Cost-parity games and Cost-Streett games

FSTTCS’2012

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Framework

This paper is about two-player games over finite graphs, as used in automata theory and verification (synthesis problem).

We are after efficient algorithms to:

- decide the winner, *and*
- synthesize a winning strategy.
This paper is about two-player games over finite graphs, as used in automata theory and verification (synthesis problem).

We are after efficient algorithms to:
- decide the winner, \textit{and}
- synthesize a winning strategy.

\textit{Starting point}: good understanding of $\omega$-regular specifications.

\textit{Objective}: add boundedness specifications.
Games

controlled by Eve
controlled by Adam
Games

controlled by Eve

controlled by Adam
Games

controlled by Eve
controlled by Adam
Games

controlled by Eve
controlled by Adam
Games

controlled by Eve

controlled by Adam
Almost all requests are answered.
Parity conditions

- Parity conditions allow to express all $\omega$-regular specifications.
- Both players have positional winning strategies.
- Deciding the winner is in $\text{NP} \cap \text{coNP}$. 

Diagram:

- States: 0, 1, 2, 3, 4
- Transitions:
  - From 0 to 2
  - From 0 to 3
  - From 2 to 1
  - From 2 to 3
  - From 3 to 1
  - From 3 to 4
Parity and finitary parity

Finitary specifications: proposed by Alur and Henzinger, games studied by Chatterjee, Henzinger and Horn.

**Parity:**
Almost all requests are answered.

**Finitary parity:**
There exists a bound $b$, s.t. almost all requests are answered *within $b$ steps.*
Parity and finitary parity

**Parity:**
Almost all requests are answered.

**Finitary parity:**
There exists a bound $b$, s.t. almost all requests are answered within $b$ steps.

Eve wins for the parity condition,
but loses for the finitary parity condition!
Parity and finitary parity

**Parity:**
Almost all requests are answered.

**Finitary parity:**
There exists a bound \( b \), s.t. almost all requests are answered *within* \( b \) steps.

- Both players have positional winning strategies.
- Deciding the winner is in \( \text{NP} \cap \text{coNP} \).

- Eve has positional winning strategies.
- Adam needs infinite memory.
- Deciding the winner is in \( \text{PTIME} \).
Cost-parity games

Parity Finitary parity

???
Cost-parity games

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Finitary parity:
There exists a bound $b$, s.t. almost all requests are answered within $b$ steps.
Cost-parity games

**Parity:**
Almost all requests are answered.

**Finitary parity:**
There exists a bound $b$, s.t.
almost all requests are answered within $b$ steps.

**Cost-parity:**
There exists a bound $b$, s.t.
almost all requests are answered with cost at most $b$. 
Costs

Parity game

Finitary parity game
Objective: **strategy optimization**

Assume $\sigma$ is a winning strategy.
How to construct a memoryless winning strategy $\sigma'$ from $\sigma$?
Positional determinacy for Eve

Objective: **strategy optimization**

Assume $\sigma$ is a winning strategy.
How to construct a memoryless winning strategy $\sigma'$ from $\sigma$?

Tool: **scoring functions**

“à la Müller and Schupp” past-oriented proof
A general framework

Consider a winning strategy \( \sigma : V^* \rightarrow V \).

Define a scoring function \( S_c : V^* \rightarrow (S, \leq) \) satisfying:
A general framework

Consider a winning strategy $\sigma : V^* \rightarrow V$.

Define a scoring function $Sc : V^* \rightarrow (S, \leq)$ satisfying:

1. $(S, \leq)$ is a total order and $Sc$ is a congruence:
   - if $Sc(w) \leq Sc(w')$, then $Sc(w \cdot v) \leq Sc(w' \cdot v)$. 

A general framework

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2. If there exists a bound $b$ such that the scores of all prefixes of a play $\rho$ are bounded by $b$, then $\rho$ is winning.
A general framework

Consider a winning strategy \( \sigma : V^* \rightarrow V \).

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1. \((S, \leq)\) is a total order and \(\text{Sc}\) is a congruence:
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2. If there exists a bound \(b\) such that the scores of all prefixes of a play \(\rho\) are bounded by \(b\), then \(\rho\) is winning.

3. Assume a play \(\rho\) is consistent with \(\sigma\), then the scores of all prefixes of \(\rho\) are (uniformly) bounded.
A general framework

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2. If there exists a bound $b$ such that the scores of all prefixes of a play $\rho$ are bounded by $b$, then $\rho$ is winning.

3. Assume a play $\rho$ is consistent with $\sigma$, then the scores of all prefixes of $\rho$ are (uniformly) bounded.

Construct a memoryless strategy $\sigma'$:

“play according to $\sigma$ assuming the worst play prefix”.
Deciding the winner in cost-parity games

\( n \): number of vertices
\( m \): number of edges
\( d \): number of colors

Theorem

*Given a parity games solver of complexity* \( T(n, m, d) \), *there exists a cost-parity games solver of complexity*

\[
O(n \cdot T(n \cdot d, m \cdot d, d + 2))
\]
## Results

<table>
<thead>
<tr>
<th>winning condition</th>
<th>complexity</th>
<th>Eve</th>
<th>Adam</th>
</tr>
</thead>
<tbody>
<tr>
<td>parity</td>
<td>$\mathsf{NP} \cap \mathsf{coNP}$</td>
<td>memoryless</td>
<td>memoryless</td>
</tr>
<tr>
<td>finitary parity</td>
<td>$\mathsf{PTIME}$</td>
<td>memoryless</td>
<td>infinite</td>
</tr>
<tr>
<td>cost-parity</td>
<td>$\mathsf{NP} \cap \mathsf{coNP}$</td>
<td>memoryless</td>
<td>infinite</td>
</tr>
<tr>
<td>Streett</td>
<td>coNP-complete</td>
<td>finite</td>
<td>memoryless</td>
</tr>
<tr>
<td>finitary Streett</td>
<td>EXPTIME-complete</td>
<td>finite</td>
<td>infinite</td>
</tr>
<tr>
<td>cost-Streett</td>
<td>EXPTIME-complete</td>
<td>finite</td>
<td>infinite</td>
</tr>
</tbody>
</table>
Towards $\omega B$-conditions

- Parity
- Finitary parity
- Cost-parity
- ???