A step up in expressiveness of decidable fixpoint logics

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Oxford OASIS Seminar
April 2016

Based on joint work with
Michael Benedikt and Pierre Bourhis

Fixpoint logics

Fixpoint logics can express dynamic, recursive properties.

Example

binary relation R, unary relation P

"from w, it is possible to R-reach some P-element"

$$[\texttt{Reach-}P](w)$$

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$$[\mathbf{lfp}_{Y,y}\,.\,\exists z(Ryz \wedge (Pz \vee Yz))](w)$$

LFP

LFP: extension of first-order logic with fixpoint formulas $[\mathbf{lfp}_{Y,\boldsymbol{y}}.\psi(\boldsymbol{y},Y)](\boldsymbol{w})$ for $\psi(\boldsymbol{y},Y)$ positive in Y (of arity $m = |\boldsymbol{y}|$).

For all structures $\mathfrak{A}$, the formula ψ induces a monotone operation

$$\begin{aligned} \mathcal{P}(A^m) &\longrightarrow \mathcal{P}(A^m) \\ V &\longmapsto \psi_{\mathfrak{A}}(V) := \{\boldsymbol{a} \in A^m : \mathfrak{A}, \boldsymbol{a}, V \models \psi\} \end{aligned}$$

$\Rightarrow$ there is a unique least fixpoint $[\mathbf{lfp}_{Y,\boldsymbol{y}}.\psi(\boldsymbol{y},Y)]_{\mathfrak{A}} := \bigcup_\alpha \psi^\alpha_{\mathfrak{A}}$

$$\begin{aligned} \psi^0_{\mathfrak{A}} &:= \varnothing \\ \psi^{\alpha+1}_{\mathfrak{A}} &:= \psi_{\mathfrak{A}}(\psi^\alpha_{\mathfrak{A}}) \\ \psi^\lambda_{\mathfrak{A}} &:= \bigcup_{\alpha<\lambda} \psi^\alpha_{\mathfrak{A}} \end{aligned}$$

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Semantics of fixpoint operator: $\mathfrak{A}, \boldsymbol{a} \models [\mathbf{lfp}_{Y,\boldsymbol{y}}.\psi(\boldsymbol{y}, Y)](\boldsymbol{w})$ iff $\boldsymbol{a} \in \bigcup_\alpha \psi_{\mathfrak{A}}^\alpha$

Examples

"from w, it is possible to R-reach some P-element"

$$[\mathbf{lfp}_{Y,y}\,.\,\exists z(Ryz \wedge (Pz \vee Yz))](w)$$

$$a_1 \longrightarrow a_2 \longrightarrow a_3 \cdots\cdots\to a_k \longrightarrow a_{k+1}$$

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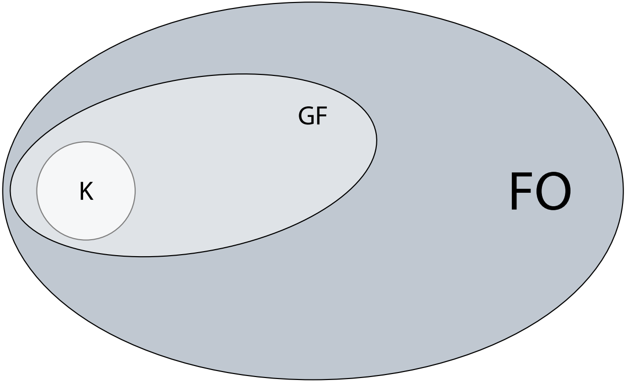
"from w, it is possible to R-reach x", i.e. "(w, x) is in the transitive closure of R"

$$[\mathbf{lfp}_{Y,y} \, . \, \exists z(Ryz \wedge (z = x \vee Yz))](w)$$

(Free first-order variable x in the fixpoint predicate is called a parameter.)

