Finitary Languages
Presentation for LATA 2011

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Introduction: system specification

- non-terminating (e.g. web server);
- discrete time;
- non-deterministic.
Introduction: system specification

- non-terminating;
- discrete time;
- non-deterministic.

- a finite alphabet $\Sigma$ represent propositions; (e.g. “available”, “waiting”, “critical error”)
- runs are infinite words $w = w_0 \cdot w_1 \ldots w_n \ldots \in \Sigma^\omega$;
- specification given as a language $L \subseteq \Sigma^\omega$. 
Introduction: system specification
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queue strategy
Introduction: system specification

- $\omega$-regular language: safety + liveness;
- liveness properties: “something good happens eventually”.
Classical liveness properties

A first example, Büchi:

- a given set of propositions appears infinitely often;
  (e.g. “job done”)
Classical liveness properties

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A second example, Streett (fairness):
- propositions are either requests $R_i$ or grants $G_i$;
- if $R_i$ is requested infinitely often, then it is serviced ($G_i$) infinitely often.
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(special case: parity)
Outline

1 Motivations

2 Characterizations

3 Expressions
A drawback of classical $\omega$-regular specifications

stack strategy
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Streett specification: for $i \in \{1, 2\}$, if $R_i$ is requested infinitely often, then it is serviced infinitely often.
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Satisfied, but the “service time” may grow unbounded!
A stronger formulation of liveness: finitary liveness [AH94]

Intuitively: there exists an unknown, fixed bound $b$ such that good things happen within $b$ transitions.
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unknown: retain independence from granularity.
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It can be expressed as a finitary operator on languages:

$$\text{fin}(L) = \bigcup \{M \mid M \text{ closed and } \omega\text{-regular, } M \subseteq L\}$$
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- closed: involves Cantor topology;
- $\omega$-regular: involves $\omega$-regularity;
- restriction operator: $\text{fin}(L) \subseteq L$. 
Back to the example

Finitary Streett specification: there exists a bound $b$, such that in the limit, for $i \in \{1, 2\}$, if $R_i$ is requested, then it is serviced within $b$ transitions.
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Satisfied!
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2. Characterizations
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Describing classical finitary objectives: Büchi

Let $F \subseteq \Sigma,$

$$\text{Büchi}(F) = \{ w \mid \text{Inf}(w) \cap F \neq \emptyset \}$$

$\text{Inf}(w)$ is the set of propositions that appear infinitely often in $w.$
Describing classical finitary objectives: Büchi

Let $F \subseteq \Sigma$, 

$$\text{Büchi}(F) = \{ w \mid \text{Inf}(w) \cap F \neq \emptyset \}$$

$$\text{next}_k(w, F) = \inf\{ k' - k \mid k' \geq k, w_{k'} \in F \}$$
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$$w = v_0 \ldots v_k \underbrace{v_{k+1} \ldots v_{k'}}_{\not\in F} \underbrace{v_{k'}_{-1} \ldots v'_{k'}}_{\in F}$$

waiting time from the $k^{th}$ position.
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Lemma

$$\text{fin}(\text{Büchi}(F)) = \{w \mid \limsup_k \text{next}_k(w, F) < \infty\}$$
Topological classification in Borel hierarchy

Theorem
fin(\text{Büchi}(F)), \text{fin}(\text{Parity}(p)) \text{ and } \text{fin}(\text{Streett}(R, G)) \text{ are } \Sigma_2\text{-complete.}
Automata-theoretic expressive power

We consider automata over infinite words, whose acceptance conditions are finitary Büchi, finitary parity or finitary Streett.
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$$\{D, N\} \cdot \{\varepsilon \text{ (classical)}\} \cdot \{F \text{ (finitary)}\} \cdot \{B \text{ (Büchi)}, P \text{ (parity)}, S \text{ (Streett)}\}$$
Figure: Expressive power classification

- \( \omega \text{-reg} \)
- \( DB \)
- \( DFB \)
- \( DFP = DFS \)
- \( NFB = NFP = NFS \)
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Regular and $\omega$-regular expressions

