The complexity of general-valued CSPs seen from the other side

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General-valued Constraint Satisfaction Problem (VCSP)

A **signature** \( \sigma \) is a set of function symbols each of which has a fixed arity \( ar(f) \).

**Valued structure** \( \mathbb{A} \) over \( \sigma \):
- (finite) universe \( A \)
- interpretations \( f^A : A^{ar(f)} \to \mathbb{Q}_{\geq 0} \cup \{\infty\} \) for each \( f \in \sigma \)

For valued structures \( \mathbb{A} \) and \( \mathbb{B} \) over \( \sigma \), the **cost** of \( h : A \to B \) is:

\[
\text{cost}(h) = \sum_{f \in \sigma, \bar{x} \in A^{ar(f)}} f^A(\bar{x}) f^B(h(\bar{x}))
\]

**VCSP**

**Instance:** Valued structures \( \mathbb{A} \) and \( \mathbb{B} \) over the same signature \( \sigma \)

**Goal:** Compute \( \text{opt}(\mathbb{A}, \mathbb{B}) = \min_{h:A \to B} \text{cost}(h) \)
VCSP: particular cases

**VCSP**

**Instance:** Valued structures $\mathbb{A}$ and $\mathbb{B}$ over the same signature $\sigma$

**Goal:** Compute $\text{opt}(\mathbb{A}, \mathbb{B}) = \min_{h: A \rightarrow B} \text{cost}(h)$

\[
\text{CSP} = \text{satisfy all constraints simultaneously} \\
\quad = \text{is there a homomorphism from } \mathbb{A} \text{ to } \mathbb{B} \text{?} \\
\quad \quad \bullet \ \{0, \infty\}\text{-valued structures}
\]

\[
\text{MinCSP} = \text{minimise unsatisfied constraints} \\
\quad \bullet \ \{0, 1\}\text{-valued structures}
\]

\[
\text{Finite-valued CSP} = \mathbb{Q}_{\geq 0}\text{-valued structures}
\]
The complexity of VCSP

VCSP

Instance: Valued structures $\mathbb{A}$ and $\mathbb{B}$ over the same signature $\sigma$

Goal: Compute $\text{opt}(\mathbb{A}, \mathbb{B}) = \min_{h: \mathbb{A} \rightarrow \mathbb{B}} \text{cost}(h)$

VCSP is **NP-hard**

Tractable restrictions:

- **Non-uniform restrictions:** VCSP($-\), \{\mathbb{B}\})
  - Finite valued (Thapper, Zivny STOC’13)
  - CSP (Bulatov FOCS ’17; Zhuk FOCS’17)
  - VCSP (Ochremiak, Kozic ICALP’15; Kolmogorov, Krokhin, Rolinek FOCS’15)

- **Structural restrictions:** VCSP($\mathbb{C}$, $-\)
  - CSP, bounded arity (Dalmau, Kolaitis, Vardi CP’02; Grohe FOCS’03)
  - CSP, unbounded arity: FPT classification (Marx STOC’10)
The complexity of VCSP

**VCSP**

*Instance:* Valued structures $\mathbb{A}$ and $\mathbb{B}$ over the same signature $\sigma$

*Goal:* Compute $\text{opt}(\mathbb{A}, \mathbb{B}) = \min_{h: \mathbb{A} \to \mathbb{B}} \text{cost}(h)$

VCSP is **NP-hard**

Tractable restrictions:

- **Non-uniform** restrictions: VCSP($-$, $\{\mathbb{B}\}$)
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- **Structural** restrictions: VCSP($\mathcal{C}$, $-$)
  - CSP, bounded arity (Dalmau, Kolaitis, Vardi CP’02; Grohe FOCS’03)

**Main Question:**
For which classes $\mathcal{C}$ of bounded arity is VCSP($\mathcal{C}$, $-$) tractable?
Contributions

• Characterisation of the tractable structural restrictions for VCSP
  (in the case of bounded arity)

