

Coalitional Games

- **Coalitional games** model scenarios where agents can *benefit by cooperating*.
- Issues in coalitional games (Sandholm et al, 1999):
 - *Coalition structure generation*.
 - *Teamwork*.
 - *Dividing the benefits of cooperation*.

<http://www.csc.liv.ac.uk/~mjw/pubs/imas/>

1

Coalition Structure Generation

- Deciding *in principle* who will work together.
- The basic question:
 - Which coalition should I join?
- The result: *partitions* agents into disjoint *coalitions*.
The overall partition is a *coalition structure*.

<http://www.csc.liv.ac.uk/~mjw/pubs/imas/>

2

Solving the optimization problem of each coalition

- Deciding *how* to work together.
- Solving the “joint problem” of a coalition
- Finding how to maximise the utility of the coalition itself.
- Typically involves joint planning etc.

<http://www.csc.liv.ac.uk/~mjw/pubs/imas/>

3

Formalising Cooperative Scenarios

Dividing the Benefits

- Deciding “who gets what” in the payoff.

- Coalition members cannot ignore each other’s preferences, because members can **defect**, if you try to give me a bad payoff, I can always walk away.
- We might want to consider issues such as **fairness** of the distribution.

<http://www.csc.liv.ac.uk/~mjt/pubs/jimas/>

4

A coalitional game:

$$\langle A_g, \nu \rangle$$

where:

- $A_g = \{1, \dots, n\}$ is a set of **agents**;
- $\nu : 2^{A_g} \rightarrow \mathbb{R}$ is the **characteristic function** of the game.

Usual interpretation: if $\nu(C) = k$, then coalition C can cooperate in such a way they will obtain utility k , which may then be distributed amongst team members.

<http://www.csc.liv.ac.uk/~mjt/pubs/jimas/>

5

Which Coalition Should I Join?

- Most important question in coalitional games:

is a coalition stable?

that is,

is it rational for all members of coalition to stay with the coalition, or could they benefit by defecting from it?

- (There is no point in me trying to join a coalition with you unless you want to form one with me, and vice versa.)
- Stability is a **necessary** but not **sufficient** condition for coalitions to form.

<http://www.csc.liv.ac.uk/~mjt/pubs/jimas/>

6

The Core

- The **core** of a coalitional game is the set of **feasible** distributions of payoff to members of a coalition that

no sub-coalition can reasonably object to.

- An **outcome** for a coalition C in game $\langle A_g, \nu \rangle$ is a vector of payoffs to members of C , $\langle x_1, \dots, x_k \rangle$ which represents a **feasible distribution of payoff to members of A_g** .

“Feasible” means:

$$\nu(C) \geq \sum_{i \in C} x_i$$

<http://www.csc.liv.ac.uk/~mjt/pubs/jimas/>

7

- **Example:** if $\nu(\{1, 2\}) = 20$, then possible outcomes are $\langle 20, 0 \rangle, \langle 19, 1 \rangle, \langle 18, 2 \rangle, \dots, \langle 0, 20 \rangle$.
(Actually there will be infinitely many!)

Objections

- Intuitively, a coalition C **objects** to an outcome if there is some outcome **for them** that makes **all of them** strictly better off.
- Formally, $C \subseteq Ag$ objects to an outcome $\langle x_1, \dots, x_n \rangle$ for the grand coalition if there is some outcome $\langle x'_1, \dots, x'_k \rangle$ for C such that

$$x'_i > x_i \quad \text{for all } i \in C$$

- The idea is that an outcome is not going to happen if somebody objects to it!

The Core

- The **core** is the set of outcomes for the *grand coalition* to which **no** coalition objects.

- If the core is **non-empty** then the **grand coalition** is **stable**, since nobody can benefit from defection.
- Thus, asking

is the grand coalition stable?

is the same as asking:

- *Why unfair? Because the agents are identical!*

Problems with the Core

- Sometimes, the core is empty; what happens then?
- Sometimes it is non-empty but isn't "fair".

Suppose $Ag = \{1, 2\}$, $\nu(\{1\}) = 5$, $\nu(\{2\}) = 5$,
 $\nu(\{1, 2\}) = 20$.

