Weak Cost Monadic Logic over Infinite Trees

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Cost monadic second-order logic (cost MSO)

Syntax

First-order logic with second-order quantification over sets
$+$ new predicates $|X| \leq N$ appearing positively ($N \in \mathbb{N}$)

Example

$$\begin{aligned}\text{block}(X) := \quad & \forall x, y.(x \in X \wedge y \in X \wedge x < y) \rightarrow \\ & (\forall Z.(x \in Z \wedge \text{Z closed under successor in } X) \rightarrow y \in Z)\end{aligned}$$

$$\varphi := \quad \exists X.\ \text{block}(X) \wedge a(X) \wedge X \text{ surrounded by } b\text{'s} \wedge |X| \leq N$$

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Semantics

$[\![\varphi]\!]$: structures $\rightarrow \mathbb{N} \cup \{\infty\}$
$[\![\varphi]\!](u) := \min\{n : u \text{ satisifes } \varphi \text{ when } N \text{ takes value } n\}$

By convention: $\min \emptyset = \infty$

Boundedness relation

"$[\![\varphi]\!] = n$" : decidable
"$[\![\varphi]\!] = [\![\psi]\!]$": undecidable [Krob 1994]

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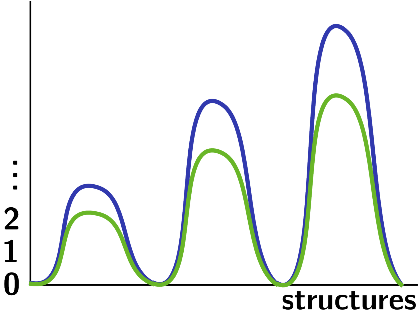

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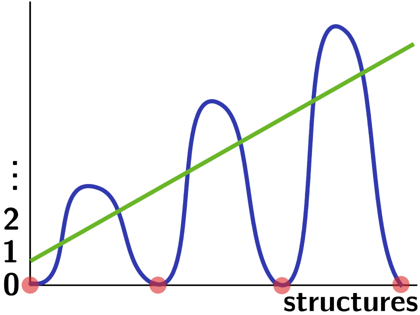

$[\![\varphi]\!] \not\approx [\![\psi]\!]$

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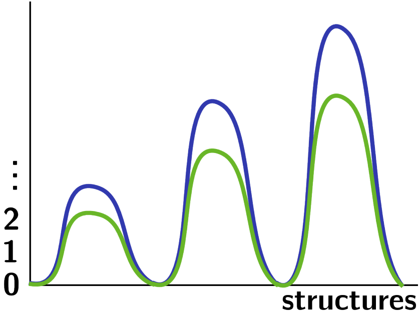

A **cost function** is an equivalence class of $\approx$.

B-automata

Nondeterministic finite-state automaton $\mathcal{A}$

$+$ finite set of counters

(initialized to 0, values range over $\mathbb{N}$)

$+$ counter operations on transitions

(increment $\mathtt{I}$, reset $\mathtt{R}$, no change ε)

Semantics

$$\mathit{val}(\rho) := \text{max value achieved by any counter during run } \rho$$

$$[\![\mathcal{A}]\!](u) := \min\{\mathit{val}(\rho) : \rho \text{ is an accepting run of } \mathcal{A} \text{ on } u\}$$

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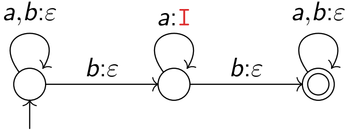
Example

$[\![\mathcal{A}]\!](u) =$ min length of block of a's surrounded by b's in u



Theory of regular cost functions over finite words

Cost MSO

$\min, \max, \pi_{\inf}, \pi_{\sup}$

B/S
Automata

Regular Cost Functions

B/S
Expressions

$f \approx g$ decidable
over finite words
[Colcombet '09]

Stabilization
Monoids

Theory of regular cost functions over finite trees

Cost MSO

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B/S Tree Automata

Regular Cost Functions

B/S Expressions

$f \approx g$ decidable over finite trees

[Colcombet+Löding '10]

Cost Games

Applications

Many problems for a regular language L can be reduced to deciding $\approx$:

- **Finite power property**
  [Simon '78, Hashiguchi '79]:

  is there some n such that $(L+\epsilon)^n = L^*$?

- **Star-height problem**
  [Hashiguchi '88, Kirsten '05, Colcombet+Löding '08]:

  given n, is there a regular expression for L with at most n nestings of Kleene star operations?

- **Parity-index problem**
  [reduction in Colcombet+Löding '08, decidability open]:

  given $i < j$, is there a parity automaton for L which uses only priorities $\{i, i+1, \ldots, j\}$?

