Interpolation with decidable fixpoint logics

Michael Benedikt[1], Balder ten Cate[2], **Michael Vanden Boom**[1]

[1]University of Oxford [2]LogicBlox and UC Santa Cruz

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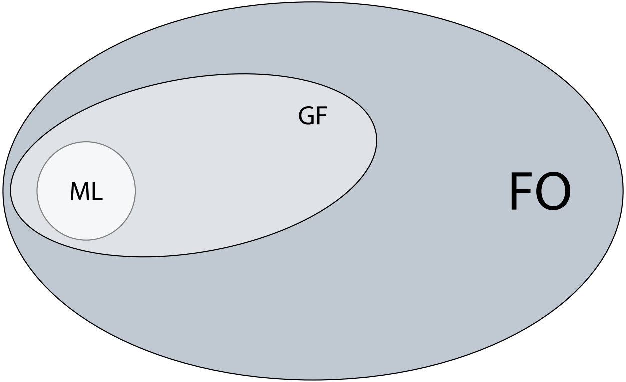
Some decidable fragments of first-order logic



constrain
quantification

$\exists \boldsymbol{x}(G(\boldsymbol{xy}) \wedge \psi(\boldsymbol{xy}))$
$\forall \boldsymbol{x}(G(\boldsymbol{xy}) \rightarrow \psi(\boldsymbol{xy}))$

[Andréka, van Benthem, Németi ’95-’98]

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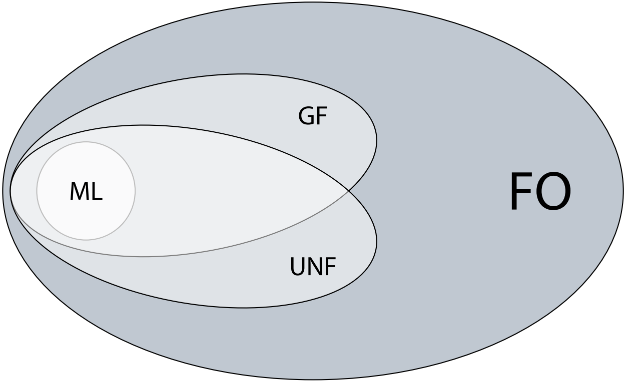

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constrain negation

$\exists \boldsymbol{x}(\psi(\boldsymbol{xy}))$
$\neg\psi(x)$

[ten Cate, Segoufin ’11]

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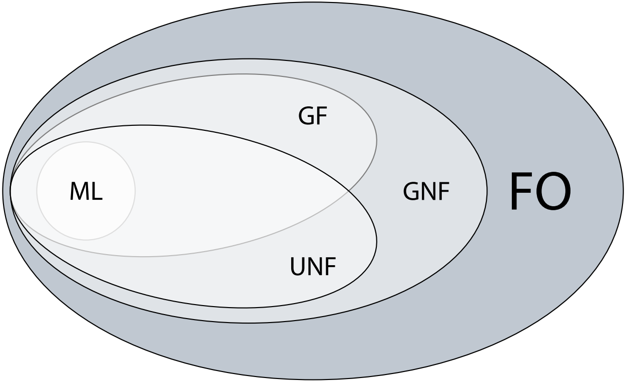

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$\exists \boldsymbol{x}(G(\boldsymbol{xy}) \land \psi(\boldsymbol{xy}))$
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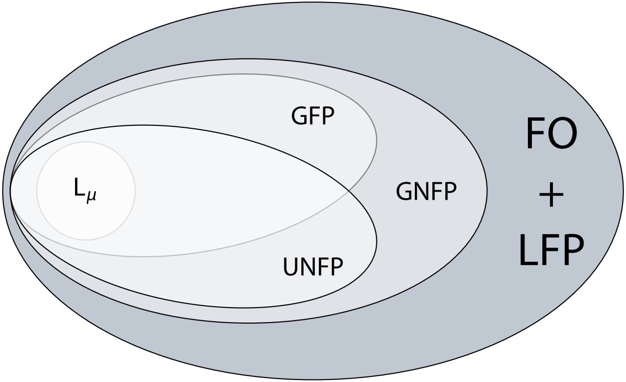
Some decidable fragments of FO+LFP



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Why study guarded logics?

