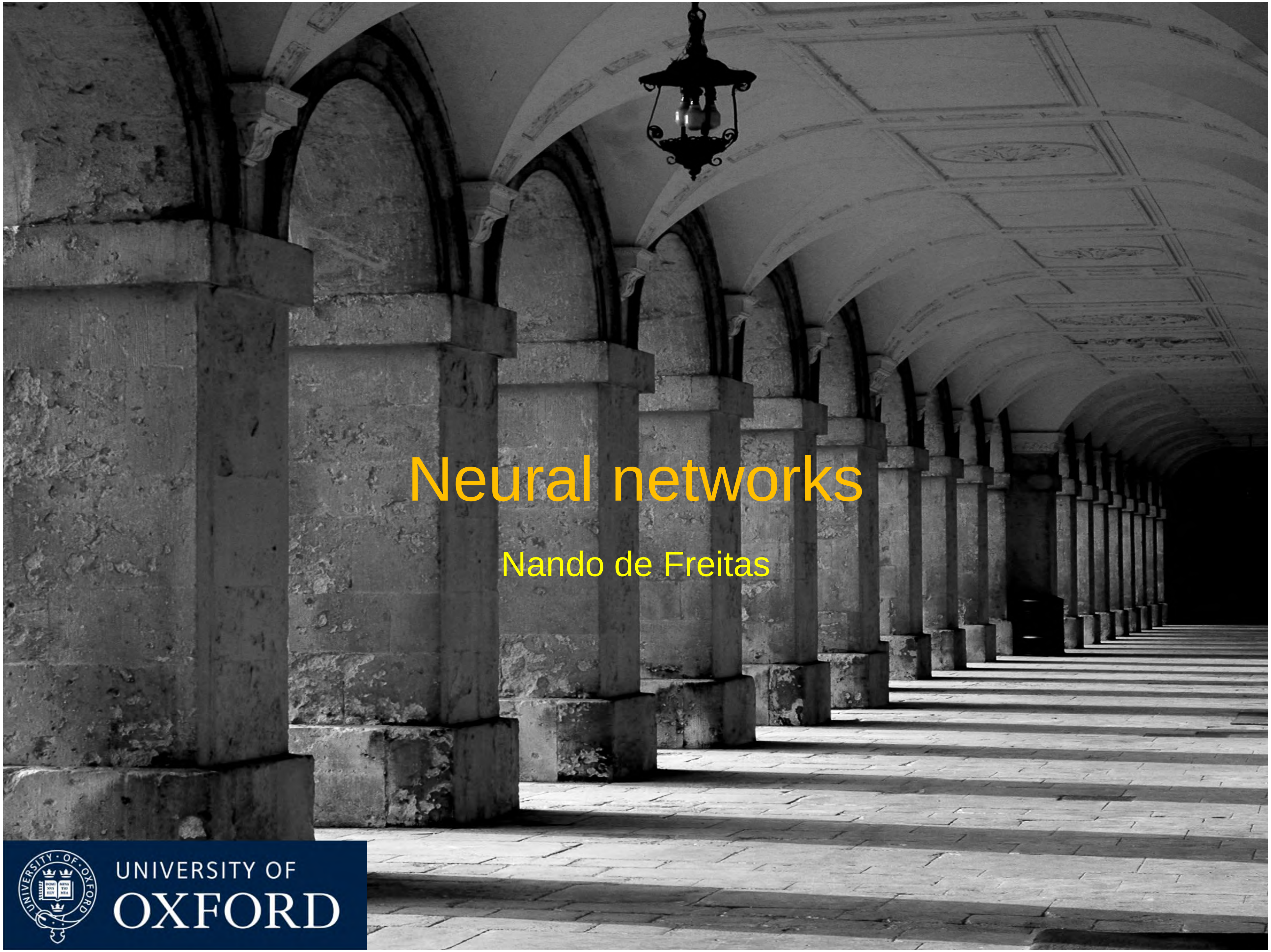

Neural networks

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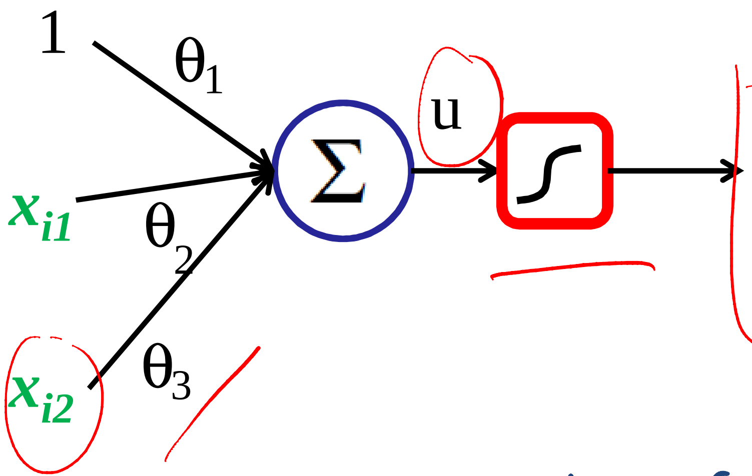
Outline of the lecture

This lecture introduces you to the fascinating subject of classification and regression with artificial neural networks. In particular, it

- ❑ Introduces multi-layer perceptrons (MLPs)
- ❑ Teaches you how to combine probability with neural networks so that the nets can be applied to regression, binary classification and multivariate classification.
- ❑ Discusses the modular approach to backpropagation and neural network construction in Torch, which was introduced in the previous lecture.

MLP – 1 neuron

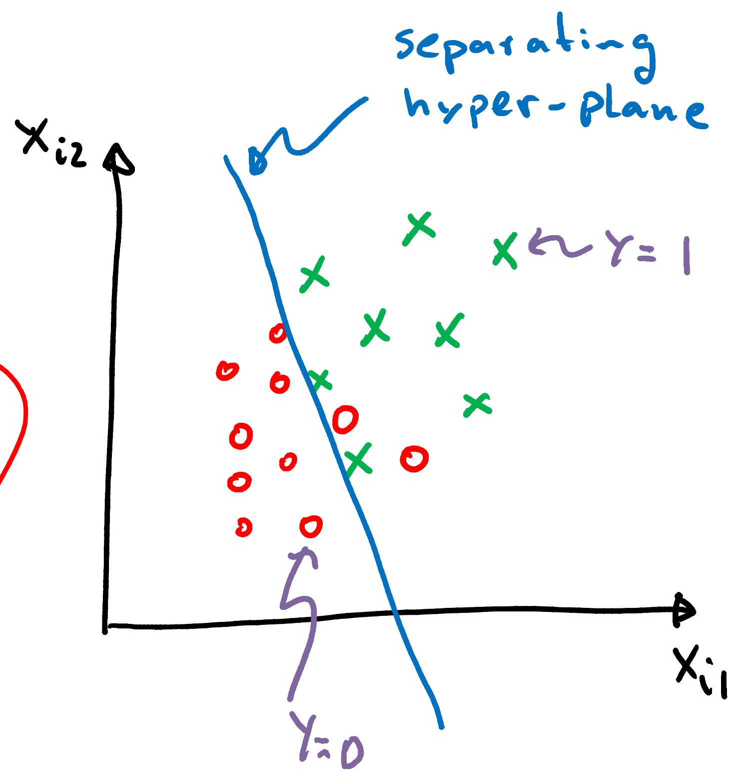


$$\hat{y}_i = P(y_i = 1 \mid x_i, \theta)$$

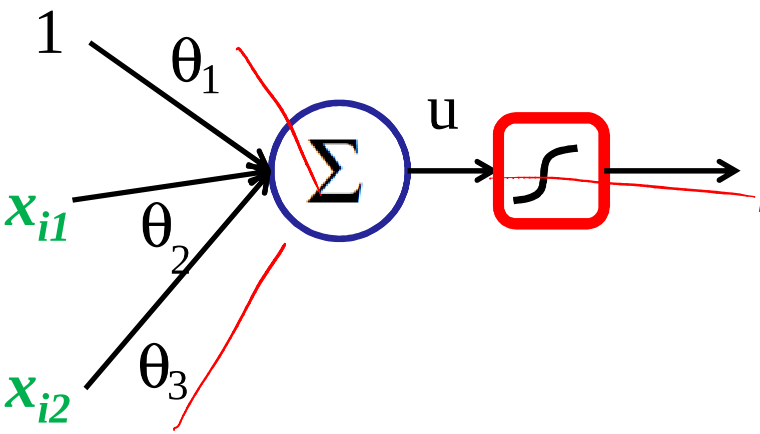
We are given the data $\{x_i, y_i\}_{i=1}^{n}$

eg.