Then outcome $\langle 20, 0 \rangle$ (i.e., agent 1 gets everything) is **not** in the core, since the coalition $\{2\}$ can object. (He can work on his own and do better.) However, outcome $\langle 15, 5 \rangle$ **is** in the core: even though this seems unfair to agent 2, this agent has no objection.

- Why unfair? Because the agents are **identical!**

How To Share Benefits of Cooperation?

- The **Shapley value** is best known attempt to define how to divide benefits of cooperation fairly. It does this by taking into account *how much an agent contributes*.
- The Shapley value of agent i is the average amount that i is expected to contribute to a coalition.
- Axiomatically: a value which satisfies axioms: *symmetry*, *dummy player*, and *additivity*.

<http://www.csc.liv.ac.uk/~mjt/pubs/jmas/>

12

Shapley Defined

- Let $\delta_i(S)$ be the amount that i adds by joining $S \subseteq Ag$:
$$\delta_i(S) = \nu(S \cup \{i\}) - \nu(S)$$

$$\dots \text{the } \textit{marginal contribution of } i \text{ to } S.$$
- Then the Shapley value for i , denoted φ_i , is:

$$\varphi_i = \frac{\sum_{r \in R} \delta_i(S_i(r))}{|Ag|!}$$

where R is the set of all orderings of Ag and $S_i(r)$ is the set of agents preceding i in ordering r .

<http://www.csc.liv.ac.uk/~mjt/pubs/jmas/>

13

Chapter 13
An Introduction to Multiagent Systems 2e

- so large that it renders comparisons to this input size meaningless: stating that we have an algorithm that runs in (say) time *linear* in the size of such a representation means it runs in time *exponential* in the size of Ag !

Representing Coalitional Games

- It is important for an agent to know (eg) whether the core of a coalition is non-empty ... so, how hard is it to decide this?
- Problem: naive, obvious representation of coalitional game is **exponential** in the size of Ag !
- Now such a representation is:
 - *utterly* infeasible in practice; and

<http://www.csc.liv.ac.uk/~mjt/pubs/jmas/>

14

How to Represent Characteristic Functions?

Two approaches to this problem:

- try to find a **complete** representation that is succinct in “most” cases
- try to find a representation that is not complete but is always succinct
- A common approach: interpret characteristic function over combinatorial structure.

<http://www.csc.liv.ac.uk/~mjt/pubs/jmas/>

16

Representation 1: Induced Subgraph

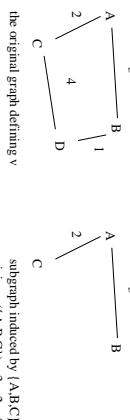
- Represent ν as an undirected graph on Ag , with integer weights $w_{i,j}$ between nodes $i,j \in Ag$.
- Value of coalition C then:

$$\nu(C) = \sum_{\{i,j\} \subseteq Ag} w_{i,j}$$

i.e., the value of a coalition $C \subseteq Ag$ is the weight of the subgraph induced by C .

<http://www.csc.liv.ac.uk/~mjt/pubs/jmas/>

17



the original graph defining ν
subgraph induced by $\{A, B, C\}$
giving $\nu(\{A, B, C\}) = 3 + 2 = 5$

Representation 1: Induced Subgraph

(Deng & Papadimitriou, 94)

- Computing Shapley: in polynomial time.
- Determining emptiness of the core: NP-complete
- Checking whether a specific distribution is in the core co-NP-complete

But this representation is not **complete**.

<http://www.csc.liv.ac.uk/~mjt/pubs/jmas/>

18

Representation 2: Weighted Voting Games

- For each agent $i \in Ag$, assign a weight w_i , and define an overall **quota**, q .

$$\nu(C) = \begin{cases} 1 & \text{if } \sum_{i \in C} w_i \geq q \\ 0 & \text{otherwise.} \end{cases}$$

- Shapley value: #P-complete, and “hard to approximate” (Deng & Papadimitriou, 94).
- Core non-emptiness: in polynomial time.
- Not a complete representation.