Some decidable fragments of first-order logic



constrain
quantification

$\exists \boldsymbol{x}(G(\boldsymbol{xy}) \wedge \psi(\boldsymbol{xy}))$
$\forall \boldsymbol{x}(G(\boldsymbol{xy}) \rightarrow \psi(\boldsymbol{xy}))$

[Andréka, van Benthem, Németi '95-'98]

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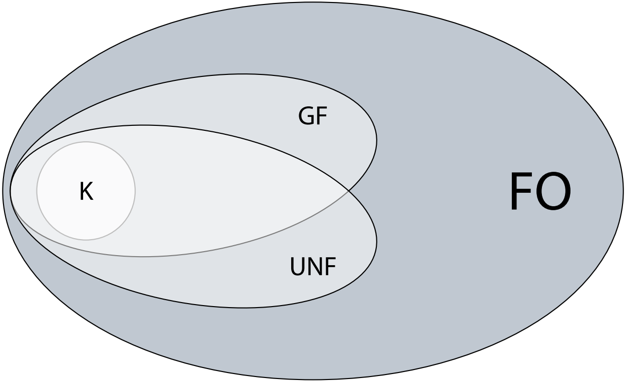

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$\exists \boldsymbol{x}(\psi(\boldsymbol{xy}))$
$\neg\psi(x)$

[ten Cate, Segoufin '11]

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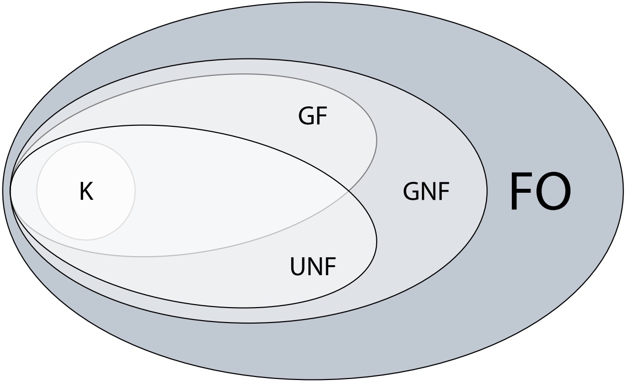

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Guarded negation fragment of first-order logic

Let σ be a signature of relations and constants.

Syntax of GNF$[\sigma]$

$$\varphi \ ::= \ R\,\boldsymbol{t} \ \mid \ Y\,\boldsymbol{t} \ \mid \ \varphi \wedge \varphi \ \mid \ \varphi \vee \varphi \ \mid \ \exists y(\psi(\boldsymbol{x}y)) \ \mid \ G(\boldsymbol{x}) \wedge \neg\psi(\boldsymbol{x})$$

where R and G are relations in σ or $=$, and $\boldsymbol{t}$ is a tuple over variables and constants.

"There is an R-cycle of length 3"

$$\exists xyz(Rxy \wedge Ryz \wedge Rzx)$$

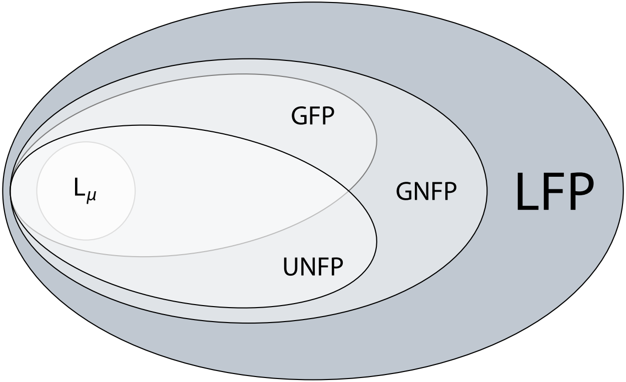
"R is symmetric"

$$\forall xy(Rxy \rightarrow Ryx) \ \equiv \ \neg\exists xy(Rxy \wedge \neg Ryx)$$

"Every element has an R-successor "

$$\forall x(\exists y(Rxy)) \ \equiv \ \neg\exists x(\neg\exists y(Rxy)) \ \equiv \ \neg\exists x(x = x \wedge \neg\exists y(Rxy))$$

Some decidable fragments of LFP (fixpoint extension of FO)