	x_{i1}	x_{i2}	y_i
i=1	0.2	6	0
i=2	0.3	22	1
i=3	0.6	-0.6	1
i=4	-0.4	58	0
⋮			



MLP – 1 neuron



$\hat{y}_i = P(y_i = 1 | x_i, \theta)$

$$u = \Theta_1 + \Theta_2 x_{i1} + \Theta_3 x_{i2}$$

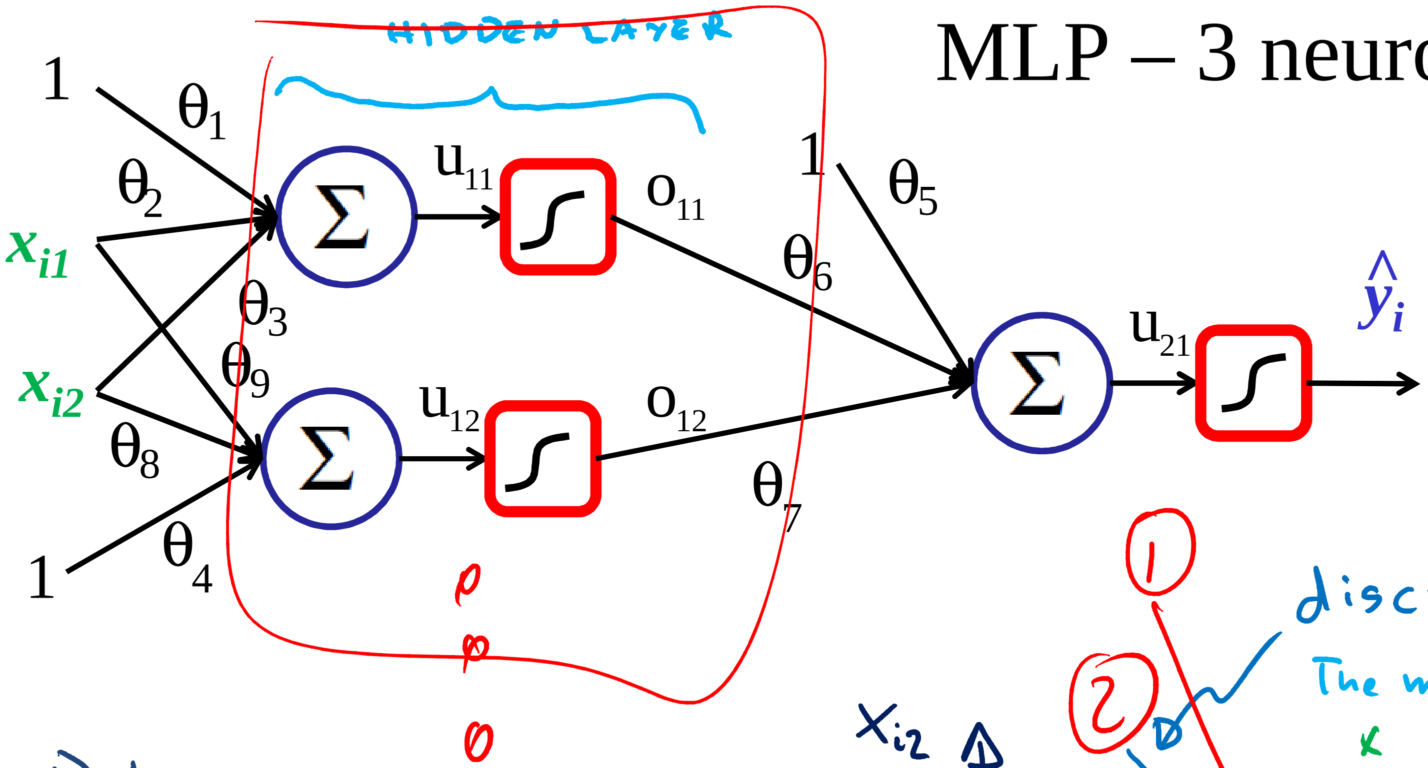
$$\hat{y}_i = \frac{1}{1+e^{-u}} = \frac{1}{1+e^{-\Theta_1 - \Theta_2 x_{i1} - \Theta_3 x_{i2}}} = P(y_i = 1 \mid \underline{x}_i, \underline{\Theta})$$

$$P(y_i \mid \underline{x}_i, \underline{\Theta}) = \hat{y}_i^{\,y_i}(1-\hat{y}_i)^{1-y_i} = \begin{cases} \hat{y}_i & \text{when } y_i = 1 \\ 1-\hat{y}_i & \text{otherwise} \end{cases}$$

For n independent observations (Bernoulli)

$$P(\underline{y} \mid \underline{\underline{x}}, \underline{\Theta}) = \prod_{i=1}^{n} P(y_i \mid \underline{x}_i, \underline{\Theta})$$

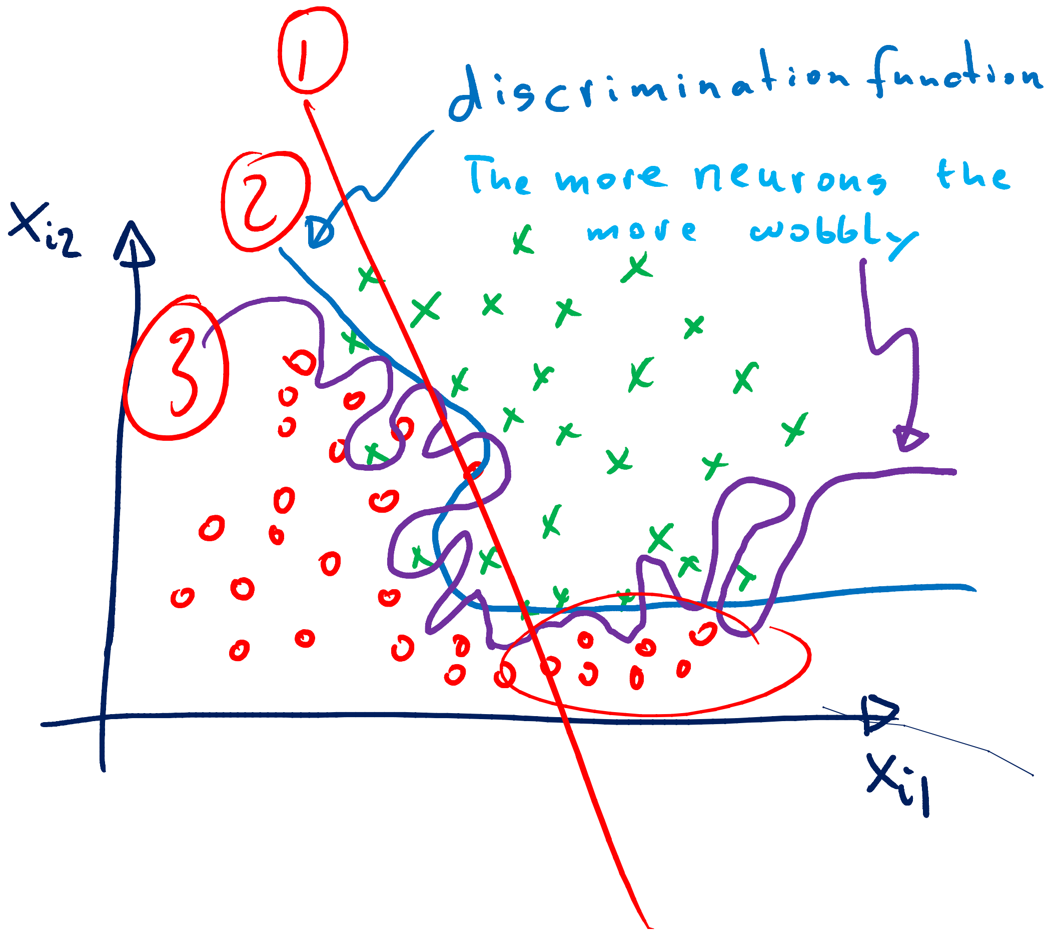
MLP – 3 neurons, 2 layers



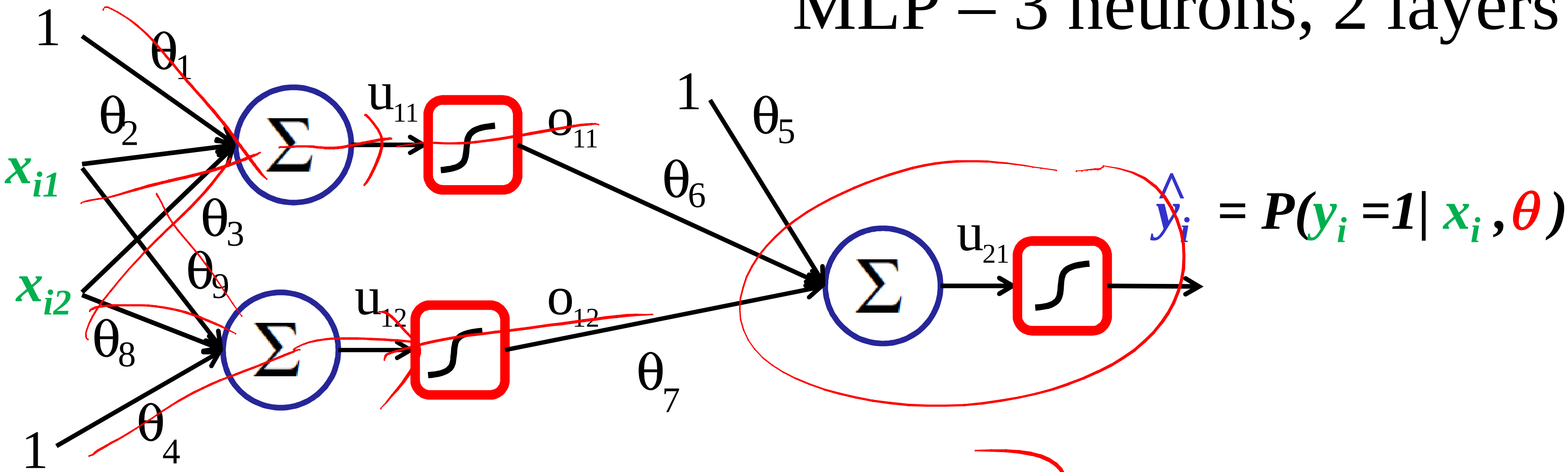
$$\hat{y}_i = P(y_i = 1 \mid x_i, \theta)$$