<http://www.csc.liv.ac.uk/~mjw/pubs/jamas/>

20

Representation 3: Marginal Contribution Nets

(leong & Shoham, 2005)

- Characteristic function represented as rules: pattern \longrightarrow value.

<http://www.csc.liv.ac.uk/~mjw/pubs/jamas/>

21

- Pattern is conjunction of agents, a rule **applies** to a group of agents C if C is a superset of the agents in the pattern.

Value of a coalition is then sum over the values of all the rules that apply to the coalition.

Example:

$$\begin{array}{l} a \wedge b \longrightarrow 5 \\ b \longrightarrow 2 \end{array}$$

We have: $\nu(\{a\}) = 0$, $\nu(\{b\}) = 2$, and $\nu(\{a, b\}) = 7$.

- We can also allow negations in rules (agent not present).

A complete representation, but not necessarily succinct.

Qualitative Coalitional Games

- Often not interested in utilities, but in *goals* – either the goal is satisfied or not
- QCGs** are a type of coalitional game in which each agent has a set of goals, and wants one of them to be achieved (doesn't care which)
 - Agents cooperate in QCGs to achieve mutually satisfying sets of goals.
 - Coalitions have *sets of choices* representing the different ways they could cooperate
 - Each choice is a set of goals.

<http://www.csc.liv.ac.uk/~mjt/pubs/jamas/>

24

QCGs

A **Qualitative Coalitional Game** (QCG) is a structure:

$$\Gamma = \langle G, Ag, G_1, \dots, G_n, V \rangle$$

where

- $G = \{g_1, \dots, g_m\}$ is a set of *possible goals*;
- $Ag = \{1, \dots, n\}$ is a set of *agents*;
- $G_i \subseteq G$ is a set of goals for each agent $i \in Ag$, the intended interpretation being that any of G_i would satisfy i ;

<http://www.csc.liv.ac.uk/~mjt/pubs/jamas/>

25

- $V : 2^{Ag} \rightarrow 2^{2^G}$ is a *characteristic function*, which for every coalition $C \subseteq Ag$ determines a set $V(C)$ of *choices*, the intended interpretation being that if $G' \in V(C)$, then one of the choices available to coalition C is to bring about *all* the goals in G' simultaneously.

Feasible/Satisfying Goal Sets

- Goal set $G' \subseteq G$ *satisfies* an agent i iff $G_i \cap G' \neq \emptyset$.
- Goal set $G' \subseteq G$ satisfies a coalition $C \subseteq Ag$ iff $\forall i \in C, G_i \cap G' \neq \emptyset$
- A goal set G' is *feasible* for C if $G' \in V(C)$.

Representing QCGs

- So, how do we represent the function $V : 2^{Ag} \rightarrow 2^{2^G}$?
- We use a formula Ψ_V of propositional logic over propositional variables Ag, G , such that:

$$\Psi[C, G'] = \top \text{ if and only if } G' \in V(C)$$

- “Often” permits **succinct** representations of V .
- Note that given Ψ_V, C, G' , determining whether $G' \in V(C)$ can be done in time polynomial in size of C, G', Ψ_V .

<http://www.csc.liv.ac.uk/~mjt/pubs/jimas/>

28

Fourteen QCG Decision Problems (AIJ, Sep 2004)

Problem	Description	Complexity	q^{mono}
SC	SUCCESSFUL COALITION	NP-complete	NP-complete
SSC	SELFISH SUCCESSFUL COALITION	NP-complete	NP-complete
UGS	UNATTAINABLE GOAL SET	NP-complete	NP-complete
MC	MINIMAL COALITION	co-NP-complete	co-NP-complete
CM	CORE MEMBERSHIP	co-NP-complete	co-NP-complete
CNE	CORE NON-EMPTYNESS	D^P -complete	D^P -complete
VP	VETO PLAYER	co-NP-complete	-
MD	MUTUAL DEPENDENCE	co-NP-complete	-
GR	GOAL REALISABILITY	NP-complete	P
NG	NECESSARY GOAL	co-NP-complete	-
EG	EMPTY GAME	co-NP-complete	co-NP-complete
TG	TRIVIAL GAME	Π_2^P -complete	Π_2^P -complete
GU	GLOBAL UNATTAINABILITY	Σ_2^P -complete	NP
IG	INCOMPLETE GAME	D_2^P -complete	-

<http://www.csc.liv.ac.uk/~mjt/pubs/jimas/>

29

Coalitional Resource Games (CRGs)

- Problem:

Where does characteristic function come from?