Weak cost monadic logic over infinite trees

Cost WMSO: interpret second-order quantification over finite sets

$$[\![\varphi]\!](t) = \begin{cases} \text{max } a\text{'s on single branch} & \text{if } t \text{ has infinitely many } b\text{'s} \\ \infty & \text{otherwise} \end{cases}$$

for trees t over finite alphabet $\Sigma = \{a, b, c\}$

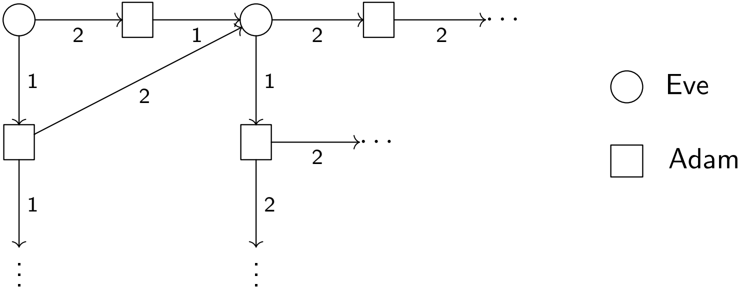
$$\begin{aligned} \varphi \quad := \quad & \forall X. \exists x. \big(\neg(x \in X) \wedge b(x)\big) \quad \wedge \\ & \forall Z. \big((\forall z. (z \in Z \rightarrow a(z)) \wedge \mathit{chain}(Z)) \rightarrow |Z| \leq N\big) \end{aligned}$$

where $\mathit{chain}(Z)$ asserts Z is totally ordered

Weak automata and games

Alternating parity automaton with priorities $\{1, 2\}$
$+$ no cycle in transition function which visits both priorities

Game $(\mathcal{A}, t)$



Semantics

A strategy σ for Eve is winning if every play in σ stabilizes in priority 2
$\mathcal{A}$ accepts t if Eve has a winning strategy from the initial position

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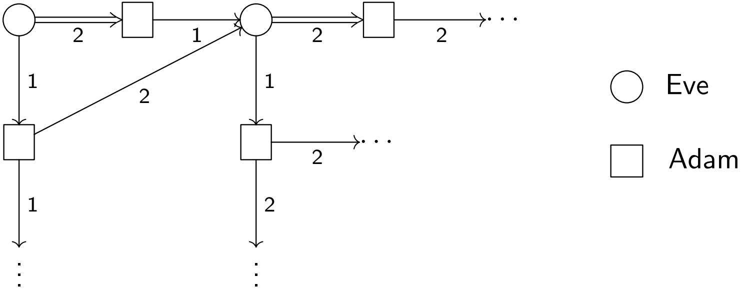

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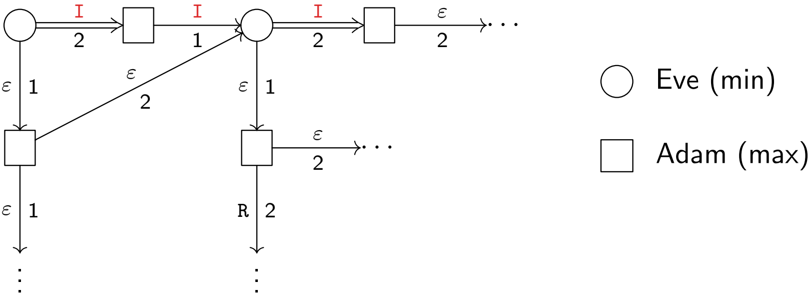

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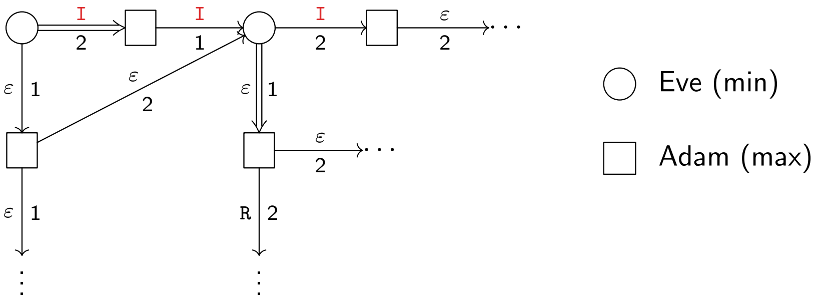

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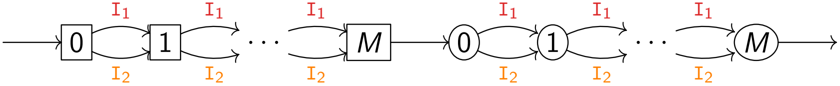
Strategies in cost games

Do **finite-memory strategies** guarantee the same value? no



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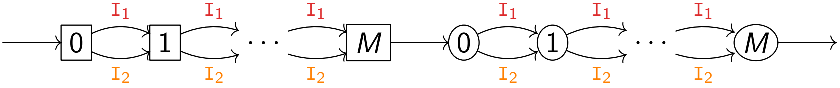

Do **finite-memory strategies** guarantee an $\approx$-equivalent value?

