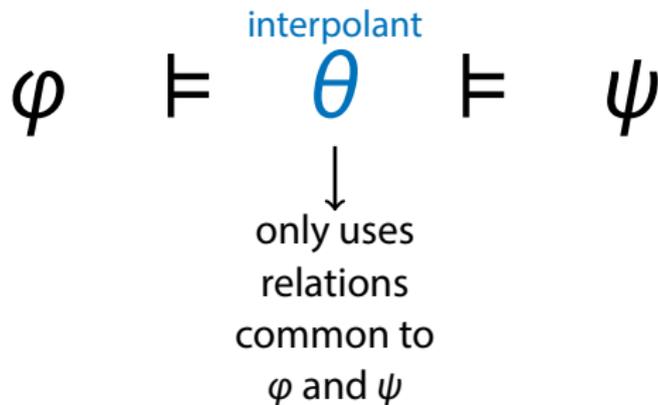
...what about **interpolation**?

$\varphi \quad \vDash \quad \psi$

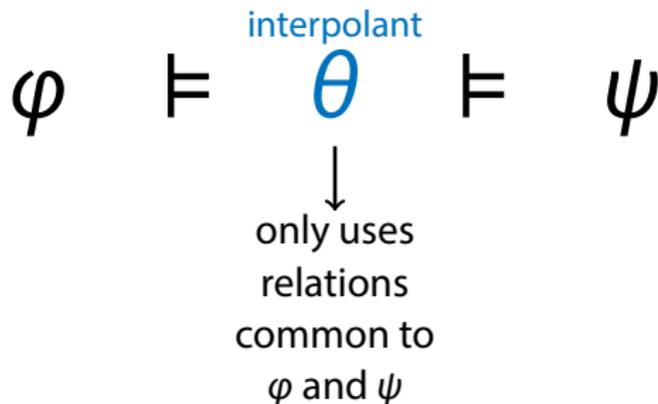
Interpolation



Interpolation



Craig interpolation: θ depends on φ and ψ



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Uniform interpolation: θ depends only on φ and common signature (not on a particular ψ)

Theorem (D'Agostino, Hollenberg '00)

L_μ has effective uniform interpolation.

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Let UNFP^k denote the k -variable fragment of UNFP (in normal form...).

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Proof strategy: [Bootstrap from modal world](#), making use of results/ideas of [Grädel, Walukiewicz '99], [Grädel, Hirsch, Otto '00], [D'Agostino, Hollenberg '00].

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Proof structure:

Relational
structures

Coded structures
(tree decompositions of
width k)

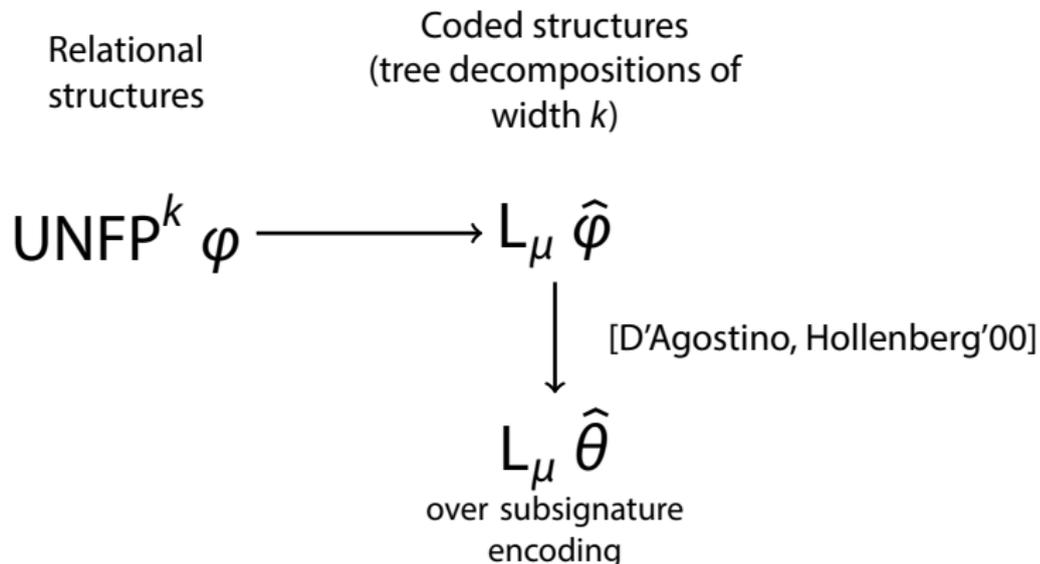
$$\text{UNFP}^k \varphi \longrightarrow L_\mu \hat{\varphi}$$

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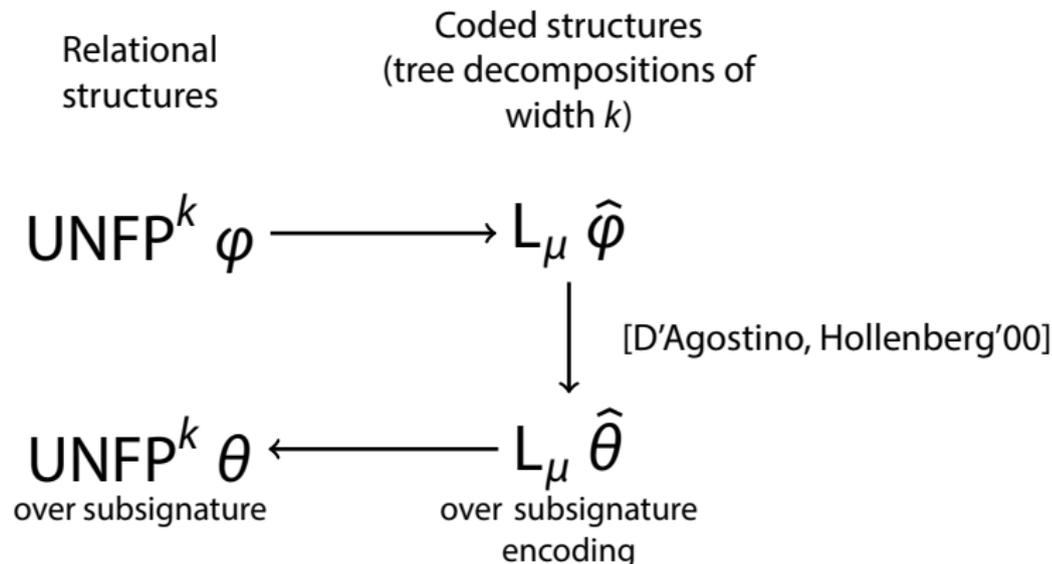


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UNFP is an expressive, decidable fixpoint logic
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Can this result about UNFP help us answer any interesting query rewriting problems?

Uniform interpolation example

“ S holds at x , and from every position y where S holds, there is an R -neighbor z where S holds”

$$\begin{aligned}\varphi(x) &:= Sx \wedge \forall y(Sy \rightarrow \exists z(Ryz \wedge Sz)) \\ &\equiv Sx \wedge \neg \exists y(Sy \wedge \neg \exists z(Ryz \wedge Sz))\end{aligned}$$

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Uniform interpolant of φ over subsignature $\{R\}$

“there is an infinite R -path from x ”

$$\begin{aligned}&[\mathbf{gfp}_{Y,y} . \exists z(Ryz \wedge Yz)](x) \\ \equiv &\neg[\mathbf{lfp}_{Y,y} . \neg \exists z(Ryz \wedge \neg Yz)](x)\end{aligned}$$