Data:

	x_{i1}	x_{i2}	y_i
i=1	6	8	1
i=2	0.2	-5	0
	-100	3.1	1
	6	9	0
	5	8	0



MLP – 3 neurons, 2 layers



$$u_{11} = \theta_1 + \theta_2 x_{i1} + \theta_3 x_{i2}$$

$$u_{12} = \theta_4 + \theta_9 x_{i1} + \theta_8 x_{i2}$$

$$o_{11} = 1/_{1+e^{-u_{11}}}$$

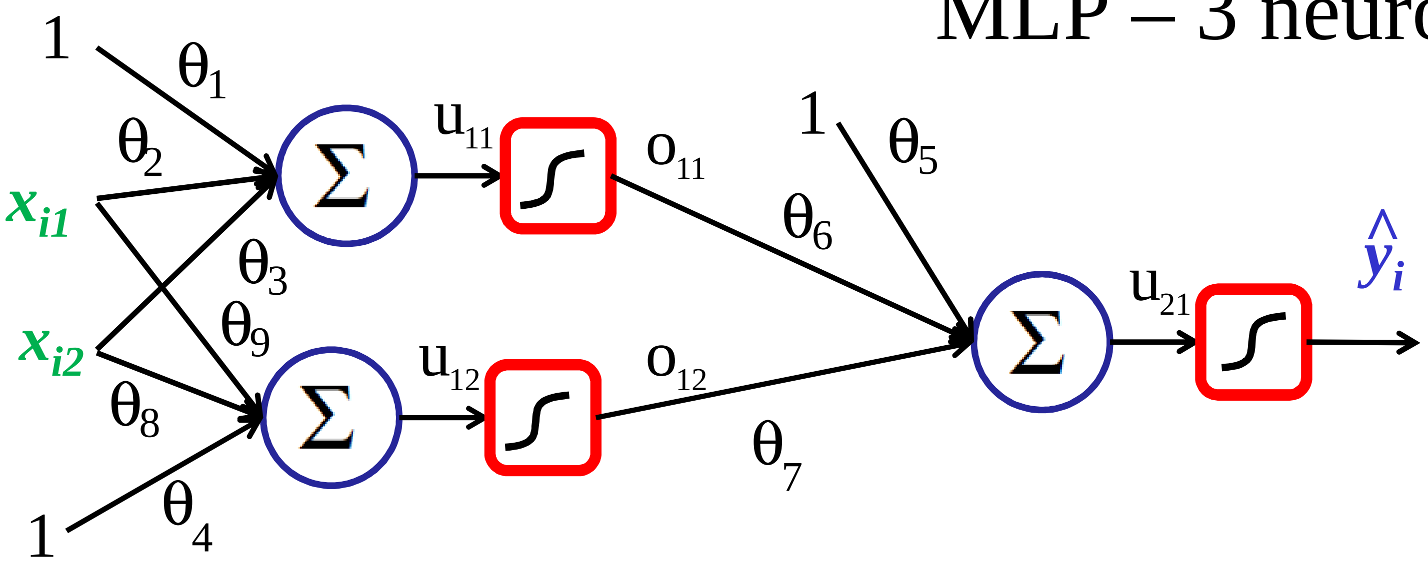
$$o_{12} = 1/_{1+e^{-u_{12}}}$$

$$\hat{y}_i = \frac{1}{1+e^{-u_{21}}}$$

$$u_{21} = \theta_5 + \theta_6 o_{11} + \theta_7 o_{12}$$

$$\boxed{P(y_i | \underline{x}_i, \underline{\theta}) = \hat{y}_i^{\,y_i} (1 - \hat{y}_i)^{1-y_i}}$$

MLP – 3 neurons, 2 layers



$$\hat{y}_i = P(y_i = 1 \mid x_i, \theta)$$

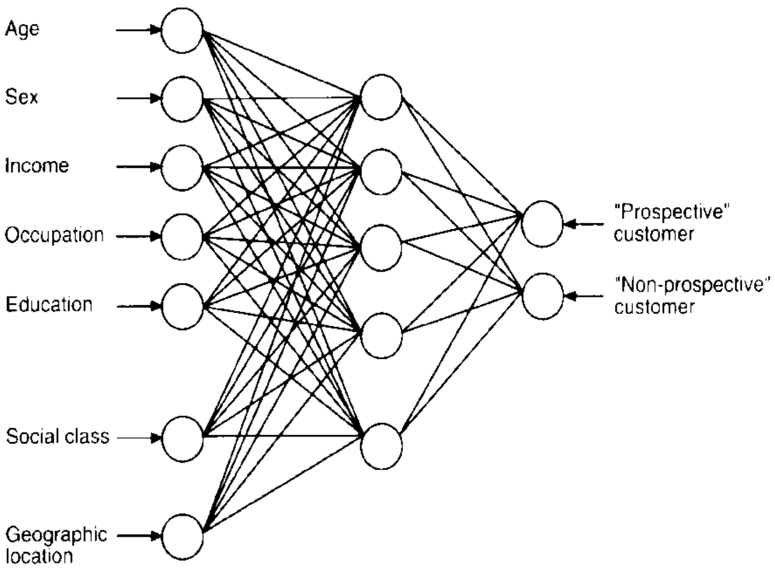
For n independent observations.

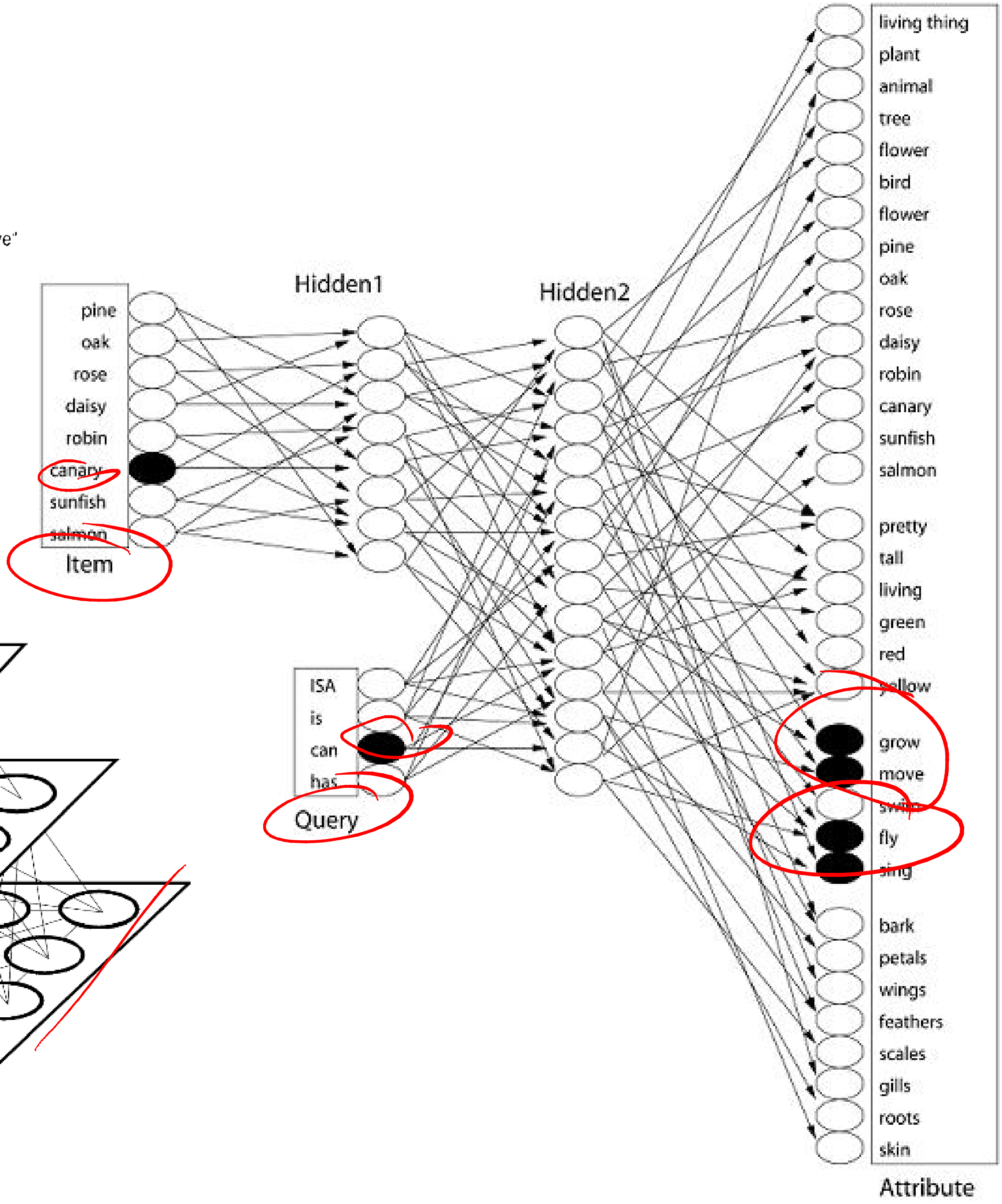
$$P(\underline{y} \mid \underline{\underline{x}}, \underline{\theta}) = \prod_{i=1}^{n} \hat{y}_i^{\,y_i} (1-\hat{y}_i)^{1-y_i} = \prod_{i=1}^{n} P(y_i \mid x_i, \theta)$$

