Effective interpolation for guarded logics

Michael Benedikt[1], Balder ten Cate[2], **Michael Vanden Boom**[1]

[1]University of Oxford [2]LogicBlox and UC Santa Cruz

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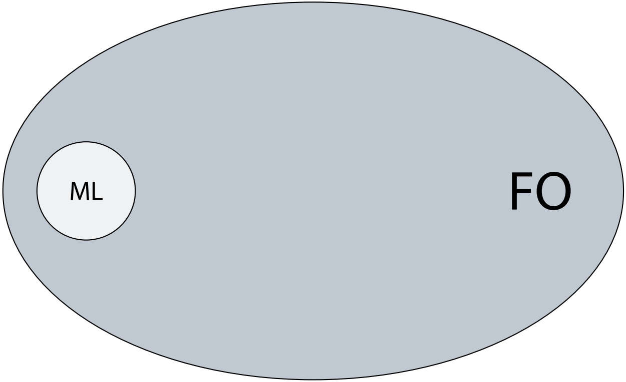
Some decidable fragments of first-order logic



	ML
finite model property	✓
tree-like model property	✓
Craig interpolation	✓

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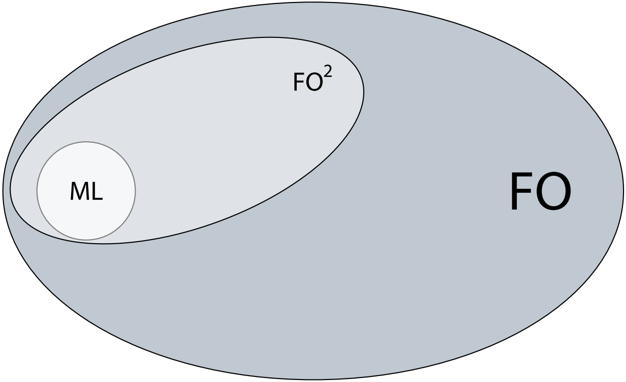

constrain
number of variables

	ML	FO^2
finite model property	✓	✓
tree-like model property	✓	✗
Craig interpolation	✓	✗

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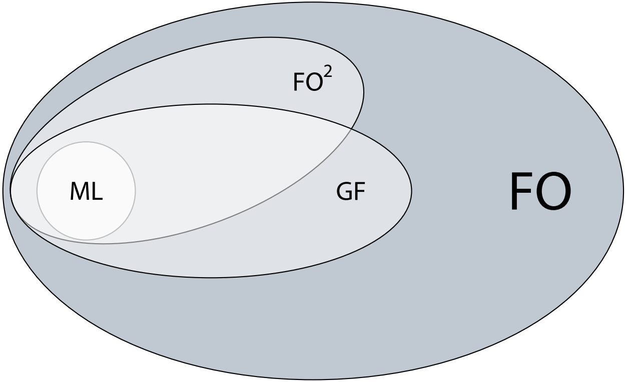

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quantification
[Andréka, van Benthem, Németi ’95-’98]

$\exists \boldsymbol{x}\, (G(\boldsymbol{xy}) \wedge \psi(\boldsymbol{xy}))$
$\forall \boldsymbol{x}\, (G(\boldsymbol{xy}) \rightarrow \psi(\boldsymbol{xy}))$

	ML	FO^2	GF
finite model property	✓	✓	✓
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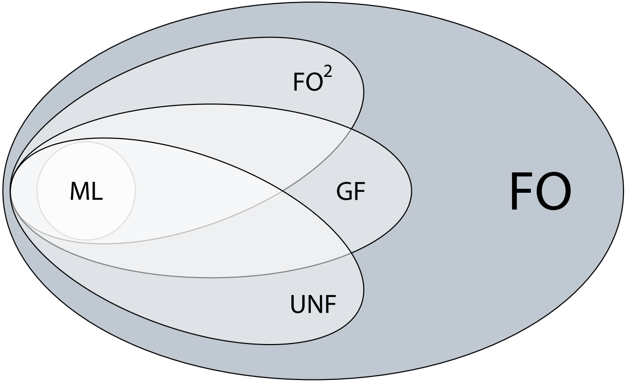

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negation
[ten Cate, Segoufin '11]