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Restrictions on fixpoint operator:

- must define a guarded relation
  (tuples in the fixpoint must be guarded by an atom from σ)
- cannot use parameters

Examples

In GNFP:

$$[\mathbf{lfp}_{Z,xy}\,.\,Sxy \wedge \exists uv(Rxu \wedge Ryv \wedge (Zuv \vee (Pu \wedge Pv)))](xy)$$

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a_1 \arrow[r] \arrow[d] & a_2 \arrow[r] \arrow[d] & a_3 \arrow[r, dotted] \arrow[d] & a_k \arrow[r] \arrow[d] & a_{k+1} \\
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Not in GNFP:

$$[\mathbf{lfp}_{Y,y} \,.\, \exists z(Ryz \wedge (z = x \vee Yz))](w)$$

Some nice computational properties for guarded fixpoint logics

Decidable satisfiability and finite satisfiability
(2EXPTIME in general, EXPTIME for fixed-width formulas in GFP)

[Grädel, Walukiewicz ’99 ; Bárány, Segoufin, ten Cate ’11; Bárány, Bojańczyk ’12]

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Decidable boundedness
(given $\psi(\boldsymbol{y}, Y)$ positive in Y, is there $n \in \mathbb{N}$ such that for all $\mathfrak{A}$, $\psi^n_{\mathfrak{A}} = \psi^{n+1}_{\mathfrak{A}}$?)

[Blumensath, Otto, Weyer '14 ; Bárány, ten Cate, Otto '12 ; Benedikt, ten Cate, Colcombet, VB. '15]

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Constructive interpolation for UNFP
[Benedikt, ten Cate, VB. '15]

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These guarded logics all have tree-like models (bounded tree width).

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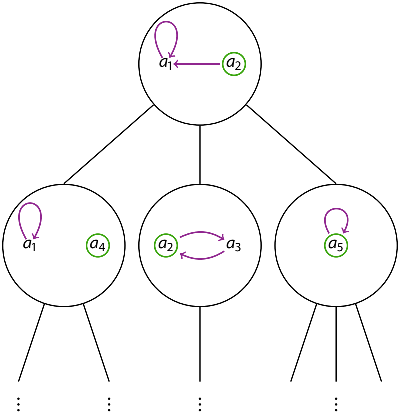
A structure $\mathfrak{A}$ has tree width $k-1$ if it can be covered by (overlapping) bags of size at most k, arranged in a tree t s.t.

- every fact appears in some bag in t;
- for each element, the set of bags with this element is connected in t.



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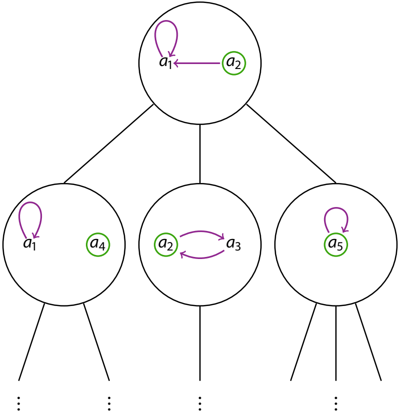

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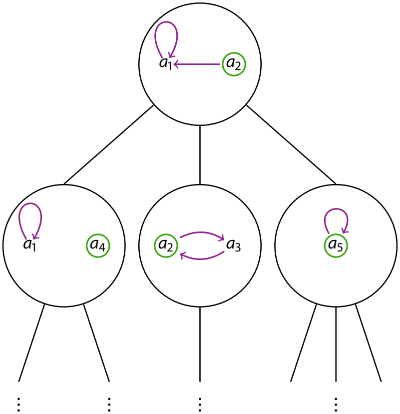

There is a natural encoding of these tree-like models (of some bounded tree width) as trees over a finite alphabet.

$\Rightarrow$ We can reason about tree encodings rather than relational structures.