Guarded logics are expressive. For instance, UNFP captures:

- mu-calculus, even with backwards modalities;
- positive existential FO (i.e. unions of conjunctive queries);
- description logics including $\mathcal{ALC}$, $\mathcal{ALCHIO}$, $\mathcal{ELI}$;
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Some guarded logics have interpolation...

Interpolation

$$\varphi \quad \models \quad \psi$$

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$$\varphi \models \underset{\text{only uses relations in both } \varphi \text{ and } \psi}{\overset{\text{interpolant}}{\chi}} \models \psi$$

Interpolation example

$$\exists xyz(Txyz \wedge Rxy \wedge Ryz \wedge Rzx) \quad \models \quad \exists xy(Rxy \wedge ((Sx \wedge Sy) \vee (\neg Sx \wedge \neg Sy)))$$

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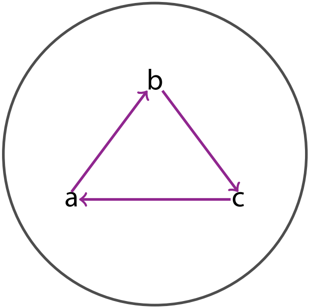

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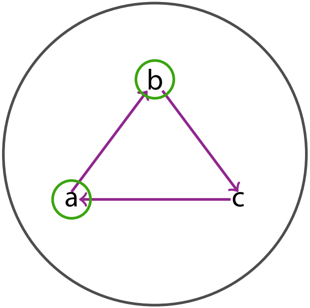

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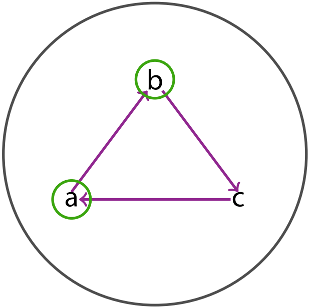

interpolant $\chi := \exists xyz(Rxy \wedge Ryz \wedge Rzx)$

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- Interpolation is related to query rewriting over views.
- Interpolation is related to modularity in description logics.

Interpolation results

Very little is known about interpolation for fixpoint logics
over general relational structures, where relations can have arbitrary arity.

	ML	GF	UNF	GNF	L_μ	GFP	UNFP	GNFP
Craig interpolation	✓	✗	✓	✓	✓	?	?	?

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Contribution: bootstrapping from ML / L_μ extended to interpolation

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Theorem (D'Agostino, Hollenberg '00)

L_μ has effective uniform interpolation.

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> *Given φ and a sub-signature σ,*
> *there is a formula χ over σ such that*
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Let UNFP^k denote the k-variable fragment of UNFP (in normal form...).

Theorem (Benedikt, ten Cate, VB. '15)

UNFP^k has effective uniform interpolation.
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Uniform interpolation example

"S holds at x, and from every position y where S holds,
there is an R-neighbor z where S holds"

$$\begin{aligned}\varphi(x) &:= Sx \wedge \forall y\big(Sy \to \exists z(Ryz \wedge Sz)\big) \\ &\equiv Sx \wedge \neg\exists y\big(Sy \wedge \neg\exists z(Ryz \wedge Sz)\big)\end{aligned}$$

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Uniform interpolant of φ over subsignature $\{R\}$
"there is an infinite R-path from x"

$$[\mathbf{gfp}_{Y,y} \,.\, \exists z(Ryz \wedge Yz)](x)$$
$$\equiv \neg[\mathbf{lfp}_{Y,y} \,.\, \neg\exists z(Ryz \wedge \neg Yz)](x)$$

Uniform interpolation for UNFP^k

Theorem (Benedikt, ten Cate, VB. ’15)

UNFP^k has effective uniform interpolation.