$\exists \boldsymbol{x}\,(\psi(\boldsymbol{xy}))$
$\neg\psi(x)$

	ML	FO^2	GF	UNF
finite model property	✓	✓	✓	✓
tree-like model property	✓	✗	✓	✓
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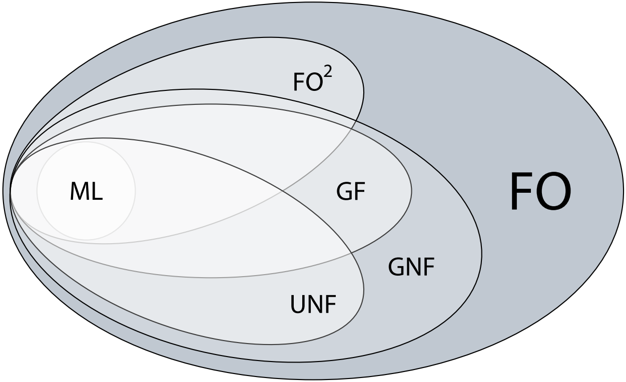

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	ML	FO^2	GF	UNF	GNF
finite model property	✓	✓	✓	✓	✓
tree-like model property	✓	✗	✓	✓	✓
Craig interpolation	✓	✗	✗	✓	✓

Interpolation

$$\varphi \quad \models \quad \psi$$

Interpolation

$$\varphi \quad \models \quad \overset{\text{interpolant}}{\chi} \quad \models \quad \psi$$

$\downarrow$

only uses
relations in
both φ and ψ

Interpolation example

$$\exists xyz(Txyz \land Rxy \land Ryz \land Rzx) \quad \models \quad \exists xy(Rxy \land ((Sx \land Sy) \lor (\neg Sx \land \neg Sy)))$$

"there is a T-guarded
3-cycle using R"

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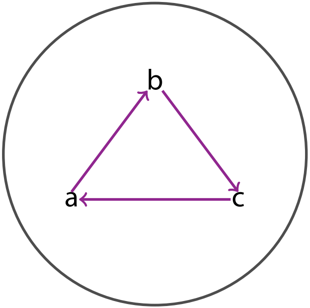

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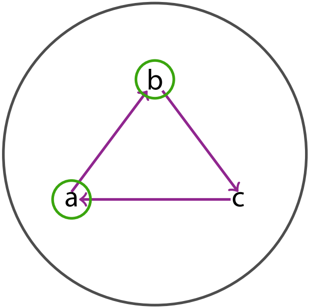

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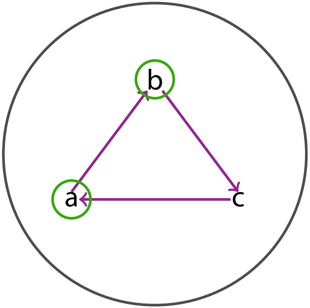

interpolant $\chi := \exists xyz(Rxy \wedge Ryz \wedge Rzx)$

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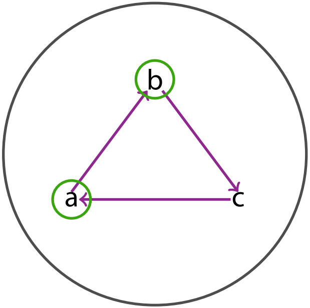

GNF interpolant $\chi := \exists xyz(Rxy \land Ryz \land Rzx)$

"there is a 3-cycle using R"

Interpolation

$$\varphi \quad \models \quad \overset{\text{interpolant}}{\chi} \quad \models \quad \psi$$

$\downarrow$

only uses
relations in
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Theorem (Bárány+Benedikt+ten Cate '13)

Given GNF formulas φ and ψ such that $\varphi \models \psi$, there is a GNF interpolant χ (but model theoretic proof implies no bound on size of χ).

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Given GNF formulas φ and ψ such that $\varphi \models \psi$, there is a GNF interpolant χ (but model theoretic proof implies no bound on size of χ).

Even when input is in GF, no idea how to **compute** interpolants (or other rewritings related to interpolation).

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Theorem (Bárány+Benedikt+ten Cate '13)

Given GNF formulas φ and ψ such that $\varphi \models \psi$, there is a GNF interpolant χ (but model theoretic proof implies no bound on size of χ).