Automata for satisfiability

We can use tree automata to analyze these tree encodings.

$$\underset{\text{GNFP}}{\varphi} \quad \rightarrow \quad \underset{\substack{\text{2-way alternating} \\ \text{parity tree automaton}}}{\mathcal{A}_{\varphi}}$$

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GNFP

2-way alternating parity tree automaton

φ satisfiable? $\leftarrow$ $L(\mathcal{A}_\varphi)$ empty?

[Vardi '98]

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φ GNFP	$\rightarrow$	$\mathcal{A}_\varphi$ 2-way alternating parity tree automaton
φ satisfiable?	$\leftarrow$	$L(\mathcal{A}_\varphi)$ empty? [Vardi '98]

Theorem (Bárány, ten Cate, Segoufin '11)

Satisfiability is decidable for $\varphi \in$ GNFP in 2EXPTIME.

Game for least fixpoint

Game for testing if
$\mathfrak{A}, \boldsymbol{a} \models [\mathbf{lfp}_{Y,\boldsymbol{y}}.G(\boldsymbol{y}) \land \psi(\boldsymbol{y}, Y)](\boldsymbol{x})$.

Initial position $\boldsymbol{y} := \boldsymbol{a}$.

Game from position $\boldsymbol{y}$:

- Eve chooses Y such that
  $\mathfrak{A} \models G(\boldsymbol{y}) \land \psi(\boldsymbol{y}, Y)$
  (if it is not possible, she loses).
- Adam chooses some new $\boldsymbol{y}' \in Y$
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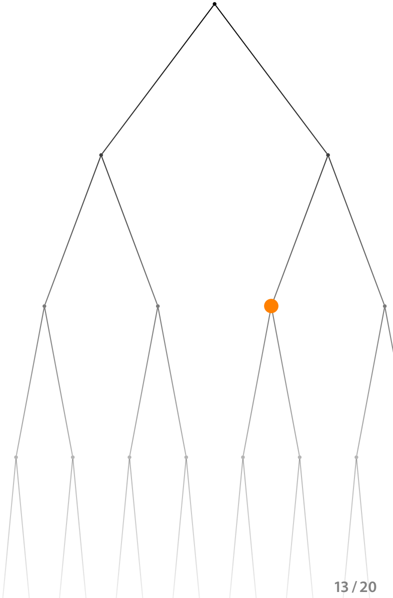
Adam wins if the game continues forever.



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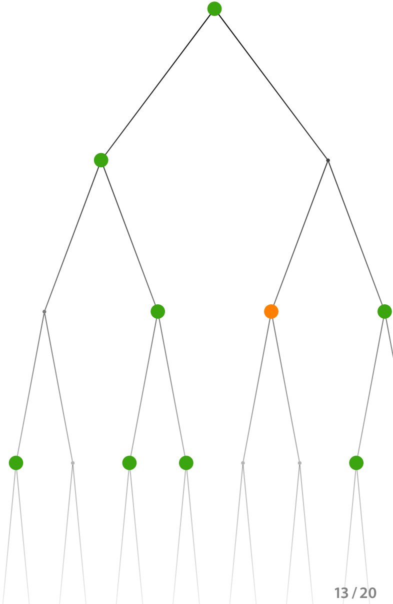

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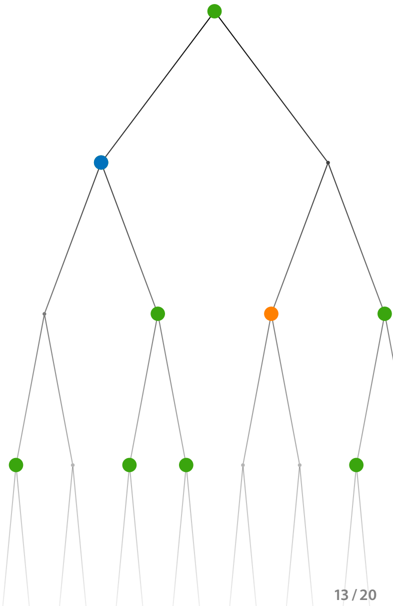

