The surprizing complexity of generalized reachability games

GAMES’2010 workshop

Nathanaël Fijalkow ¹,² & Florian Horn ¹

LIAFA
CNRS & Université Denis Diderot - Paris 7, France
florian.horn@liafa.jussieu.fr

ÉNS Cachan
École Normale Supérieure de Cachan, France
nathanael.fijalkow@gmail.com

September 20th, 2010
Outline

1 Generalized reachability games
   - Games
   - Reachability
   - Generalized

2 Complexity
   - PSPACE-hardness: encoding QBF
   - Memory requirements
Outline

1. Generalized reachability games
   - Games
     - Reachability
     - Generalized

2. Complexity
   - PSPACE-hardness: encoding QBF
   - Memory requirements
Players

Two players: Eve and Adam.
Playing
Generalized reachability games

Games

Playing
Generalized reachability games

Games

Playing
Generalized reachability games

Games

Playing
Generalized reachability games

Playing
Generalized reachability games

Playing
Outline

1. Generalized reachability games
   - Games
   - Reachability
   - Generalized

2. Complexity
   - PSPACE-hardness: encoding QBF
   - Memory requirements
Generalized reachability games

Reachability

Solving reachability objectives

Given $F \subseteq Q$
Solving reachability objectives

Given $F \subseteq Q$
Solving reachability objectives

Given $F \subseteq Q$
Solving reachability objectives

Given $F \subseteq Q$
Outline

1. Generalized reachability games
   - Games
   - Reachability
   - Generalized

2. Complexity
   - PSPACE-hardness: encoding QBF
   - Memory requirements
Generalized reachability games

Generalized reachability objectives

- Reachability objectives: given $F \subseteq Q$, reach at least one vertex in $F$;
- Generalized reachability objectives: given $F_1, F_2, \ldots, F_p \subseteq Q$, reach at least one vertex in each $F_i$. 
Example

Given $F_1$ and $F_2 \subseteq Q$
Example

Given $F_1$ and $F_2 \subseteq Q$
Example

Given $F_1$ and $F_2 \subseteq Q$
Example

Given $F_1$ and $F_2 \subseteq Q$
Outline

1 Generalized reachability games
   - Games
   - Reachability
   - Generalized

2 Complexity
   - PSPACE-hardness: encoding QBF
   - Memory requirements
Outline

1. Generalized reachability games
   - Games
   - Reachability
   - Generalized

2. Complexity
   - PSPACE-hardness: encoding QBF
   - Memory requirements
Generalized reachability games

PSPACE-hardness: encoding QBF

Reduction from QBF to generalized reachability games

\[ \phi = \forall x \exists y \forall z (x \lor \neg y) \land (\neg y \lor z) \]
Reduction from QBF to generalized reachability games

\[ \phi = \forall x \exists y \forall z \ (x \lor \neg y) \land (\neg y \lor z) \]

\[ F_1 = \{x, \neg y\} \quad F_2 = \{\neg y, z\} \]
PSPACE-hardness: encoding QBF

Reduction from QBF to generalized reachability games

$$\phi = \forall x \exists y \forall z \ (x \lor \neg y) \land (\neg y \lor z)$$

$$F_1 = \{x, \neg y\} \quad F_2 = \{\neg y, z\}$$

Note that the number of literals in a clause is the size of the corresponding reachability set.
Results

Theorem (Lower bounds for generalized reachability games)

- Solving two players generalized reachability games is PSPACE-hard;
- Solving one player (Eve) generalized reachability games is NP-hard.
Theorem (Lower bounds for generalized reachability games)

- Solving two players generalized reachability games is PSPACE-hard;
- Solving one player (Eve) generalized reachability games is NP-hard.

Special cases where reachability sets have size less than 3 might be easier...
An easier case

Lemma (A polynomial special case)

Solving two players generalized reachability games where reachability sets are singletons is in P.
An easier case

Lemma (A polynomial special case)

Solving two players generalized reachability games where reachability sets are singletons is in P.

Why bigger reachability sets is harder to handle?

\[ F_1 = \{v_1, u_1\} \]
\[ F_2 = \{v_2, u_2\} \]
Generalized reachability games

PSPACE-hardness: encoding QBF

An easier case

Lemma (A polynomial special case)

Solving two players generalized reachability games where reachability sets are singletons is in P.

If reachability sets are singletons, then Eve can predict the objectives vertices appearance order.
Outline

1. Generalized reachability games
   - Games
   - Reachability
   - Generalized

2. Complexity
   - PSPACE-hardness: encoding QBF
   - Memory requirements
Exponential lower bound for Eve, reachability sets of size 2
Florian’s piece of art; exponential lower bound for Eve
Conclusion and further work

- Conjunctions of easy objectives may be way harder to solve;
- Open case: reachability sets of size 2.
The end.

Thank for your attention!