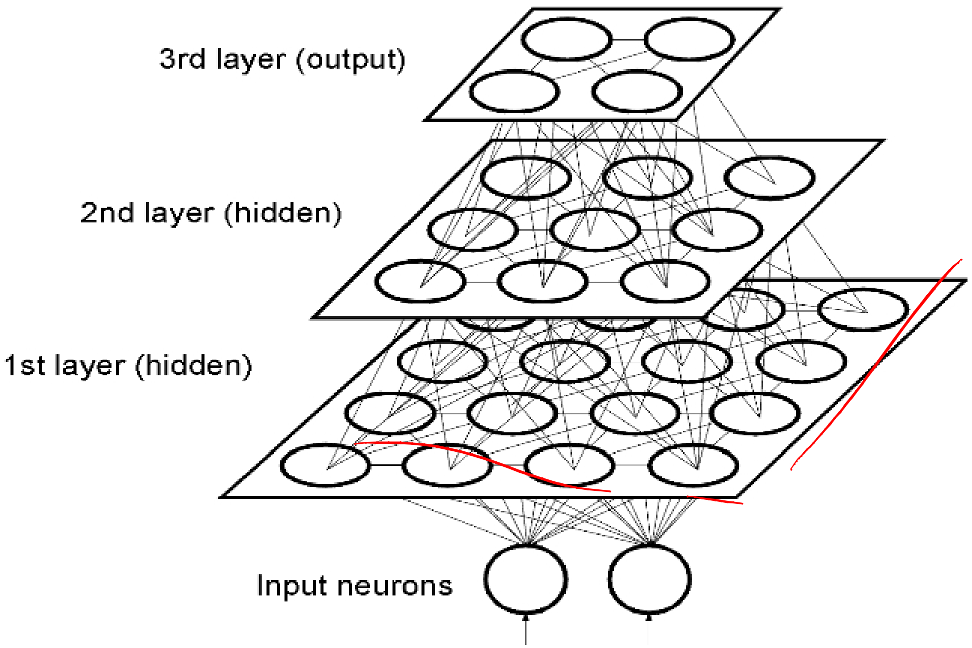
Cost:

$$c(\theta) = -\log P(\underline{y} \mid \underline{\underline{x}}, \underline{\theta}) = -\sum_{i=1}^{n} y_i \log \hat{y}_i + (1-y_i) \log (1-\hat{y}_i)$$

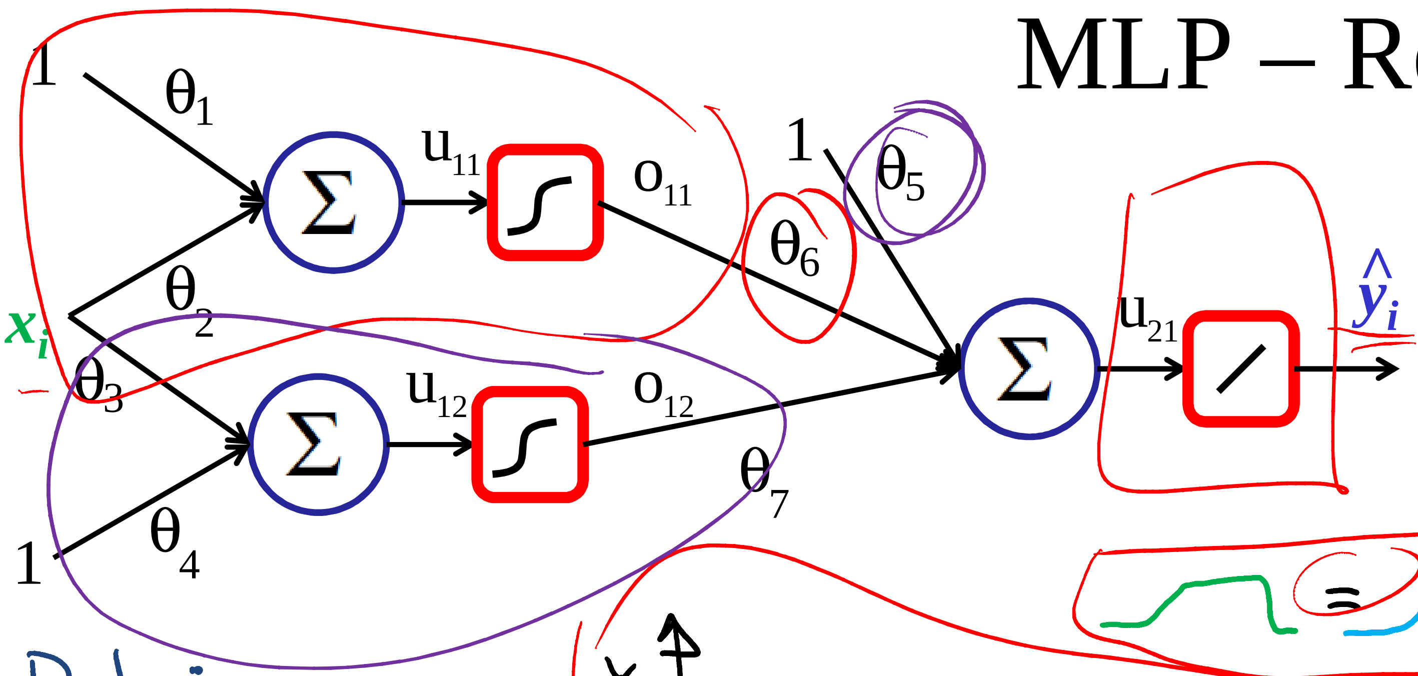
i.e minimize the cross-entropy error.


Age
Sex
Income
Occupation
Education
Social class
Geographic location
"Prospective" customer
"Non-prospective" customer

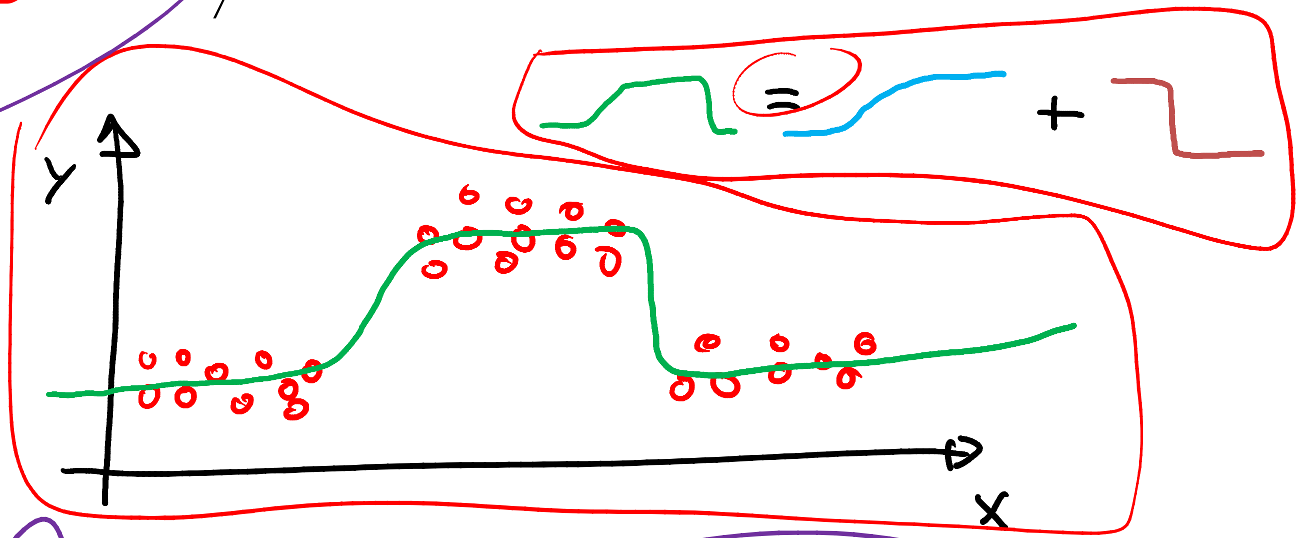

pine
oak
rose
daisy
robin
canary
sunfish
salmon
Item
Hidden1
Hidden2
ISA
is
can
has
Query
living thing
plant
animal
tree
flower
bird
flower
pine
oak
rose
daisy
robin
canary
sunfish
salmon
pretty
tall
living
green
red
yellow
grow
move
swim
fly
sing
bark
petals
wings
feathers
scales
gills
roots
skin
Attribute