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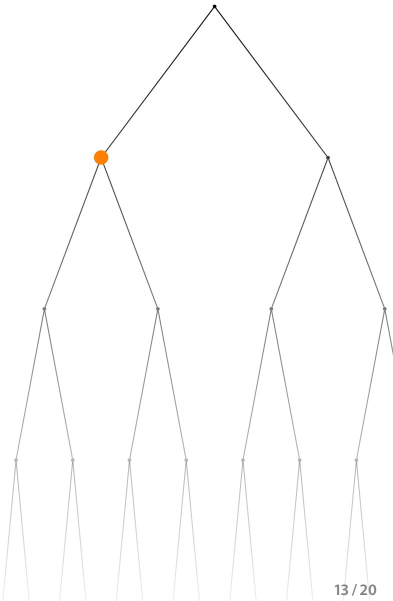

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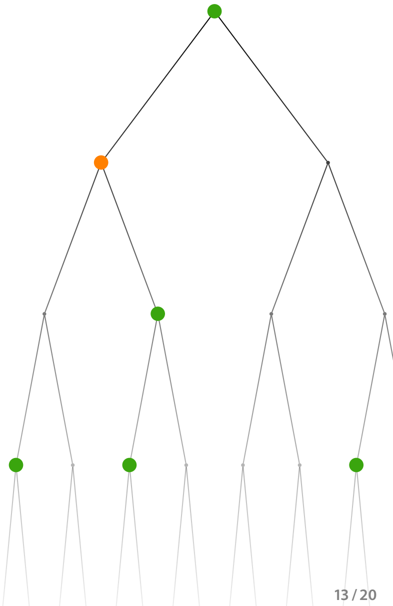

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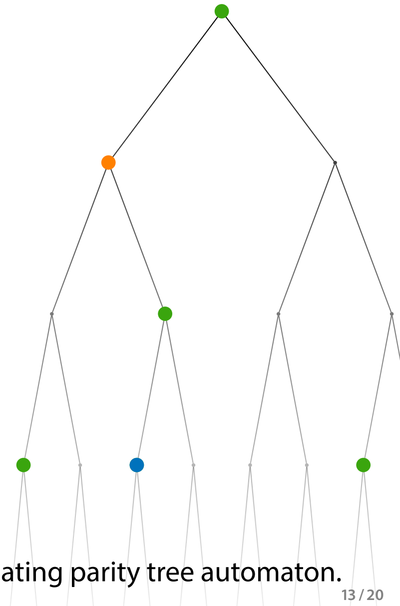
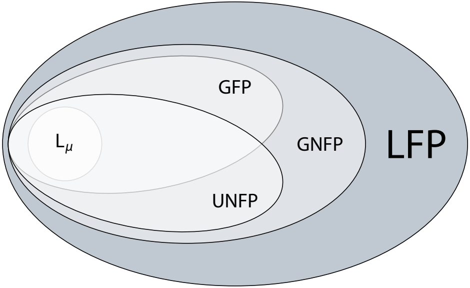


Game can be implemented by a 2-way alternating parity tree automaton.

Can we go further?



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- must define a guarded relation
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Which of these restrictions are essential for decidability?

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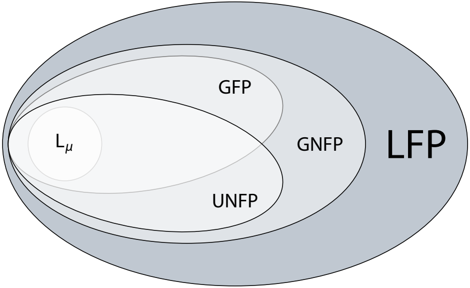

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Answer: only first one!