Regular expressions defines regular languages over finite words:

$$L := \emptyset | \varepsilon | \sigma | L \cdot L | L^* | L + L; \quad \sigma \in \Sigma$$

concatenation  star  union

$\omega$-regular languages are finite union of $L_1 \cdot L_2^\omega$, where $L_1$ and $L_2$ are regular languages over finite words.
The bound operator $B$ [BC06]

$$L^\omega = \{ u_0 \cdot u_1 \cdot \ldots \cdot u_k \ldots \mid u_0, u_1, \ldots, u_k, \ldots \in L \}$$
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**Example:** $(a^* \cdot b)^\omega$ expresses “infinitely many $b$’s”.
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$$L^\omega = \{u_0 \cdot u_1 \cdot \ldots \cdot u_k \ldots \mid u_0, u_1, \ldots, u_k, \ldots \in L\}$$

**Example:** $(a^* \cdot b)^\omega$ expresses “infinitely many $b$’s”.

**Example:** $(a^B \cdot b)^\omega$ expresses “infinitely many $b$’s with an upper bound on the length of $a$’s blocks”.
Star-free $\omega B$-regular expressions

$B$-regular languages are described by the grammar:

$$M := \emptyset \mid \varepsilon \mid \sigma \mid M \cdot M \mid M^* \mid M^B \mid M + M; \quad \sigma \in \Sigma$$

$\omega B$-regular languages are finite union of $L \cdot M^\omega$, where

- $L$ is a regular language over finite words;
- $M$ is a $B$-regular language over infinite words.
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- $L$ is a regular language over finite words;
- $M$ is a **star-free** $B$-regular language over infinite words.

“no star operator under the $\omega$-operator”.
Equivalence

**Theorem**

*NFB (non-deterministic finitary Büchi automata) has exactly the same expressive power as star-free $\omega B$-regular expressions.*
First example: $c^* \cdot (a^B \cdot b)^{\omega}$ is a star-free $\omega B$-regular expression,
Examples

First example: $c^* \cdot (a^B \cdot b)\omega$ is a star-free $\omega B$-regular expression, it expresses “a finite number of $c$’s followed by an infinite word over alphabet $\{a, b\}$, with infinitely many $b$’s and an upper bound on the length of $a$’s blocks”.
Examples

First example: \(c^* \cdot (a^B \cdot b)^\omega\) is a star-free \(\omega B\)-regular expression, it expresses “a finite number of \(c\)’s followed by an infinite word over alphabet \(\{a, b\}\), with infinitely many \(b\)’s and an upper bound on the length of \(a\)’s blocks”.

Second example: \((a^B \cdot b \cdot (a^* \cdot b)^*)^\omega\) is not a star-free \(\omega B\)-regular expression,
Examples

First example: \( c^* \cdot (a^B \cdot b)^\omega \) is a star-free \( \omega B \)-regular expression, it expresses “a finite number of \( c \)’s followed by an infinite word over alphabet \( \{a, b\} \), with infinitely many \( b \)’s and an upper bound on the length of \( a \)’s blocks”.

Second example: \( (a^B \cdot b \cdot (a^* \cdot b)^*)^\omega \) is not a star-free \( \omega B \)-regular expression, it expresses “words of the form \( a^{n_0} \cdot b \cdot a^{n_1} \cdot b \ldots \) such that \( \lim \inf n_i < \infty \)”.
Conclusion

- finitary objectives is a refinement for specification purposes;
- for $\omega$-regular languages, topological, logical and automata-theoretic studies are well-known;
- for finitary languages, all were missing; we established:
  - topological classification;
  - automata-theoretic characterization, comparison to $\omega$-regular languages, closure properties;
  - characterization using by $\omega B$-regular expressions.
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Future work:

- games (work in progress);
- a finitary logic, Myhill-Nerode equivalence relations, ...
Bibliography


The end

Thank you for your attention!