Theorem (Benedikt+ten Cate+VB. '14)

Given GNF formulas φ and ψ such that $\varphi \models \psi$, we can **construct** a GNF interpolant χ of doubly exponential DAG-size (in size of input).

Mosaics

A **mosaic** $\tau(\boldsymbol{a})$ for φ is a collection of subformulas of φ over some guarded set $\boldsymbol{a}$ of parameters.

$\tau_1(ab)$	$\tau_2(bc)$	$\tau_3(d)$
Raa	Sb	Sd
$\neg Sa$	$\neg Rbb$	$\neg Sd$
$\exists z(Rbz \wedge Sz)$	$Rbc \wedge Sc$	$\exists yz(Ryz \wedge Sz)$
Sb	Rcb	$\forall z(Rdz)$
Rba	Sc	$Rdd \vee Sd$
$\dots$	$\dots$	$\dots$

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$\tau_1(ab)$	$\tau_2(bc)$	$\tau_3(d)$
Raa	Sb	**Internally inconsistent**
$\neg Sa$	$\neg Rbb$	(e.g. S d & $\neg S$ d)
$\exists z(Rbz \wedge Sz)$	$Rbc \wedge Sc$	
Sb	Rcb	
Rba	Sc	
…	…	

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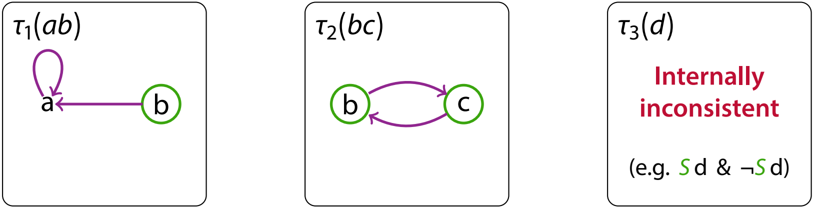
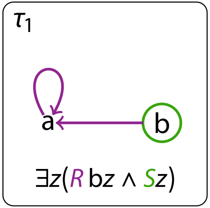


Internally consistent mosaics are windows into a (guarded) piece of a structure.

Linking mosaics

Mosaics can be **linked** together to fulfill an existential requirement if they agree on all formulas that use only shared parameters.



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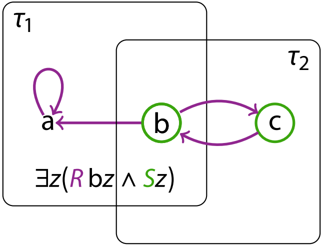

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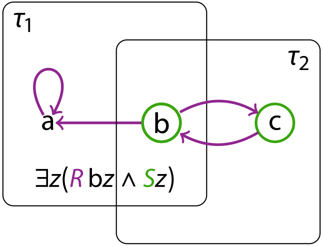

We say a set S of mosaics is **saturated** if every existential requirement in a mosaic $\tau \in S$ is fulfilled in τ or in some linked $\tau' \in S$.

Mosaics

Fix some set P of size $2 \cdot \mathrm{width}(\varphi)$ and let $\mathscr{M}_\varphi$ be the set of mosaics for φ over parameters P. The size of $\mathscr{M}_\varphi$ is doubly exponential in the size of φ.

Theorem

φ is satisfiable iff there is a saturated set S of internally consistent mosaics from $\mathscr{M}_\varphi$ that contains some τ with $\varphi \in \tau$.

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$$\boxed{\tau_3}$$

$$S = \left\{ \boxed{\tau_1}, \boxed{\tau_2}, \boxed{\tau_3}, \boxed{\tau_4} \right\}$$

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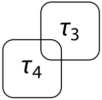

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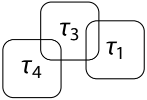

