LECTURE 15: RELATIONS

Software Engineering

Mike Wooldridge

1 Introduction

- We saw in earlier lectures that a function is just a set of maplets, for example:

  $tel == \{mikew \mapsto 1531, eric \mapsto 1489\}.$

  A maplet is just an ordered pair, so another way of writing this is

  $tel == \{(mikew, 1531), (eric, 1489)\}$

  Formally, if $f : T_1 \rightarrow T_1$, then f is a subset of the cartesian product of $T_1 \times T_1$:

  $f \subseteq T_1 \times T_2.$

  But functions can't be *defined* in this way, as they must have the uniqueness property — the following is not a function:

  $\{(mike, 1531), (eric, 1531)\}.$
- A more general way of capturing a relationship between two sets is to use a *relation*.
- Relations are similar to functions, but do not have the uniquness property.

- Just as we have a variety of function construction arrows, so we have the relation constructor arrow, '$\leftrightarrow$'.
- **Definition:** If T_1 and T_2 are arbitrary types, then

  $T_1 \leftrightarrow T_2$

  is an expression giving the set of all relations between T_1 and T_2. It may be defined:

  $T_1 \leftrightarrow T_2 == \mathbb{P}(T_1 \times T_2).$
- EXAMPLE. Let

  $A == \{a, b, c\}$
  $B == \{1, 2\}$

  then

  $\{(a, 1), (a, 2)\} \in A \leftrightarrow B.$

2 Operations on Relations

- Everything you can do with a function, you can do with a relation:

$\mathrm{dom}\, R$	domain of R
$\mathrm{ran}\, R$	range of R
$S \lhd R$	R with domain restricted to S
$R \rhd S$	R with range restricted to S
$S ⩤ R$	R with S subtracted from domain of R
$R ⩥ S$	R with S subtracted from range of R

- EXAMPLES. Let

  $R == \{(a,1),(a,2),(b,1),(c,3)\}$

  then

$$
\begin{aligned}
\operatorname{dom} R &= \{a,b,c\} \\
\operatorname{ran} R &= \{1,2,3\} \\
\{a\} \lhd R &= \{(a,1),(a,2)\} \\
\{a,b\} \lhd R &= \{(a,1),(a,2),(b,1)\} \\
R \rhd \{1\} &= \{(a,1),(b,1)\} \\
R \rhd \{1,3\} &= \{(a,1),(b,1),(c,3)\} \\
\{a\} \mathbin{⩤} R &= \{(b,1),(c,3)\} \\
\{a,b\} \mathbin{⩤} R &= \{(c,3)\} \\
R \mathbin{⩥} \{1\} &= \{(a,2),(b,3)\} \\
R \mathbin{⩥} \{1,2,3\} &= \emptyset.
\end{aligned}
$$

2.1 The Inverse of a Relation

- Additionally, we can take the *inverse* of a relation.
- **Definition:** If

  $R : T_1 \leftrightarrow T_2$

  then

  $R^{\sim} : T_1 : T_2$

  such that $(y,x) \in R^{\sim}$ iff $(x,y) \in R$. Formally:

$$
\begin{aligned}
R^{\sim} \;==\; &\{x : T_1;\ y : T_1 \mid \\
&(x,y) \in R \bullet (y,x)\}
\end{aligned}
$$

- EXAMPLE. If

  $R == \{(a,1),(a,2),(b,1),(c,3)\}$

  then

  $R^{\sim} = \{(1,a),(1,2),(1,b),(3,c)\}.$

- Note that you can always take the inverse of a function, since functions are just special kinds of relation, but you do not always get a function as a result.

- EXAMPLE. Suppose we had

  $f == \{a \mapsto 1, b \mapsto 1\}$

  Now f is certainly a function (though it is not one-to-one.) But take the inverse of f:

  $f^{\sim} = \{(1,a),(1,b)\}.$

  Although $f^{\sim}$ is a relation, it is *not* a function.

2.2 Relational Image

- Can we do relation application ...?
- EXAMPLE. If

  $R == \{(a,1),(a,2),(b,1),(c,3)\}$

  then

  $R(a) = ?$

- No, we can't, since it is not defined what the value of R should be here — should it be 1 or 2 or both?
- However, we can do something a bit more general — we can get the *relation image* of a value, using the image brackets '⦇ ⦈'.