- One answer provided by **Coalitional Resource Games (CRGs)**.

- Key ideas:
 - achieving a goal requires **expenditure of resources**;
 - each agent **endowed** with a profile of resources;
 - coalitions form to pool resource so as to achieve mutually satisfactory set of goals.

CRGs

A **coalitional resource game** Γ is an $(n + 5)$ -tuple:

$$\Gamma = \langle Ag, G, R, G_1, \dots, G_n, \text{en}, \text{req} \rangle$$

where:

- $Ag = \{a_1, \dots, a_n\}$ is a set of **agents**;
- $G = \{g_1, \dots, g_m\}$ is a set of **possible goals**;
- $R = \{r_1, \dots, r_t\}$ is a set of **resources**;
- for each $i \in Ag$, $G_i \subseteq G$ is a set of goals, as in QCGs;
- $\text{en} : Ag \times R \rightarrow \mathbb{N}$ is an **endowment function**,
- $\text{req} : G \times R \rightarrow \mathbb{N}$ is a **requirement function**.

<http://www.csc.liv.ac.uk/~mjt/pubs/jimas/>

30

Nine Decision Problems for CRGs

Problem	Complexity
SUCCESSFUL COALITION	NP-complete
MAXIMAL COALITION	co-NP-complete
NECESSARY RESOURCE	co-NP-COMPLETE
STRICTLY NECESSARY RESOURCE	D ^P -complete
(C, G', r)-OPTIMAL	NP-complete
R-PARETO OPTIMALITY	co-NP-complete
SUCCESSFUL COALITION WITH RESOURCE BOUNDS	NP-complete
CONFLICTING COALITIONS	co-NP-complete
ACHIEVABLE GOAL SET	in P

<http://www.csc.liv.ac.uk/~mjt/pubs/jamas/>

32

QCG and CRG Equivalence

- We can define a notion of “equivalence” (\equiv) between QCGs and CRGs:
 - $\Gamma_1 \equiv \Gamma_2$ means that QCG Γ_1 and CRG Γ_2 agree on the goal sets that are feasible for coalitions
- Given a QCG Γ_1 and CRG Γ_2 , the problem of determining whether $\Gamma_1 \equiv \Gamma_2$ is co-NP-complete.

<http://www.csc.liv.ac.uk/~mjt/pubs/jamas/>

33

Can we translate between QCGs and CRGs?

Four questions suggest themselves:

1. Given a *crg*, Γ , is there always a QCG, Q_Γ such that $Q_\Gamma \equiv \Gamma$?
2. Given a *qcg*, Q , is there always a CRG, Γ_Q such that $\Gamma_Q \equiv Q$?
3. How “efficiently” can a given CRG be expressed as an equivalent QCG in those cases where such an equivalent structure exists?
4. How “efficiently” can a given QCG be expressed as an equivalent CRG in those cases where such an equivalent structure exists?

<http://www.csc.liv.ac.uk/~mjt/pubs/jamas/>

34

Translating CRGs \rightarrow QCGs

- We can always translate a CRG into an equivalent QCG.
- More interestingly, we can do this *efficiently*:
 - for every CRG Γ_1 there exists an equivalent QCG Γ_2 such that $|\Gamma_2| \leq |\Gamma_1|^2$.

<http://www.csc.liv.ac.uk/~mjt/pubs/jamas/>

35

Translating QCGs to CRGs

- We **cannot** always translate QCGs to equivalent CRGs.
- Moreover, even when we **can** translate, we can't always do it efficiently:
there exist QCGs Γ for which equivalent CRGs exist but for which the size of the **smallest** equivalent CRG is at least $2^{|\Gamma|}$