Topological, Automata-Theoretic and Logical Characterizations of Finitary Languages
Presentation for YR-CONCUR 2010

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Introduction: system specification

- non-terminating (e.g. web server);
- discrete time;
- non-deterministic.
Introduction: system specification

- non-terminating
- discrete time
- non-deterministic

- a finite alphabet $\Sigma$ represent propositions (e.g. ”available”, ”waiting”, ”critical error”)
Introduction: system specification

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- discrete time;
- non-deterministic.

- a finite alphabet $\Sigma$ represent propositions;
- runs are infinite words: $w = w_0 \cdot w_1 \ldots w_n \ldots \in \Sigma^\omega$;
- specification given as a language $L \subseteq \Sigma^\omega$;
Introduction: system specification

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- discrete time;
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- a finite alphabet $\Sigma$ represent propositions;
- runs are infinite words: $w = w_0 \cdot w_1 \ldots w_n \ldots \in \Sigma^\omega$;
- specification given as a language $L \subseteq \Sigma^\omega$;
- $\omega$-regular language: safety + liveness;
- liveness properties: ”something good happens eventually”. 
Outline

Motivations

Characterizations

Expressions
Classical liveness properties

A first example, Büchi:

- a given set of propositions appears infinitely often;
  (e.g. "job done")
Classical liveness properties

A first example, Büchi:
- a given set of propositions appears infinitely often;

A second example, parity:
- integers are assigned to propositions, representing a priority;
- along an execution, some integers appear infinitely often;
- parity specifies that the least priority appearing infinitely often is even.
A drawback of classical $\omega$-regular specifications
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Specification: Büchi with $F = \{v_{2k} \mid k \in \mathbb{N}\}$. 
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Satisfied, but the time until something good happens may grow unbounded!
A stronger formulation of liveness: finitary liveness [AH94]

Intuitively: there exists an unknown, fixed bound $b$ such that something good happens within $b$ transitions.
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unknown: retain independence from granularity.
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$$\text{fin}(L) = \bigcup \{M \mid M \text{ closed and } \omega\text{-regular}, M \subseteq L\}$$
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- closed: involves Cantor topology;
- $\omega$-regular: involves $\omega$-regularity;
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- Closed: involves Cantor topology;
- $\omega$-regular: involves $\omega$-regularity;
- Restriction operator: $\text{fin}(L) \subseteq L$. 
Back to the example

Finitary Büchi: \( F = \{ \nu_{2k} \mid k \in \mathbb{N} \} \)

Not satisfied!
Outline

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Describing classical finitary objectives: Büchi

Let $F \subseteq \Sigma$,

$$\text{Büchi}(F) = \{w \mid \text{Inf}(w) \cap F \neq \emptyset\}$$

Inf$(w)$ is the set of propositions that appear infinitely often in $w$. 
Describing classical finitary objectives: Büchi

Let $F \subseteq \Sigma$, 

$$\text{Büchi}(F) = \{w \mid \text{Inf}(w) \cap F \neq \emptyset\}$$

We define:

$$\text{next}_k(w, F) = \inf\{k' - k \mid k' \geq k, w_{k'} \in F\}$$
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\[
w = v_0 \ldots v_k \underbrace{v_{k+1} \ldots v_{k'}}_{\notin F} \underbrace{v_{k'} - 1}_{\in F}
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Lemma

$$\text{fin}(\text{Büchi}(F)) = \{ w | \limsup_k \text{next}_k(w, F) < \infty \}$$
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Lemma

$$\text{fin}(\text{Büchi}(F)) = \{w \mid \exists B \in \mathbb{N}, \exists n \in \mathbb{N}, \forall k \geq n, \text{next}_k(w, F) \leq B\}$$
Describing classical finitary objectives: parity

Let $p : \Sigma \to \mathbb{N}$ a priority function,

$$\text{Parity}(p) = \{ w | \min(p(\text{Inf}(w))) \text{ is even} \}$$
Describing classical finitary objectives: parity

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\text{Parity}(p) = \{ w \mid \min(p(\text{Inf}(w))) \text{ is even} \}
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We define:

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\text{dist}_k(w, p) = \inf \{ k' - k \mid k' \geq k, p(w_{k'}) \text{ even and } p(w_{k'}) \leq p(w_k) \}
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**Lemma**

\[
\text{fin(Parity}(p)) = \{ w \mid \limsup_{k} \text{dist}_k(w, p) < \infty \}
\]
Topological characterization

Theorem

Büchi($F$) is $\Pi_2$-complete.