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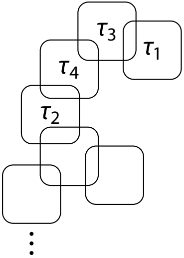
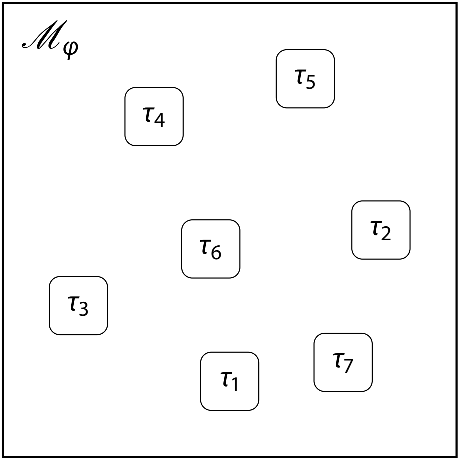
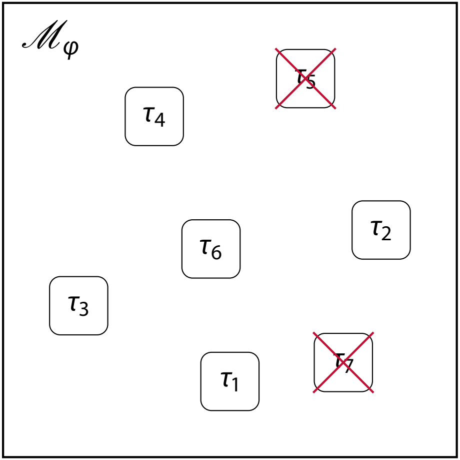


Mosaic elimination algorithm for satisfiability testing



Mosaic elimination algorithm for satisfiability testing

Stage 1.
Eliminate mosaics with internal inconsistencies.



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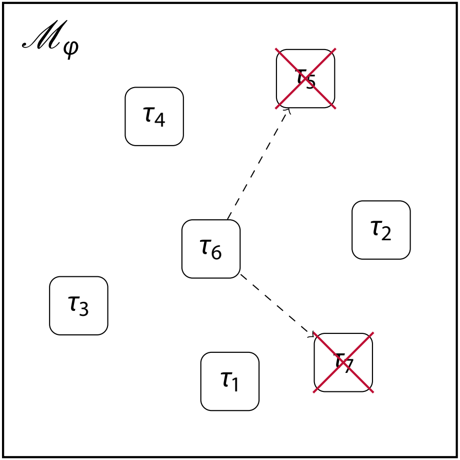
Stage $i+1$.
Eliminate mosaics with existential requirements that can only be fulfilled using mosaics eliminated in earlier stages.

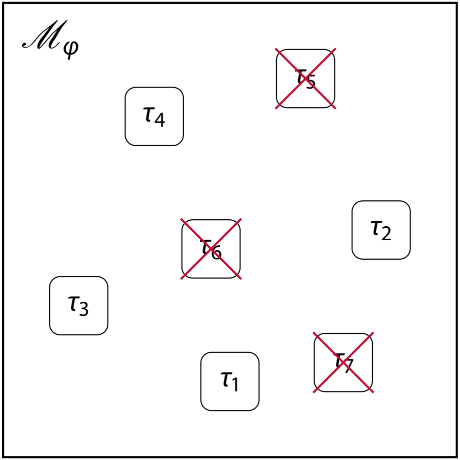


Mosaic elimination algorithm for satisfiability testing

Stage 1.
Eliminate mosaics with internal inconsistencies.

Stage $i + 1$.
Eliminate mosaics with existential requirements that can only be fulfilled using mosaics eliminated in earlier stages.

Continue until fixpoint $\mathscr{M}'$ reached. The set $\mathscr{M}'$ is a saturated set of internally consistent mosaics.



Theorem

φ is satisfiable iff there is some mosaic $\tau \in \mathscr{M}'$ with $\varphi \in \tau$.

Mosaics for interpolation

Assume $\varphi_{\mathtt{L}} \models \varphi_{\mathtt{R}}$.

Idea: Construct interpolant from proof that $\varphi_{\mathtt{L}} \wedge \neg\varphi_{\mathtt{R}}$ is unsatisfiable.

Mosaics for interpolation

Assume $\varphi_{\mathtt{L}} \models \varphi_{\mathtt{R}}$.

Idea: Construct interpolant from proof that $\varphi_{\mathtt{L}} \wedge \neg\varphi_{\mathtt{R}}$ is unsatisfiable.