• Characterisation of the power of Sherali-Adams relaxations for VCSP
Outline

The case of CSP and bounded arity

Tractable structural restrictions for VCSP

Power of Sherali-Adams relaxations

Open questions
Outline

The case of CSP and bounded arity

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Open questions
The case of CSP and bounded arity

**Theorem (Freuder AAAI ’90):**
\[ \text{CSP}(\mathcal{C}, -) \text{ is in PTIME if the treewidth of } \mathcal{C} \text{ is bounded} \]

\( A \) and \( A' \) are homomorphically equivalent
= there is homomorphism from \( A \) to \( A' \), and from \( A' \) to \( A \)

Treewidth modulo homomorphic equivalence of \( A \)
= minimum treewidth over all \( A' \) homo. equiv. to \( A \)
= treewidth of the **core** of \( A \)

**Theorem (Dalmau, Kolaitis, Vardi CP’02):**
\[ \text{CSP}(\mathcal{C}, -) \text{ is in PTIME if the treewidth modulo homomorphic equivalence of } \mathcal{C} \text{ is bounded} \]
The case of CSP and bounded arity

**Theorem (Grohe FOCS ’03)**
Suppose $\mathcal{C}$ is recursively enumerable and has bounded arity. If the treewidth modulo homo. equiv. of $\mathcal{C}$ is unbounded, then $p$-CSP$(\mathcal{C}, -)$ is $W[1]$-hard

$p$-CSP$(\mathcal{C}, -)$: parameter $|A|$

Reduction from $p$-CLIQUES
The case of CSP and bounded arity

Complete classification:

**Theorem (Dalmau, Kolaitis, Vardi CP ’02; Grohe FOCS ’03)**
Assume $\text{FPT} \neq \text{W}[1]$. For every $\mathcal{C}$ recursively enum. and of **bounded arity**, TFAE:
1. $\text{CSP}(\mathcal{C}, -)$ is in PTIME
2. $\text{p-CSP}(\mathcal{C}, -)$ is in FPT
3. The treewidth modulo homo. equiv. of $\mathcal{C}$ is bounded
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Open questions
VCSP and treewidth

**Theorem (Folklore):**

$\text{VCSP}(C, -)$ is in PTIME if the treewidth of $C$ is bounded

Treewidth of a valued structure $\mathfrak{A} = \text{treewidth of the positive part } \text{Pos}(\mathfrak{A})$
VCSP: example beyond treewidth

\[ \sigma = \{ \phi(\cdot, \cdot), \mu(\cdot) \} \]
VCSP: example beyond treewidth

\[ \sigma = \{ \phi(\cdot, \cdot), \mu(\cdot) \} \]

\[ \mathbb{A}_3 \]

\[ C = \{ \mathbb{A}_n \mid n \geq 2 \} \]

The treewidth of \( C \) is unbounded but VCSP(\( C, - \)) is in PTIME
The tractability frontier for VCSP(C, —)?

- Bounded treewidth modulo homomorphic equivalence (of the positive parts)
- W[1]-hard
- Bounded treewidth
- PTIME

????
Classification for VCSP(C,−)

\( \mathcal{A} \) and \( \mathcal{A}' \) over \( \sigma \) are valued equivalent if

\[
\text{opt}(\mathcal{A}, \mathcal{B}) = \text{opt}(\mathcal{A}', \mathcal{B}) \quad \text{for all valued structures } \mathcal{B} \text{ over } \sigma
\]

**Theorem (Classification for VCSP(C,−))**
Assume FPT \( \not\in \text{W}[1] \).
For every \( \mathcal{C} \) recursively enum. and of **bounded arity**, TFAE:
1. \( \text{VCSP}(\mathcal{C}, -) \) is in PTIME
2. \( \text{p-VCSP}(\mathcal{C}, -) \) is in FPT
3. The treewidth modulo valued equivalence of \( \mathcal{C} \) is bounded

(1) \( \Rightarrow \) (2) : trivial
(3) \( \Rightarrow \) (1) : Sherali-Adams relaxations
(2) \( \Rightarrow \) (3) : Grohe’s reduction from p-CLIQUE + new tools