3rd layer (output)
2nd layer (hidden)
1st layer (hidden)
Input neurons

MLP – Regression



$\hat{y}_i = \mathbf{E}(y_i \mid x_i, \theta) = u_{21}$

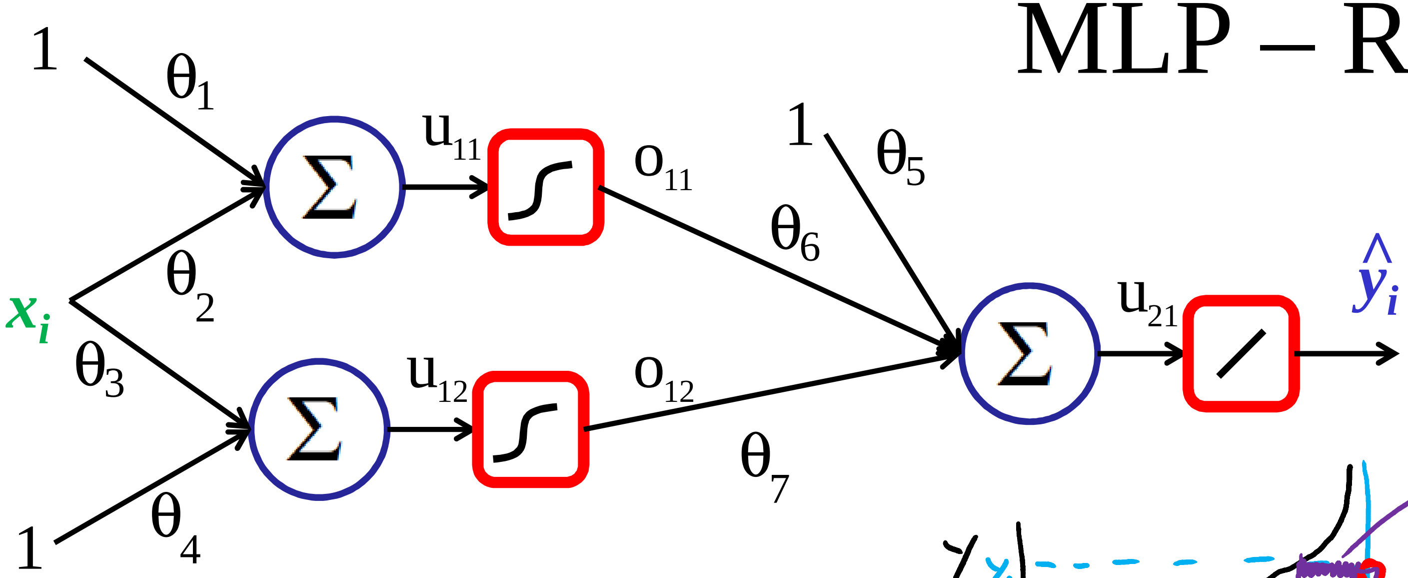


Data:

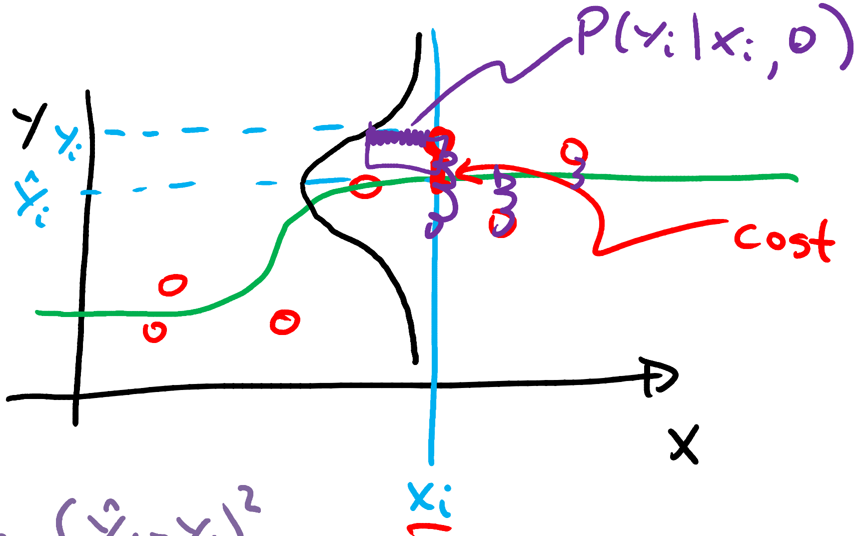
x_i	y_i
0.2	0.6
0.9	0.4
-0.6	5
-0.8	6.2

$$\hat{y}_i = \theta_5 + \frac{\theta_6}{1+e^{-\theta_1-\theta_2 x_i}} + \frac{\theta_7}{1+e^{-\theta_4-\theta_3 x_i}}$$

MLP – Regression



$$\hat{y}_i = \mathbf{E}(y_i \mid x_i, \theta) = u_{21}$$

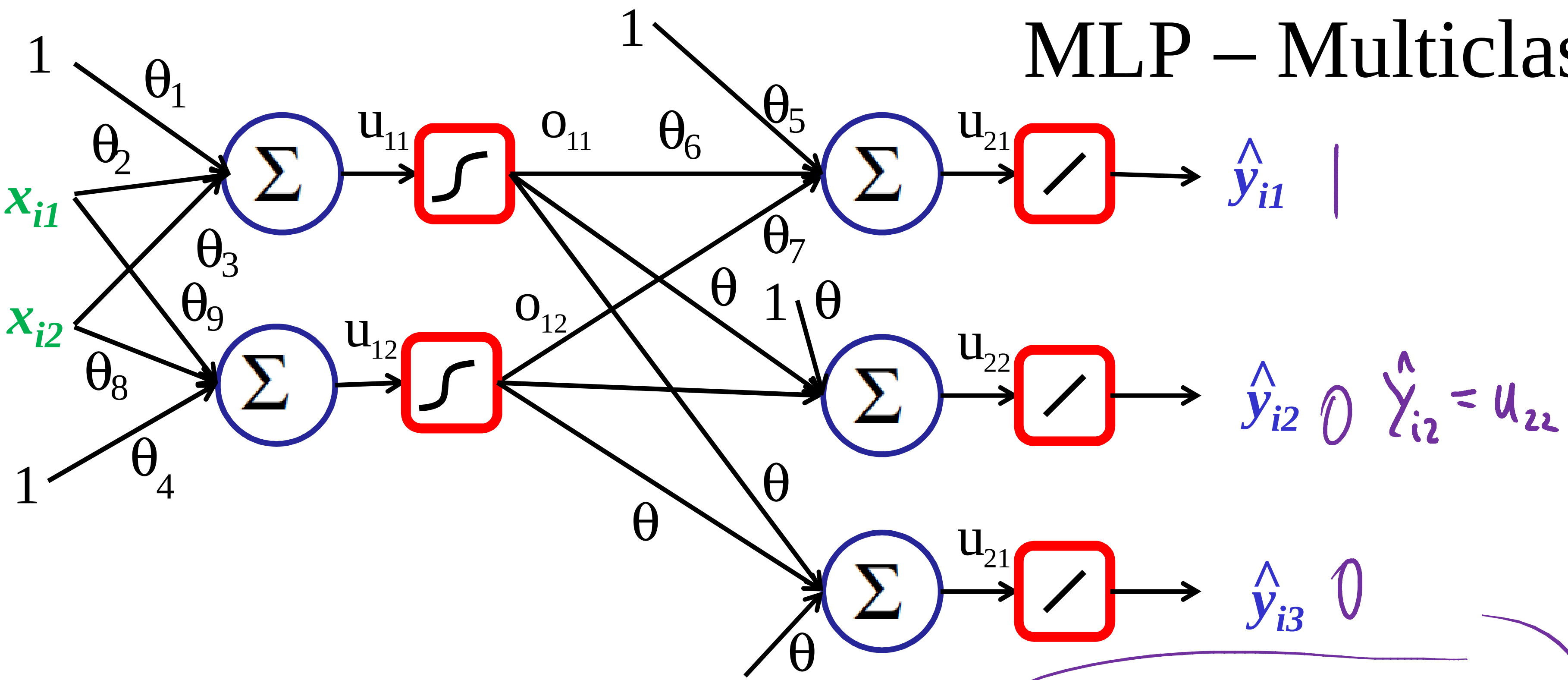


$$P(y_i \mid x_i, \theta) = (2\pi\sigma^2)^{-1/2} e^{-\frac{1}{2\sigma^2}(\hat{y}_i - y_i)^2}$$