Consider mosaics for $\varphi_{\mathtt{L}} \wedge \neg\varphi_{\mathtt{R}}$.
Annotate each mosaic and each formula with a **provenance** $\mathtt{L}$ or $\mathtt{R}$.

$\mathtt{L} : \tau_1(ab)$	$\mathtt{R} : \tau_2(bc)$	$\mathtt{L} : \tau_3(d)$
$\mathtt{L} : Raa$	$\mathtt{L} : Sb$	$\mathtt{L} : Sd$
$\mathtt{R} : \neg Sa$	$\mathtt{R} : \neg Rbb$	$\mathtt{R} : \neg Sd$
$\mathtt{R} : \exists z(Rbz \wedge Sz)$	$\mathtt{R} : Rbc \wedge Sc$	$\mathtt{R} : Rdd \wedge Sd$
$\mathtt{R} : \neg Rbb$	$\mathtt{R} : Rbc$	$\mathtt{R} : \exists yz(Ryz \wedge Sz)$
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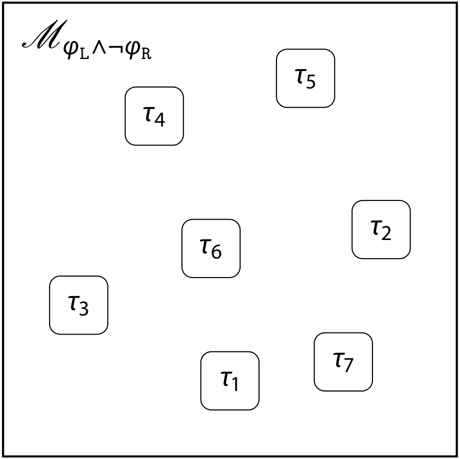
...	$\mathtt{R} : Sc$	...
	...	

Linking must respect the provenance annotations.

Mosaics for interpolation

Assume $\varphi_{\mathrm{L}} \models \varphi_{\mathrm{R}}$.

Test satisfiability of $\varphi_{\mathrm{L}} \wedge \neg\varphi_{\mathrm{R}}$ using mosaic elimination.



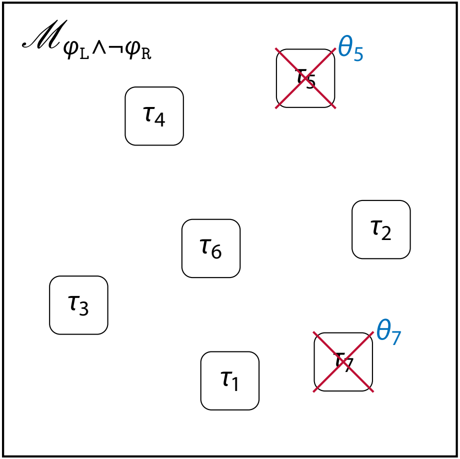
Mosaics for interpolation

Assume $\varphi_{\mathrm{L}} \vDash \varphi_{\mathrm{R}}$.

Test satisfiability of $\varphi_{\mathrm{L}} \wedge \neg\varphi_{\mathrm{R}}$ using mosaic elimination.

Assign a **mosaic interpolant** θ_τ to each eliminated mosaic τ such that $\tau_{\mathrm{L}} \vDash \theta_\tau$ and $\theta_\tau \vDash \neg\tau_{\mathrm{R}}$.

Mosaic interpolants θ_τ describe why the mosaic τ was eliminated.



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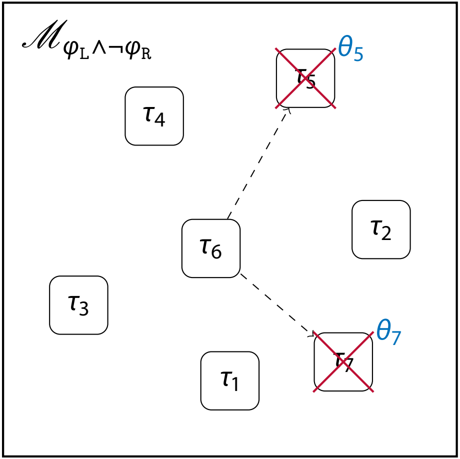