Parity lies in the boolean closure of $\Sigma_2$ and $\Pi_2$. 
Topological characterization

Theorem

\[ \text{Büchi}(F) \text{ is } \Pi_2\text{-complete.} \]

Parity lies in the boolean closure of \( \Sigma_2 \) and \( \Pi_2 \).

Theorem

1. For all \( p : \Sigma \rightarrow \mathbb{N} \), we have \( \text{fin}(\text{Parity}(p)) \in \Sigma_2 \).

2. For all \( \emptyset \subset F \subset \Sigma \), we have that \( \text{fin}(\text{Büchi}(F)) \) is \( \Sigma_2\)-complete.

3. There exists \( p : \Sigma \rightarrow \mathbb{N} \) such that \( \text{fin}(\text{Parity}(p)) \) is \( \Sigma_2\)-complete.
Automata-theoretic characterization

We consider automata on infinite words, whose acceptance conditions are finitary Büchi and finitary parity.
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\[
\{ \begin{array}{c} D \\ N \end{array} \} \cdot \{ \begin{array}{c} \varepsilon \\ F(\text{finitary}) \end{array} \} \cdot \{ \begin{array}{c} B(\text{Büchi}) \\ P(\text{parity}) \end{array} \}
\]
$NFB = NFP$
Outline

Motivations

Characterizations

Expressions
ω-regular expressions

Regular expressions defines regular languages over finite words:

\[ L := \emptyset \mid \varepsilon \mid \sigma \mid L \cdot L \mid L^* \mid L + L; \quad \sigma \in \Sigma \]

ω-regular languages are finite union of \( L \cdot L'\omega \), where \( L \) and \( L' \) are regular languages of finite words.
The bound operator $B$ [BC06]

\[ L^\omega = \{u_0 \cdot u_1 \cdots u_k \cdots \mid u_0, u_1, \ldots, u_k, \ldots \in L\} \]
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\[ L^B = \{ u_0 \cdot u_1 \cdot \ldots \cdot u_k \ldots \mid u_0, u_1, \ldots, u_k, \ldots \in L \text{ and } (|u_n|)_{n \in \mathbb{N}} \text{ is bounded} \} \]
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(complete definitions require the use of infinite sequences of finite words)
Star-free $\omega B$-regular expressions

Star-free $\omega B$-regular languages are finite union of $L \cdot M^\omega$, where

- $L$ is a regular language over finite words;
- $M$ is a star-free $B$-regular language over infinite words.
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Star-free $B$-regular languages are described by the grammar:

$$M ::= \emptyset \mid \varepsilon \mid \sigma \mid M \cdot M \mid M^B \mid M + M; \quad \sigma \in \Sigma$$
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Theorem

NFP (non-deterministic finitary parity) has exactly the same expressive power as star-free $\omega B$-regular expressions.
Conclusion

- finitary objectives is a refinement for specification purposes;
- for $\omega$-regular languages, topological, logical and automata-theoretic characterizations were known;
- for finitary languages, all were missing; we established:
  - topological characterization;
  - automata-theoretic characterization, comparison to $\omega$-regular languages, closure properties;
  - logical characterization using by $\omega B$-regular expressions.
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Future work:

- algorithmic issues: equivalence with $\omega B$-regular expressions, emptiness problem of finitary automata;
- a tool for finitary objectives...
R. Alur and T.A. Henzinger.  
Finitary fairness.  

Mikolaj Bojańczyk and Thomas Colcombet.  
Bounds in $\omega$-regularity.  
The end

Thank you for your attention!