$$c(\theta) = \sum_{i=1}^{n} \underline{(y_i - \hat{y}_i)^2}$$

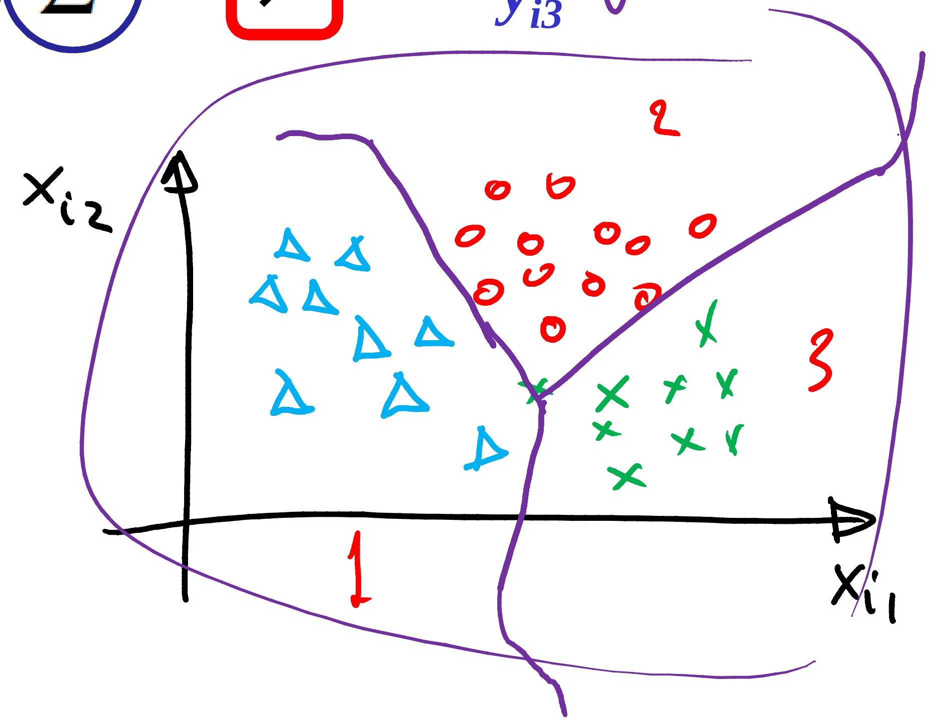
$$\hat{y}_i = f(\theta, x_i)$$

MLP – Multiclass

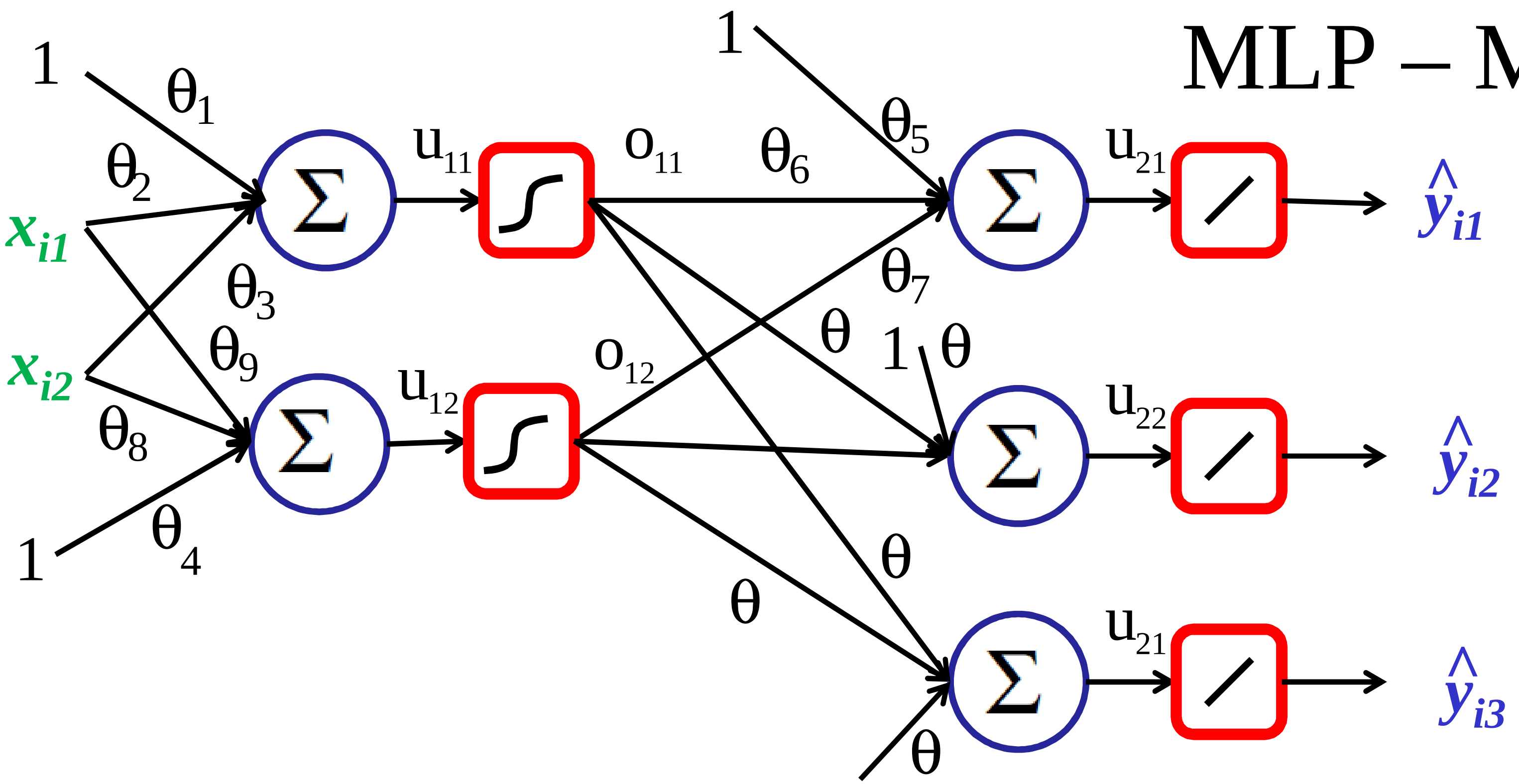


Data :

x_{i1}	x_{i2}	y_{i1}	y_{i2}	y_{i3}	
0.2	0.3	0	1	0	Class 2
-5	-6	1	0	0	Class 1
-20	4	1	0	0	" 1
42	6.8	0	0	1	Class 3



MLP – Multiclass



To get a probabilistic model, define:

SOFTMAX

$$P(y_i = (0\,1\,0) \mid x_i, \theta) = P(y_i = 2 \mid x_i, \theta) = \boxed{\frac{e^{\hat{y}_2}}{e^{\hat{y}_1} + e^{\hat{y}_2} + e^{\hat{y}_3}}}$$

MLP – Multiclass

$$\mathbb{I}_2(y_i) = \begin{cases} 1 & y_i = 2 \\ 0 & \text{o.w.} \end{cases}$$

Then,

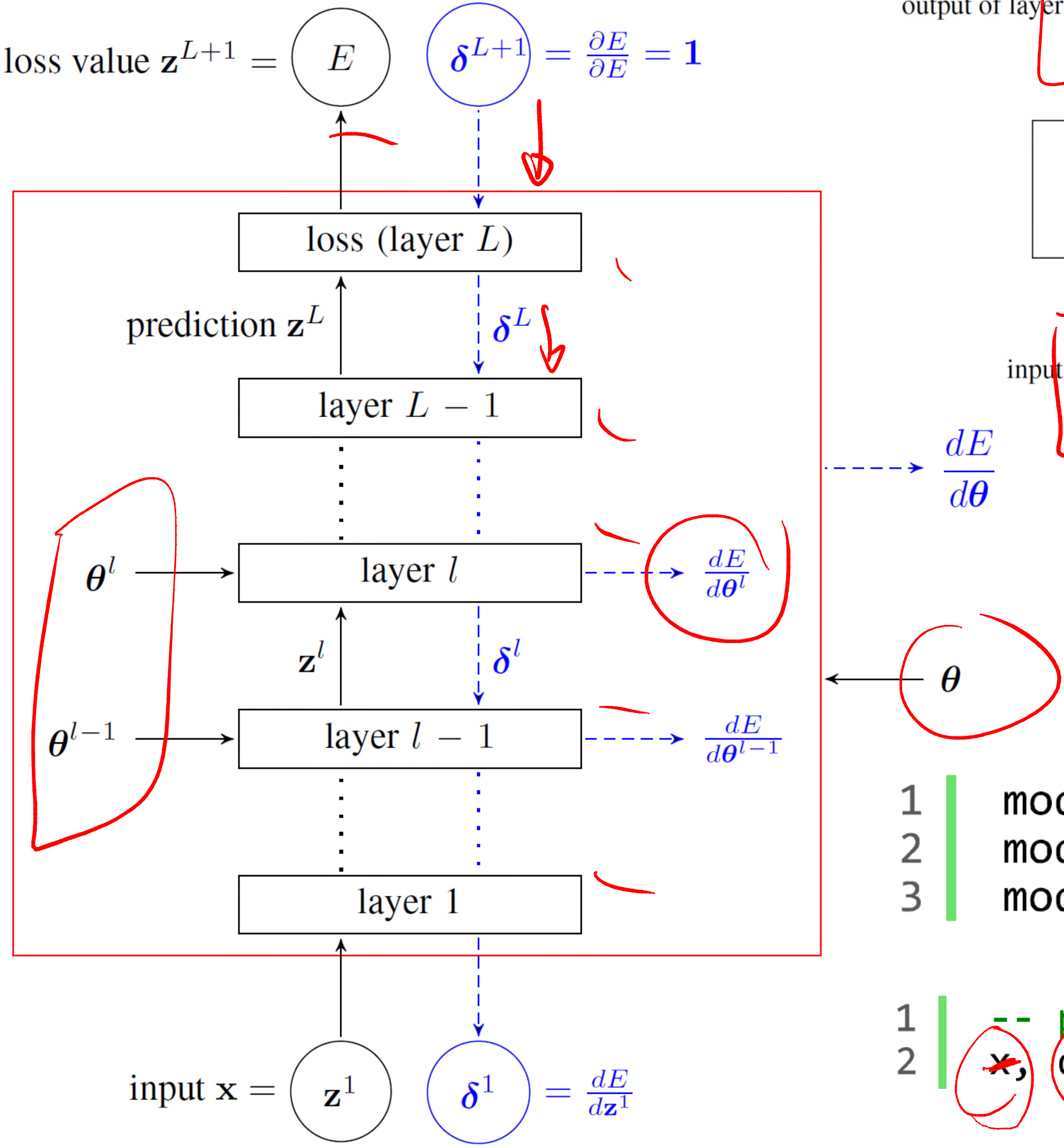
$$P(y_i | x_i, \theta) = \left[\frac{e^{\hat{y}_{i1}}}{\underbrace{e^{\hat{y}_{i1}} + e^{\hat{y}_{i2}} + e^{\hat{y}_{i3}}}_{\text{sum}}}\right]^{\mathbb{I}_1(y_i)} \left[\frac{e^{\hat{y}_{i2}}}{e^{\hat{y}_{i1}} + e^{\hat{y}_{i2}} + e^{\hat{y}_{i3}}}\right]^{\mathbb{I}_2(y_i)} \left[\frac{e^{\hat{y}_{i3}}}{e^{\hat{y}_{i1}} + e^{\hat{y}_{i2}} + e^{\hat{y}_{i3}}}\right]^{\mathbb{I}_3(y_i)}$$