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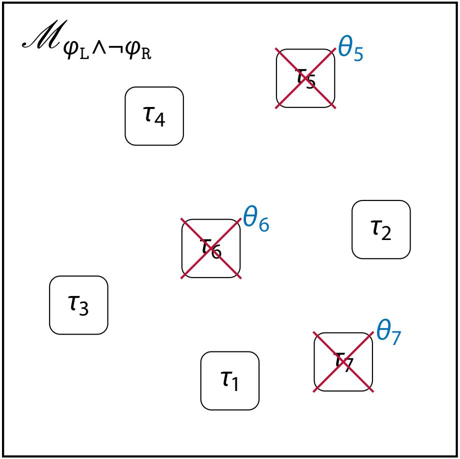

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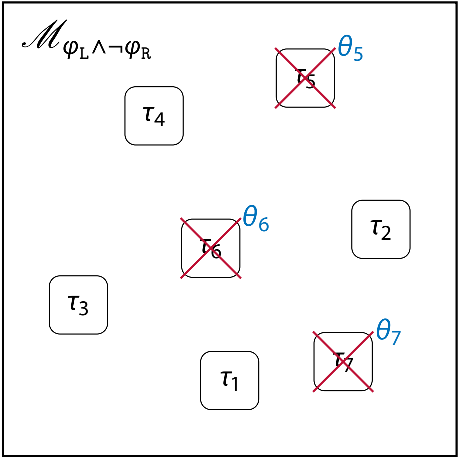

Theorem

An interpolant χ for $\varphi_{\mathrm{L}} \vDash \varphi_{\mathrm{R}}$ of at most doubly exponential DAG-size can be constructed from the mosaic interpolants.

Shape of interpolants

Mosaic interpolants θ_τ satisfy $\tau_{\mathtt{L}} \models \theta_\tau$ and $\theta_\tau \models \neg\tau_{\mathtt{R}}$.
They describe why the mosaic τ was eliminated.

Stage 1:

Internal inconsistency	$\mathtt{L} : Rab$	$\mathtt{L} : \neg Rab$	$\Rightarrow$	$\theta_\tau := \bot$
	$\mathtt{R} : Rab$	$\mathtt{R} : \neg Rab$	$\Rightarrow$	$\theta_\tau := \top$
	$\mathtt{L} : Rab$	$\mathtt{R} : \neg Rab$	$\Rightarrow$	$\theta_\tau := Rab$
	$\mathtt{R} : Rab$	$\mathtt{L} : \neg Rab$	$\Rightarrow$	$\theta_\tau := \neg Rab$

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Internal inconsistency

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\begin{array}{llcl}
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\mathtt{L} : Rab & \mathtt{R} : \neg Rab & \Rightarrow & \theta_\tau := Rab \\
\mathtt{R} : Rab & \mathtt{L} : \neg Rab & \Rightarrow & \theta_\tau := \neg Rab
\end{array}
$$

Stage $i + 1$**:**

Unfulfilled $\quad \mathtt{L} : \exists \boldsymbol{z}\,[G(\boldsymbol{bz}) \wedge \psi(\boldsymbol{bz})] \quad \Rightarrow \quad \theta_\tau := \bigvee_{\tau'(\boldsymbol{bc})} \exists \boldsymbol{z} \left[\bigwedge_{\tau'' \supseteq \tau'} \theta_{\tau''}(\boldsymbol{bz}) \right]$

"there is a mosaic τ' that can be linked to τ to fulfil the requirement, but no matter what R*-formulas are added, the resulting mosaic τ'' has already been eliminated"*

Mosaics for interpolation

Challenge: ensure interpolant χ is in GNF

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Solution: place further restrictions on the formulas in the mosaics

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Idea: in an $\mathtt{L}$-mosaic, only allow $\mathtt{R}$-formulas that are guarded by some $\mathtt{L}$-atom in the common signature.

$\mathtt{L} : \tau$	$\mathtt{R} : \tau'$
full info about $\mathtt{L}$-formulas	**full** info about $\mathtt{R}$-formulas
partial info about $\mathtt{R}$-formulas	**partial** info about $\mathtt{L}$-formulas

This makes it harder to prove completeness of mosaic method,
but makes it easier to prove properties about the mosaic interpolants.