- Characterisation of valued equivalence in terms of certain type of homomorphisms (inverse fractional homomorphisms)
- Notion of **valued core** of a valued structure
Theorem (Classification for VCSP(C,-))
Assume FPT \( \neq W[1] \).
For every \( \mathcal{C} \) recursively enum. and of bounded arity, TFAE:
1. VCSP(\( \mathcal{C}, - \)) is in PTIME
2. p-VCSP(\( \mathcal{C}, - \)) is in FPT
3. The treewidth modulo valued equivalence of \( \mathcal{C} \) is bounded

- Grohe's classification: \( \{0, \infty\} \)-valued structures
- Classification for finite-valued structures
The tractability frontier for VCSP($C, -$)

Bounded treewidth modulo homomorphic equivalence (of the positive parts) \[ \text{W}[1]\text{-hard} \]

Bounded treewidth modulo valued equivalence \[ \text{PTIME} \]

Bounded treewidth
Outline

The case of CSP and bounded arity

Tractable structural restrictions for VCSP

Power of Sherali-Adams relaxations

Open questions
Power of Sherali-Adams

Basic LP for an instance \((A, B)\):
- variables \(\lambda(x, d)\) for each \(x \in A\) and \(d \in B\)
- variables \(\phi(\tau)\) for each \(\tau : S \rightarrow B\) and scope \(S \subseteq A\)

k-th level of Sherali-Adams for \((A, B)\):
- variables \(\lambda(s)\) for each \(s : X \rightarrow B\) where \(|X| \leq k\)
- variables \(\phi(\tau)\) for each \(\tau : S \rightarrow B\) and scope \(S \subseteq A\)

**Theorem (Folklore):**
If the treewidth of \(A\) is at most \(k-1\), then the \(k\)-th level of Sherali-Adams is tight for \(A\)

For all \(B\), the \(k\)-th level is tight for \((A, B)\)
Power of Sherali-Adams

Theorem:
If the treewidth modulo valued equivalence of $\mathbb{A}$ is at most $k-1$, then the $k$-th level of Sherali-Adams is tight for $\mathbb{A}$.

treewidth of the valued core

Theorem:
Fix $k \geq 1$. Let $\mathbb{A}$ be a valued structure and $\mathbb{A}'$ its valued core. Suppose that $r \leq k$, where $r$ is the maximum arity of $\mathbb{A}$. TFAE:

1. The $k$-th level of Sherali Adams is tight for $\mathbb{A}$
2. The treewidth of $\mathbb{A}'$ is at most $k-1$
Theorem (Power of Sherali-Adams):
Fix $k \geq 1$. Let $\mathbb{A}$ be a valued structure and $\mathbb{A}'$ its valued core. TFAE:

1. The $k$-th level of Sherali Adams is tight for $\mathbb{A}$
2. The treewidth modulo scopes of $\mathbb{A}$ is at most $k-1$ and the overlap of $\mathbb{A}$ is at most $k$

$$\text{tw mod scopes} \leq k - 1$$
$$\text{overlap} \leq k : |S \cap S'| \leq k$$
Theorem (Power of Sherali-Adams):
Fix \( k \geq 1 \). Let \( \mathbb{A} \) be a valued structure and \( \mathbb{A}' \) its valued core. TFAE:

1. The k-th level of Sherali Adams is tight for \( \mathbb{A} \)
2. The treewidth modulo scopes of \( \mathbb{A} \) is at most k-1 and the overlap of \( \mathbb{A} \) is at most k

(1) \( \Rightarrow \) (2):

Results on the power of k-consistency for CSP +

(Atserias, Bulatov, Dalmau ICALP’07)

Inverse fractional homomorphism and valued cores
Open problems

• Unbounded arity?
• MinCSP/MaxCSP ($\{0, 1\}$-valued structures)
• Classification for approximation of $\text{VCSP}(\mathcal{C}, -)$?

Thank you!