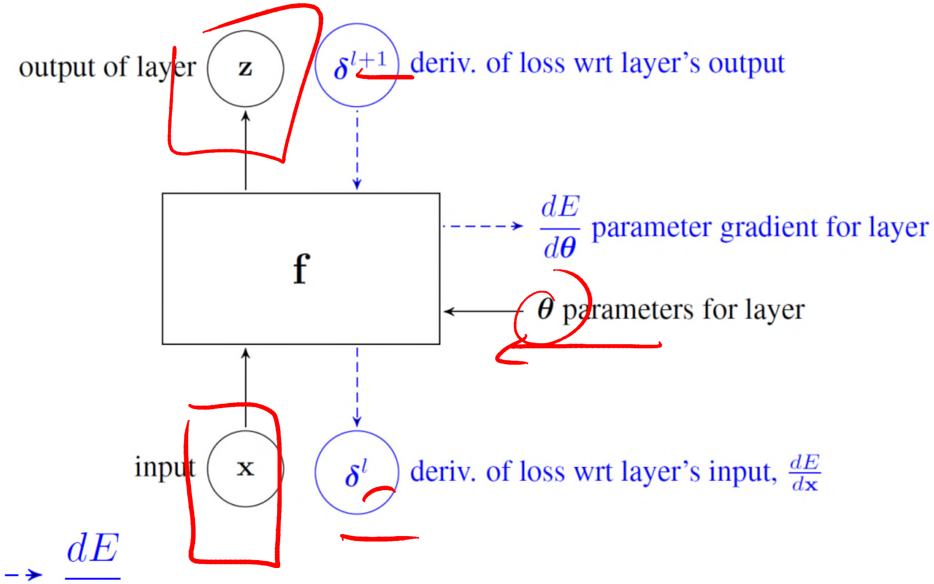
$$= \begin{cases} e^{\hat{y}_{i1}}/\text{sum} & y_i = 1 \\ e^{\hat{y}_{i2}}/\text{sum} & y_i = 2 \\ e^{\hat{y}_{i3}}/\text{sum} & y_i = 3 \end{cases}$$

Cost:

$$C(\theta) = -\log P(\underset{\sim}{y} | \underset{\sim}{X}, \underset{\sim}{\theta}) = -\sum_{i=1}^{n} \sum_{j=1}^{3} \mathbb{I}_j(y_i) \log \frac{e^{\hat{y}_{ij}}}{\text{sum}}$$

Deep learning & backprop

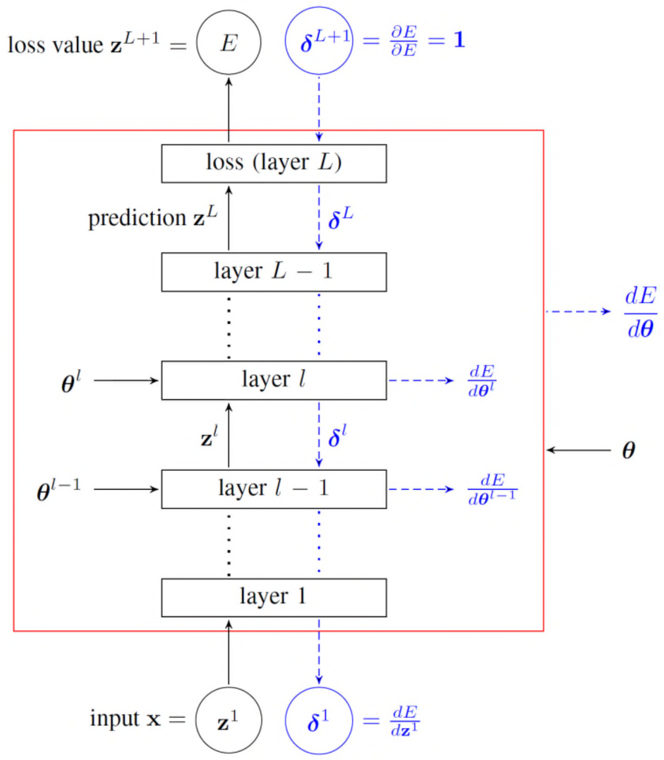





```
model = nn.Sequential()
model:add( nn.Linear(2,3) )
model:add( nn.LogSoftMax() )
```

```
-- params/gradients
x, dl_dx = model:getParameters()
```

Deep learning & backprop



$$\mathbf{z}^{l+1} = \mathbf{f}^l(\mathbf{z}^l;\boldsymbol{\theta}^l)$$

forward

Jacobian

$$\boldsymbol{\delta}^l := \frac{\partial E}{\partial \mathbf{z}^l} = \frac{\partial E}{\partial \mathbf{z}^{l+1}} \frac{\partial \mathbf{z}^{l+1}}{\partial \mathbf{z}^l} = \boldsymbol{\delta}^{l+1} \frac{\partial \mathbf{f}^l(\mathbf{z}^l;\boldsymbol{\theta}^l)}{\partial \mathbf{z}^l}$$

$$\delta_i^l = \sum_j \frac{\partial E}{\partial z_j^{l+1}} \frac{\partial z_j^{l+1}}{\partial z_i^l} = \sum_j \delta_j^{l+1} \frac{\partial f_j^l(\mathbf{z}^l;\boldsymbol{\theta}^l)}{\partial z_i^l}$$

$$\frac{\partial E}{\partial \boldsymbol{\theta}^l} = \frac{\partial E}{\partial \mathbf{z}^{l+1}} \frac{\partial \mathbf{z}^{l+1}}{\partial \boldsymbol{\theta}^l} = \boldsymbol{\delta}^{l+1} \frac{\partial \mathbf{f}^l(\mathbf{z}^l;\boldsymbol{\theta}^l)}{\partial \boldsymbol{\theta}^l}$$

$$\frac{\partial E}{\partial \theta_i^l} = \sum_j \frac{\partial E}{\partial z_j^{l+1}} \frac{\partial z_j^{l+1}}{\partial \theta_i^l} = \sum_j \delta_j^{l+1} \frac{\partial f_j^l(\mathbf{z}^l;\boldsymbol{\theta}^l)}{\partial \theta_i^l}$$

Reverse auto-diff: why it works

$$f^L(\mathbf{f}^{L-1}(\cdots\mathbf{f}^3(\mathbf{f}^1(\mathbf{x}))))$$

$$\frac{\partial f^L(\mathbf{f}^{L-1}(\cdots\mathbf{f}^2(\mathbf{f}^1(\mathbf{x}))))}{\partial\mathbf{x}} = \frac{\partial f^L}{\partial\mathbf{f}^{L-1}}\cdot\frac{\partial\mathbf{f}^{L-1}}{\partial\mathbf{f}^{L-2}}\cdots\frac{\partial\mathbf{f}^2}{\partial\mathbf{f}^1}\cdot\frac{\partial\mathbf{f}^1}{\partial\mathbf{x}}$$

$$\frac{\partial f^L(\mathbf{f}^{L-1}(\cdots\mathbf{f}^2(\mathbf{f}^1(\mathbf{x}))))}{\partial\mathbf{x}} = \left(\left(\left(\left(\frac{\partial f^L}{\partial\mathbf{f}^{L-1}}\cdot\frac{\partial\mathbf{f}^{L-1}}{\partial\mathbf{f}^{L-2}}\right)\cdots\right)\frac{\partial\mathbf{f}^2}{\partial\mathbf{f}^1}\right)\cdot\frac{\partial\mathbf{f}^1}{\partial\mathbf{x}}\right)$$