Stronger interpolation theorems for GNF

Lyndon interpolation: χ respects **polarity of relations**

A relation R occurs positively (respectively, negatively) in χ iff R occurs positively (respectively, negatively) in both φ_{L} and φ_{R}.

Relativized interpolation: χ respects **quantification pattern**

If the quantification in φ_{L} and φ_{R} is relativized to a distinguished set of unary predicates $\mathbb{U}$, then χ is $\mathbb{U}$-relativized.
i.e. quantification is of the form $\exists x\,(Ux \wedge \psi(x\boldsymbol{y}))$ for $U \in \mathbb{U}$

Bonus: effective preservation theorems

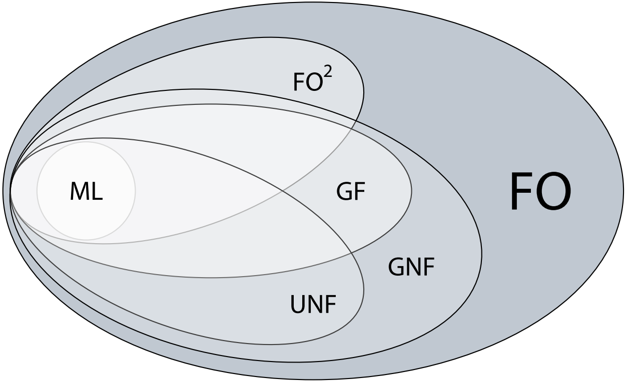
φ is **preserved under extensions** if $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \subseteq \mathfrak{B}$ implies $\mathfrak{B} \models \varphi$.

φ is in **existential GNF** if no quantifier is in the scope of a negation.

Corollary (Analog of Łoś-Tarski)

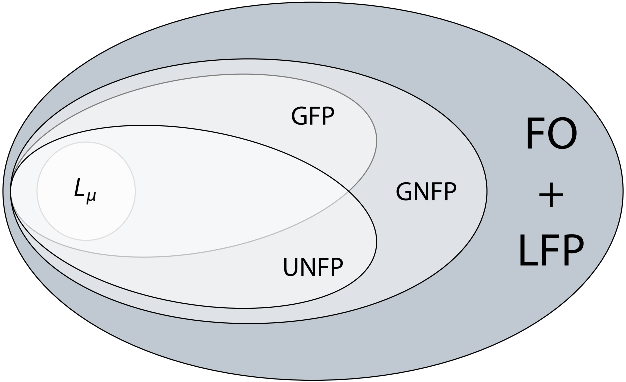
If φ is preserved under extensions and in GNF, then we can construct an equivalent existential GNF formula φ' of doubly exponential DAG-size.

Some decidable fragments of FO



	ML	FO²	GF	UNF	GNF
finite model property	✓	✓	✓	✓	✓
tree-like model property	✓	✗	✓	✓	✓
Craig interpolation	✓	✗	✗	✓	✓

Some decidable fragments of FO+LFP



	L_μ	GFP	UNFP	GNFP
finite model property	✓	✗	✗	✗
tree-like model property	✓	✓	✓	✓
Craig interpolation	✓	?	?	?

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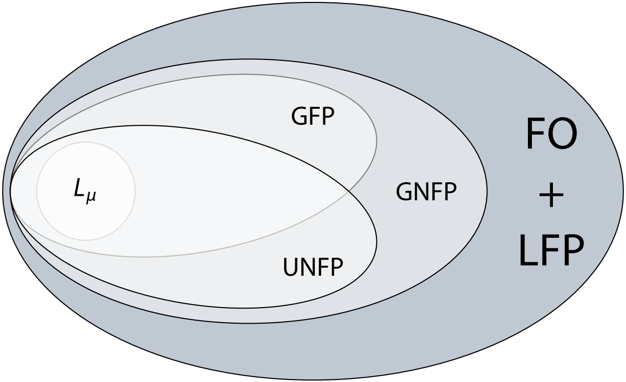

