Deciding the value 1 problem for probabilistic leaktight automata

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Probabilistic automata (Rabin, 1963)

\[ \mathbb{P}_A : A^* \rightarrow [0, 1] \]
The value 1 problem

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“are there words accepted by $\mathcal{A}$ with arbitrarily high probability?”
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“$\text{val}(A) = 1$.”
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Theorem (Gimbert, Oualhadj, 2010)

*The value 1 problem is undecidable.*
Our objective

Decide the value 1 problem for a subclass of probabilistic automata, by algebraic and non-numerical means.
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- **algebraic**: focus on the automaton structure,
- **non-numerical**: abstract away the values.
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- **algebraic**: focus on the automaton structure,
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Hence we consider non-deterministic automata:
Weighted automata using algebra (Schützenberger)

\[ \langle a \rangle = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \langle b \rangle = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ I \cdot \langle abba \rangle \cdot F = 1 \quad \text{if and only if} \quad \mathbb{P}_A(abba) > 0 \]
The stabilization operation $\#$

\[
\langle a \rangle = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

In $\langle a \rangle$, the state 1 is transient and the state 2 is recurrent.
The stabilization operation \(\#$\)

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\langle a \rangle = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \langle a^# \rangle = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}
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"$M^\# = \lim_{n} M^n$"
A saturation algorithm

Compute a monoid inside the finite monoid $\mathcal{M}_{\mathbb{Q} \times \mathbb{Q}}(\{0, 1\}, +, \times)$.

- Compute $\langle a \rangle$ for $a \in A$
- Close under product and stabilization.
A saturation algorithm

Compute a monoid inside the finite monoid $\mathcal{M}_{Q \times Q}(\{0, 1\}, +, \times)$.

- Compute $\langle a \rangle$ for $a \in A$
- Close under product and stabilization.
- If there exists a matrix $M$ such that

$$\forall t \in Q, \quad M(s_0, t) = 1 \implies t \in F$$

then “$A$ has value 1”, otherwise “$A$ does not have value 1”.
An example
An example

\[
\begin{array}{c}
\langle a \rangle \\
0 \rightarrow 1 \\
0 \rightarrow 1 \\
\end{array}
\]

\[
\begin{array}{c}
0 \rightarrow 1 \rightarrow F \\
0 \rightarrow 1 \rightarrow F \\
\end{array}
\]

\[
\begin{array}{c}
0 \rightarrow 1 \rightarrow F \\
0 \rightarrow 1 \rightarrow F \\
\end{array}
\]

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\]
An example
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\[ \langle a^\# \rangle \]

\[ \langle b \rangle \]

\[ \langle a^\# \cdot b \rangle \]
An example
An example

\[
\langle a \rangle \\
\langle a^\# \rangle \\
\langle b \rangle \\
\langle a^\# \cdot b \rangle \\
\langle (a^\# \cdot b)^\# \rangle
\]
Correct, but not complete

Theorem (Correctness)

If the algorithm answers “A has value 1” then A has value 1.
Correct, but not complete

**Theorem (Correctness)**

*If the algorithm answers “A has value 1” then A has value 1.*

But the value 1 problem is undecidable, so the converse cannot hold!
Completeness in the absence of leaks

Definition
An automaton $A$ is leaktight if it has no leak.

Theorem (Completeness)
If $A$ is leaktight and has value 1,

then the algorithm answers “$A$ has value 1”.

The proof relies on Simon’s factorization forest theorem.
A leak

Diagram:

- Circles labeled 1, 2, 3
- Arrows labeled 'a', 'b', and 'a, b'
-Node 1 has a loop labeled 'a'
- Node 2 has a loop labeled 'b'
- Node 3 has a loop labeled 'a' and an arrow labeled 'b' pointing to node 1
A leak

\[ a, b, a \cdot (a^{\#} \cdot b) \]
There is a leak from 1 to 3.
Conclusion and perspectives

- We defined a subclass of probabilistic automata which subsumes all subclasses of probabilistic automata whose value 1 problem is known to be decidable,
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- What does this algorithm actually compute?
Conclusion and perspectives

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- We defined an algebraic algorithm for the value 1 problem and proved its completeness for the class of leaktight automata.

- What does this algorithm actually compute?

- Can we use similar algorithms for other semirings?