- finite-duration B-games: yes [Colcombet + Löding '10]
- weak B-games and B-Büchi games: yes [VB]

Strategies in cost games

$\mathcal{G}$

B-Büchi game

k counters

visit priority 2
infinitely-often and
minimize cost

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winning strategy with
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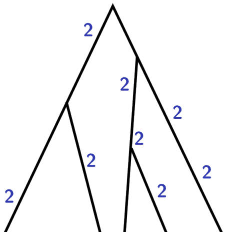

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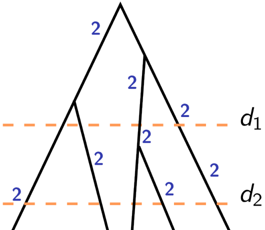


Strategies in cost games

$\mathcal{G}$ $\rightarrow$ $\mathcal{G}_{\mathcal{T}}$

B-Büchi game → *B*-safety game

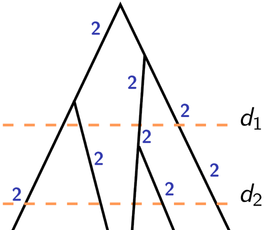
k counters

visit priority 2 infinitely-often and minimize cost

visit priority 2 between d_i and d_{i+1} and minimize cost

$\mathcal{T}$

winning strategy with $val(\tau) \leq n$



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B-Büchi game
k counters

visit priority 2
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winning strategy with
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B-safety game

visit priority 2
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σ

finite-memory winning
strategy with
$val(\sigma) \leq (n+1)^k$

Strategies in cost games

$$\mathcal{G} \quad \rightarrow \quad \mathcal{G}_{\mathcal{T}}$$

B-Büchi game
k counters

B-safety game

visit priority 2 infinitely-often and minimize cost

visit priority 2 between d_i and d_{i+1} and minimize cost

$$\sigma \quad \leftarrow \quad \sigma$$

finite-memory winning strategy with $val(\sigma) \leq (n+1)^k$

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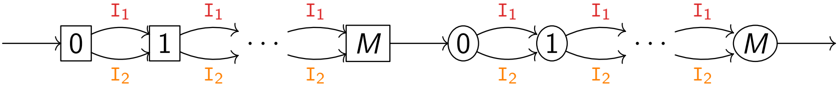

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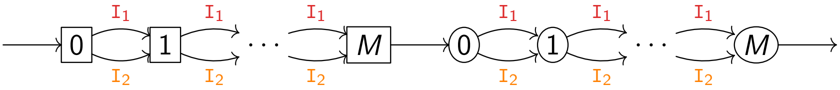

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$$\Downarrow$$

A weak B-automaton $\mathcal{A}$ can be **simulated** by a nondeterministic B-Büchi automaton $\mathcal{B}$ such that $[\![\mathcal{A}]\!] \approx [\![\mathcal{B}]\!]$.

- guess a labelling of t with a finite-memory strategy σ in $(\mathcal{A}, t)$
- run a B-Büchi automaton on each branch which computes the max value of plays from σ which stay on that branch

Theory of weak cost functions [VB]

Cost WMSO

min, max, w.$\pi_{\inf}/\pi_{\sup}$

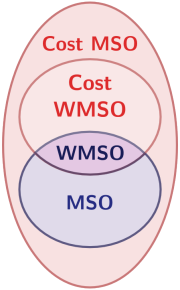
Weak Cost Functions

Weak B/S Automata

$\Downarrow$

Nondeterministic B/S-Büchi Automata

$f \approx g$ decidable over infinite trees



Weak Cost Games

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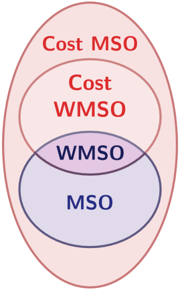

Weak Cost Games

Open Questions

- Is full cost MSO decidable over infinite trees?
- Do finite-memory strategies suffice in B-parity games?

Logic to automata

$$[\![\varphi(X)]\!](t, V) = \inf\{n : t \models \varphi[n/N, V/X]\}$$

$$\begin{aligned}
[\![|X| \leq N]\!](t, V) &= |V| \\
[\![\varphi \vee \psi]\!](t) &= \min([\![\varphi]\!](t), [\![\psi]\!](t)) \\
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Lemmas

Weak B-automata are closed under min, max, weak inf-projection.

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Weak B-automata and weak S-automata are effectively equivalent (modulo $\approx$) over infinite trees.