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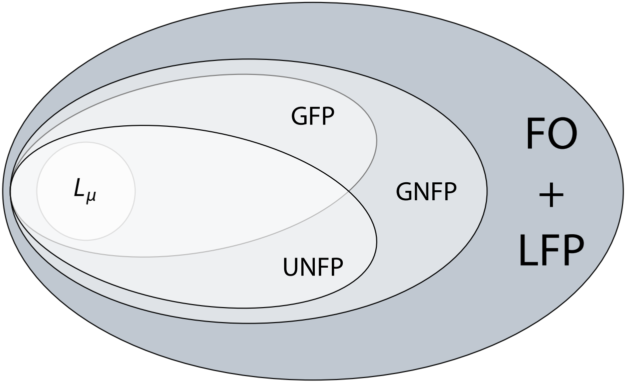

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Uniform interpolation

The modal mu-calculus (L_μ) has uniform interpolation [D'Agostino+Hollenberg '00]...

Uniform interpolation: χ depends only on antecedent and the signature of the consequent

> Given φ_{L} and a sub-signature σ,
> there is an interpolant χ over σ such that
> for all φ_{R} with $\varphi_{\mathrm{L}} \models \varphi_{\mathrm{R}}$ and common signature σ,
>
> $\varphi_{\mathrm{L}} \models \chi$ and $\chi \models \varphi_{\mathrm{R}}$

Uniform interpolation for UNFP^k

Let UNFP^k denote the k-variable fragment of UNFP
(when written in a normal form...)

Theorem (Benedikt+ten Cate+VB. unpublished)

UNFP^k has effective uniform interpolation. UNFP has Craig interpolation.

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Relational structures	Coded structures (tree decompositions of width k)

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over subsignature encoding

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over subsignature

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Summary

Guarded logics have attractive computational
and model-theoretic properties, including interpolation.

	ML	GF	UNF	GNF
Craig interpolation	✓	✗	✓	✓

adapted
mosaic method
from ML
[Benedikt,ten Cate,VB.'14]

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Guarded logics have attractive computational and model-theoretic properties, including interpolation.

	ML	GF	UNF	GNF	L_μ	GFP	UNFP	GNFP
Craig interpolation	✓	✗	✓	✓	✓	✗	✓	✗

UNF, GNF: adapted **mosaic method** from ML [Benedikt,ten Cate,VB.'14]

UNFP: used **uniform interpolation** for L_μ [Benedikt,ten Cate,VB. unpublished]

Effective preservation theorems

φ is **monotone** if $\mathfrak{A} \models \varphi$ implies that $\mathfrak{A}' \models \varphi$ for any $\mathfrak{A}'$ obtained from $\mathfrak{A}$ by adding tuples to the interpretation of some relation.

φ is **positive** if every relation appears within the scope of an even number of negations.

Corollary (Monotone = Positive)

If φ is monotone and in GNF, then we can construct an equivalent positive GNF formula φ' of doubly exponential DAG-size.

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Let φ^i be the result of replacing every relation R with a copy R^i.
The Lyndon interpolant χ for

$$\bigwedge_R \neg\exists \boldsymbol{y} \left(R^1 \boldsymbol{y} \wedge \neg R^2 \boldsymbol{y} \right) \wedge \varphi^1 \vDash \varphi^2$$

can only use relations of the form R^2, and these must all be positive.
Replacing every R^2 with R in χ yields positive φ' equivalent to φ.

Łoś-Tarski preservation theorem

φ is **preserved under extensions** if $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \subseteq \mathfrak{B}$ implies $\mathfrak{B} \models \varphi$.

φ is in **existential GNF** if no quantifier is in the scope of a negation.

Corollary (Analog of Łoś-Tarski)

If φ is preserved under extensions and in GNF, then we can construct an equivalent existential GNF formula φ' of doubly exponential DAG-size.

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Corollary (Analog of Łoś-Tarski)

If φ is preserved under extensions and in GNF, then we can construct an equivalent existential GNF formula φ' of doubly exponential DAG-size.

Let $\mathbb{U} := \{U^1, U^2\}$ be a set of fresh unary predicates.
Let φ^i be the result of relativizing every quantification to U^i.

The relativized Lyndon interpolant χ for

$$\neg\exists y \left(U^1 y \wedge \neg U^2 y\right) \wedge \varphi^1 \models \varphi^2$$

is an existential GNF formula.
Replacing every $U^2 z$ in χ with $\top$ yields the desired existential GNF φ'.