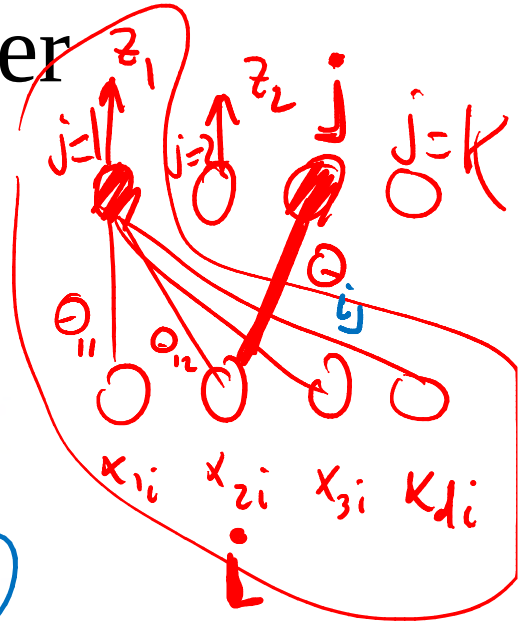
Deep learning: linear layer

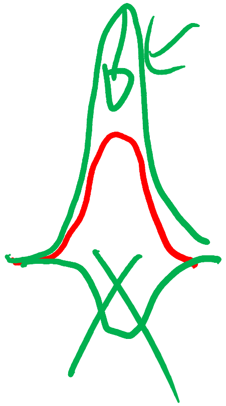
nn

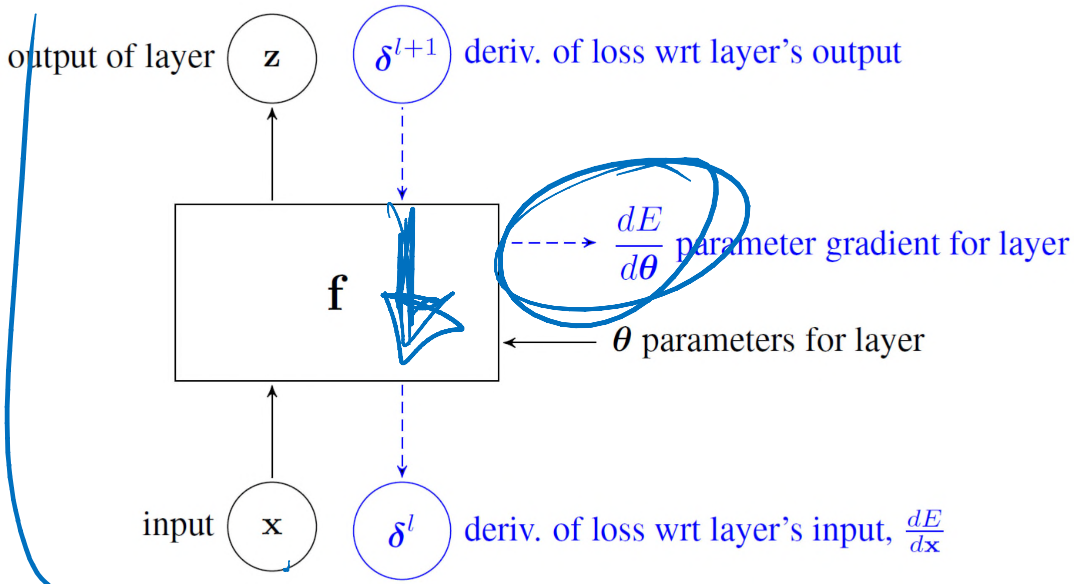
$$z_j = f_j(\mathbf{x};\boldsymbol{\theta}_j) = \sum_i x_i \theta_{ij}$$

$$\delta_i^l = \sum_j \delta_j^{l+1} \frac{\partial f_j(\mathbf{x};\boldsymbol{\theta}_j)}{\partial x_i} = \sum_j \delta_j^{l+1} \theta_{ij}$$

$$\frac{\partial E}{\partial \theta_{ij}} = \sum_j \delta_j^{l+1} \frac{\partial f_j(\mathbf{x};\boldsymbol{\theta}_j)}{\partial \theta_{ij}} = \delta_j^{l+1} x_i$$





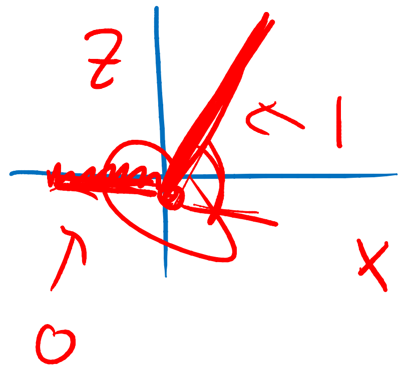


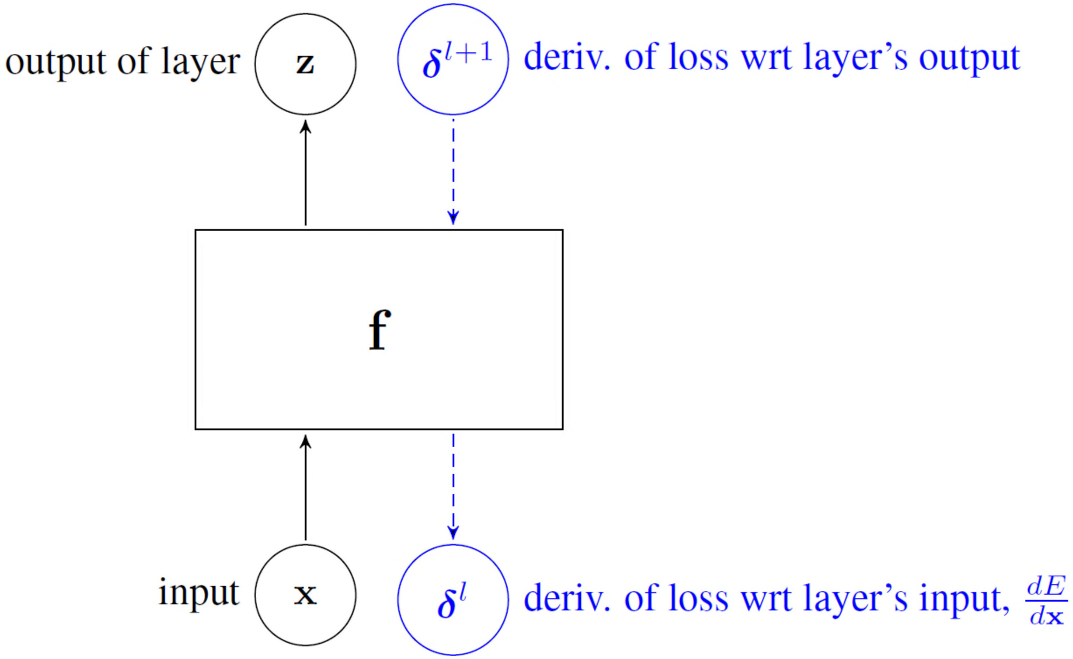
Deep learning: ReLU layer

Fukushima

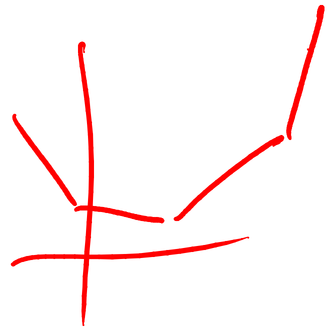
$$z_j = f_j(x_j) = \max(0, x_j)$$

$$\delta_i^l = \sum_j \delta_j^{l+1} \frac{\partial f_j(x_j)}{\partial x_i} = \delta_i^{l+1} \mathbb{I}_{[x_i > 0]}$$

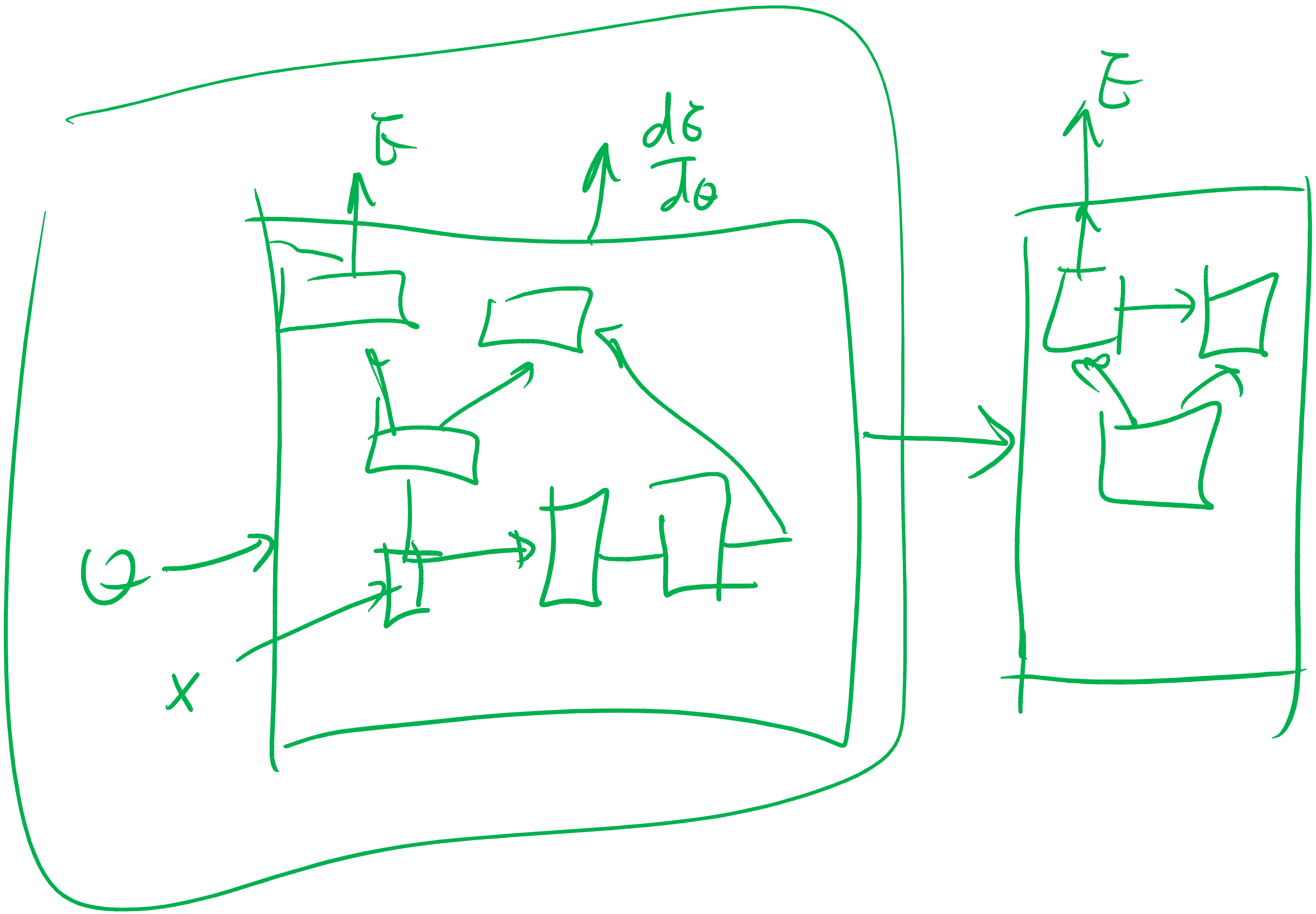




maxout



Deep learning: extremely flexible!



Next lecture

In the next lecture, we will look at a successful type of neural network that is very popular in speech and object recognition, known as a convolutional neural network.