- **Definition:** If

  $S : \mathbb{P}\, T_1$
  $R : T_1 \leftrightarrow T_2$

  then

  $R(\!|S|\!)$

  is an expression of type $\mathbb{P}\, T_2$, such that

  $y \in R(\!|S|\!) \Leftrightarrow \exists x : T_1 \bullet x \in S \wedge (x, y) \in R$

  (i.e., y is in $R(\!|S|\!)$ iff there is some x in S such that R relates x to y).

- EXAMPLE. If

  $R == \{(a, 1), (a, 2), (b, 1), (c, 3)\}$

  then

$$
\begin{aligned}
R(\!|\{a\}|\!) &= \{1, 2\} \\
R(\!|\{a, b\}|\!) &= \{1, 2\} \\
R(\!|\{a, c\}|\!) &= \{1, 2, 3\} \\
R(\!|\{b\}|\!) &= \{1\}
\end{aligned}
$$

3 Properties of Relations

- An interesting class of relations are those that are defined on just one set.
- EXAMPLE. Consider the set of binary relations on $\mathbb{N}$:

  $\mathbb{N} \leftrightarrow \mathbb{N}$.

- **Definition:** If $R : T \leftrightarrow T$, then we say R is a *binary relation on* T.
- We shall now consider the properties of these types of relation.

3.1 Reflexivity

- Suppose we have a relation R on T such that for any element $x \in (\mathrm{dom}\, R \cup \mathrm{ran}\, R)$, we have $(x, x) \in R$. Then R is said to be *reflexive*.
- EXAMPLES.
  1. $\{(1, 1), (1, 2), (2, 2), (1, 3), (3, 3)\}$
     ... *is* a reflexive relation on $\mathbb{N}$.
  2. $\{(1, 1), (1, 2), (2, 2), (1, 3)\}$
     ... *is not* a reflexive relation on $\mathbb{N}$ (because $(3, 3)$ is not a member).
  3. The subset relation, $\subseteq$, is reflexive, because $S \subseteq S$, for any set S.
  4. The 'less that' relation, $<$, is not reflexive, because it is not true that $n < n$, for any value n.
  5. The 'is the father of' relation (defined on the set of people) is *not* reflexive, because it is not true that p is the father of p, for any person p.

3.2 Symmetry

- A relation R is said to be *symmetric* if whenever $(x, y) \in R$, we have $(y, x) \in R$.
- EXAMPLES.
  1. $\{(2, 1), (1, 2), (2, 3), (3, 2)\}$
     ... *is* a symmetric relation on $\mathbb{N}$.
  2. $\{(2, 1), (1, 2), (2, 3)\}$
     ... *is not* a symmetric relation on $\mathbb{N}$ (because $(2, 3)$ is a member, but $(3, 2)$ is not).
  3. The equality relation, '=', *is* a symmetric relation, since $a = b$ implies $b = a$.
  4. The subset relation, '$\subseteq$', is not a symmetric relation, since it is not generally the case that $S \subseteq T$ implies $T \subseteq S$.
  5. The 'is father of relation' is not symmetric.

3.3 Transitivity

- A relation R is said to be *transitive* iff whenever we have $(x, y) \in R$ and $(y, z) \in R$, we also have $(x, z) \in R$.
- EXAMPLES.
  1. $\{(2, 1), (1, 2), (2, 3), (3, 2)\}$
     $\ldots$ *is not* a transitive relation, as it does not contain $(1, 3)$.
  2. The less than relation, $<$, *is* transitive, since if $a < b$ and $b < c$ then $a < c$.
  3. The 'is an ancestor of' relation is transitive.
  4. Equality is transitive.

3.4 Equivalence Relations

- **Definition:** If a relation is reflexive, symmetric, and transitive, then it is called an *equivalence* relation.
- The general idea behind equivalence relations is that they classify objects which are 'alike' in some respect.
- EXAMPLES.
  1. Equality is an equivalence relation.
  2. The relation 'is the same species as', defined on the set of all animals, is an equivalence relation.
  3. The relation 'owns the same make car as', defined on the set of people, is an equivalence relation.
  4. Neither $<$ nor $\subseteq$ are equivalence relations.
- Equivalence relation are often written $\equiv$ or $\sim$.

3.5 Reflexive and Transitive Closures

- **Definition:** If R is a relation on some set T, then the *reflexive closure* of R is the smallest reflexive relation containing R, and is given by the expression

  R^+.

- **Definition:** If R is a relationon some set T, then the *transitive closure* of R is the smallest transitive relation containing R, and is given by the expression

  R^*.