Proof strategy: Exploit tree-like model property and results from modal world.

([Grädel, Walukiewicz ’99], [Grädel, Hirsch, Otto ’00], [D’Agostino, Hollenberg ’00])

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Coded structures (tree decompositions of width k)

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$\downarrow$ [D'Agostino, Hollenberg'00]

$$\text{L}_\mu\ \widehat{\chi}$$

over subsignature encoding

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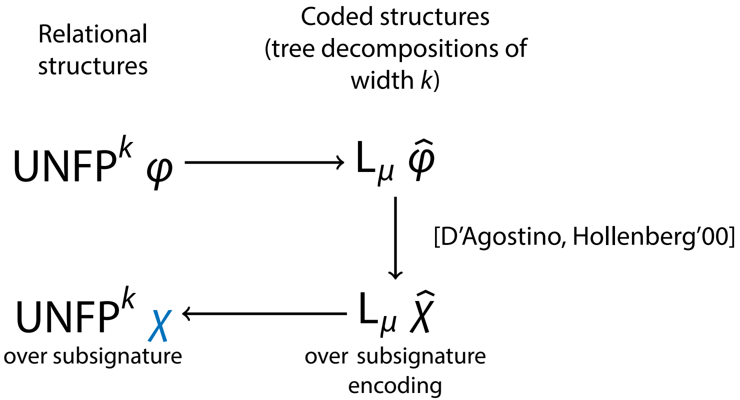


Conclusion

UNFP is an expressive, decidable fixpoint logic
with effective interpolation.

	ML	GF	UNF	GNF	L_μ	GFP	UNFP	GNFP
Craig interpolation	✓	✗	✓	✓	✓	✗	✓	✗

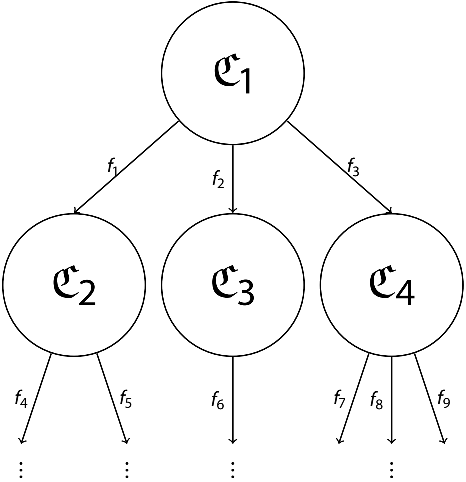
Encoding structures of tree width $k - 1$

Fix a set $K = \{1, \dots, k\}$ of names for elements.

Let $\mathbb{K}_{\sigma,k} := \{\mathfrak{C} : \mathfrak{C}$ is a σ-structure with universe $C \subseteq K$ of size at most $k\}$.

A $\mathbb{K}_{\sigma,k}$-**tree** is an unranked infinite tree with

- arbitrary branching (possibly infinite),
- node labels $\mathfrak{C} \in \mathbb{K}_{\sigma,k}$,
- edge labels are partial functions $f : K \to K$ describing relationship between names.



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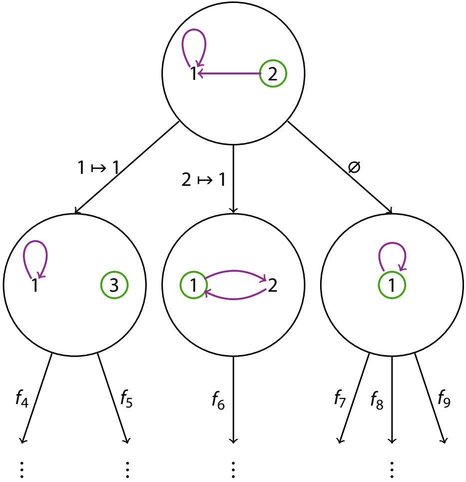

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$\mathbb{K}_{\sigma,k}$-trees are **consistent** if neighboring nodes agree on any shared names.