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Syntax of GNFP$^{\mathrm{UP}}[\sigma]$

$$\varphi \ ::= \ R\boldsymbol{t} \ | \ Y\boldsymbol{t} \ | \ \varphi \wedge \varphi \ | \ \varphi \vee \varphi \ | \ \exists y(\psi(\boldsymbol{x}y)) \ | \ G(\boldsymbol{x}) \wedge \neg\psi(\boldsymbol{x}) \ |$$
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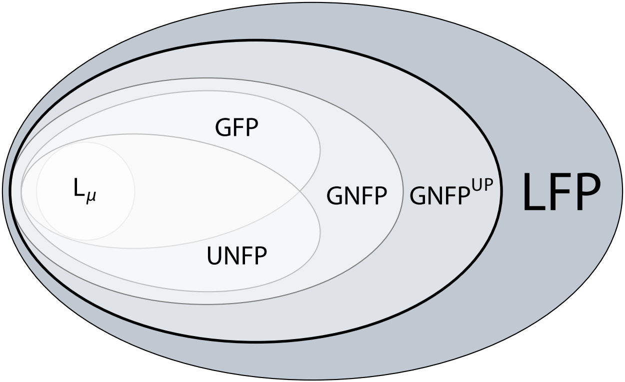
Example

$\mathrm{GNFP}^{\mathrm{UP}}$ can express the transitive closure of a binary relation R using

$$[\mathbf{lfp}_{Y,y} \,.\, \exists z(Ryz \wedge (z = x \vee Yz))](w)$$

Expressivity of GNFP^{UP}



GNFP^{UP} also subsumes

C2RPQs (conjunctive 2-way regular path queries)

$$\exists xyz\,\big(\,[R^{*}S](x,y) \wedge [S \mid R](y,z) \wedge P(z)\,\big)$$

MQs and GQs [Rudolph, Krötsch '13 ; Bourhis, Krötsch, Rudolph '15]

Satisfiability for GNFP$^{\mathrm{UP}}$

GNFP$^{\mathrm{UP}}$ still has tree-like models
$\Rightarrow$ amenable to tree automata techniques

Unlike other guarded logics, satisfiability testing for $\varphi \in$ GNFP$^{\mathrm{UP}}$ is non-elementary, with running time

$$2^{2^{\cdot^{\cdot^{\cdot^{2^{f(|\varphi|)}}}}}}$$

where the polynomial f and the height of the tower depend only on the parameter depth: the number of nested parameter changes in the formula.

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Theorem (Benedikt, Bourhis, VB. '16)

Satisfiability is decidable for $\varphi \in \mathrm{GNFP}^{\mathrm{UP}}$ in $(n+2)$-EXPTIME, where n is the parameter depth of φ.

Skirting undecidability

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$\mathrm{GNFP}^{\mathrm{UP}}$ can express the transitive closure of a binary relation R using

$$[\mathbf{lfp}_{Y,y} \,.\, \exists z(Ryz \wedge (z = x \vee Yz))](w).$$

But it cannot enforce that R is transitive.

Boundedness for GNFP$^{\mathrm{UP}}$

Theorem (Benedikt, Bourhis, VB. '16)

The following boundedness problem is decidable:

Instance: $G(\boldsymbol{y}) \wedge \psi(\boldsymbol{x}, \boldsymbol{y}, Y) \in$ GNF, positive in Y

Question: is there $n \in \mathbb{N}$ s.t. for all structures $\mathfrak{A}$, the least fixpoint $[\mathbf{lfp}_{Y,\boldsymbol{y}}\,.\, G(\boldsymbol{y}) \wedge \psi(\boldsymbol{x}, \boldsymbol{y}, Y)]_{\mathfrak{A}}$ is always reached within n iterations?

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$\Rightarrow$ For $\psi(\boldsymbol{x}, \boldsymbol{y}, Y) \in$ GNF positive in Y, it is decidable whether $[\mathbf{lfp}_{Y,\boldsymbol{y}} \,.\, G(\boldsymbol{y}) \wedge \psi(\boldsymbol{x}, \boldsymbol{y}, Y)](\boldsymbol{w})$ can be expressed in FO.

Conclusion

We can allow unguarded parameters
in guarded fixpoint logics while still retaining
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Open questions

Is finite satisfiability decidable for $\mathrm{GNFP}^{\mathrm{UP}}$?

Does $\mathrm{GNFP}^{\mathrm{UP}}$ have interpolation?